Capacity, number of ends and asymptotic planes in minimal submanifolds

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"On the Fundamental Tone of Minimal Submanifolds with Controlled Extrinsic Curvature". Potential Analysis, 2013.

Outline

Cheeger isoperimetric constant I

Given M^n a complete and non compact Riemannian manifold of dimension greater than 1 ($n \geq 2$) , the Cheeger isoperimetric constant is defined by this quotient

$$\mathcal{I}_\infty(M) := \inf_{\Omega \subset M} \frac{\mathrm{Vol}_{n-1}(\partial\Omega)}{\mathrm{Vol}_n(\Omega)}. \tag{1}$$

where Ω ranges over compact open subsets $\Omega \subset M$ with smooth boundaries $\partial\Omega$.

Cheeger constant examples

- $\mathcal{I}_\infty(\mathbb{R}^n) = 0$.
- $\mathcal{I}_\infty(\mathbb{H}^n(b)) = (n-1)\sqrt{-b}$.

Fundamental tone I

The fundamental tone $\lambda^*(M)$ of a smooth Riemannian manifold M is defined by the infimum of the quotient between the squared norm of the gradient and the squared norm of functions

$$\lambda^*(M) = \inf_{f \in L^2_{1,0}(M) \setminus \{0\}} \left\{ \frac{\int_M |\nabla f|^2 d\mu}{\int_M f^2 d\mu} \right\} \tag{2}$$

where the functions ranges in $L^2_{1,0}(M)$, the completion of smooth functions with compact support $C_0^\infty(M)$ with respect to this norm $\|\phi\|^2 = \int_M \phi^2 d\mu + \int_M |\nabla \phi|^2 d\mu$

Theorem (Cheeger)

Let M be a complete non compact manifold, then the Cheeger isoperimetric constant is a bound for the fundamental tone

$$\lambda^*(M) \geq \frac{\mathcal{I}_\infty(M)^2}{4} \tag{3}$$

And for minimal submanifolds of the Hyperbolic space

Corollary (S-T Yau, McKean, Chavel)

Let $M^n \hookrightarrow \mathbb{H}^m(b)$ be a complete, minimally immersed submanifold of $\mathbb{H}^m(b)$, then the Cheeger constant (and so the fundamental tone) are bounded from below by the following expressions

$$\begin{aligned} \mathcal{I}_\infty(M) \geq & (n-1)\sqrt{-b}, \\ \lambda^*(M) \geq & \frac{-(n-1)^2 b}{4}. \end{aligned} \tag{4}$$

Corollary

Let $M^n \hookrightarrow N$ be a complete, minimally immersed submanifold of a Cartan-Hadamard manifold N (simply connected with sectional curvatures K_N bounded above by $K_N \leq b \leq 0$), then the Cheeger constant (and so the fundamental tone) are bounded from below by the following expressions

$$\begin{aligned} \mathcal{I}_\infty(M) \geq & (n-1)\sqrt{-b}, \\ \lambda^*(M) \geq & \frac{-(n-1)^2 b}{4}. \end{aligned} \tag{5}$$

proof I

By the expression of the Hessian for submanifolds and the Hessian comparisons given by Greene-Wu for the extrinsic distance function

$$\Delta^M r \geq (n-1)\cot_b(r),$$

being

$$\cot_b(r) = \begin{cases} \frac{1}{r} \text{ if } b = 0, \\ \sqrt{-b}\ \text{cotanh}(\sqrt{-b}r) \text{ if } b < 0 \end{cases}$$

Therefore,

$$\Delta^M r \geq (n-1)\sqrt{-b},$$

Integrating on $\Omega \subset M$

$$\int_\Omega \Delta^M r dV \geq (n-1)\sqrt{-b}\,\text{Vol}_n(\Omega),$$

proof II

By the divergence theorem

$$\int_{\partial\Omega} \langle \nabla r, \nu \rangle dA \geq (m-1)\sqrt{-b}\,\mathrm{Vol}_n(\Omega),$$

Hence,

$$\mathrm{Vol}_{n-1}(\partial\Omega) \geq (n-1)\sqrt{-b}\,\mathrm{Vol}_n(\Omega),$$

¿what was known? I

Theorem A. Candel, Transactions AMS, 2007

Let M be a complete simply connected stable minimal surface in the hyperbolic space $\mathbb{H}^3(-1)$, then

$$\frac{1}{4} \leq \lambda^*(M) \leq \frac{3}{4} \quad .$$

Theorem A. Candel, Transactions AMS, 2007

The fundamental tone of the minimal catenoids (given in Do Carmo - Dajczer, Rotation hypersurfaces in spaces of constant curvature. Trans. Amer. Math. Soc. ,1983) in the hyperbolic space $\mathbb{H}^3(-1)$ is

$$\lambda^*(M) = \frac{1}{4} \quad .$$

¿what was known? II

The minimal catenoids satisfy

$$\int_M |A|^2 d\mu < \infty \quad . \tag{6}$$

Theorem K. Seo, J. Korean Math. Soc., 2011

Let M^n be a complete stable minimal hypersurface in $\mathbb{H}^{n+1}(-1)$ with $\int_M |A|^2 d\mu < \infty$. Then we have

$$\frac{(n-1)^2}{4} \leq \lambda^*(M) \leq n^2 \quad . \tag{7}$$

What I thought?

Corollary V. Gimeno (REAG-ICMAT 2012)

Given a complete submanifold $M^n \hookrightarrow N$ properly and minimaly immersed in a Cartan Hadamard N ambient manifold with sectional curvatures K_N bounded above $K_N \leq b \leq 0$, suppose moreover that the immersion has finite volume growth. Then, we obtain the following upper bound for the fundamental tone of the submanifold

$$\lambda^*(M) \leq 4\mathcal{I}_\infty^2(M) = -4(n-1)^2 b. \tag{8}$$

Extrinsic distance and extrinsic balls I

In order to understand the volume growth we need some previous concepts as the **extrinsic distance** and the **extrinsic balls**.

- The extrinsic distance is the restriction from the distance function in the ambient manifold to the submanifold.
- The extrinsic ball is the sublevel set defined by the extrinsic distance function.

Definition of extrinsic distance

Let $\varphi : M^n \to N$ be a complete, and proper immersion. Given two points $o, p \in M$, the extrinsic distance from o to p is

$$r_o(p) := \mathrm{dist}^N(\varphi(o), \varphi(p)) \tag{9}$$

where dist^N denotes the geodesic distance in N.

Extrinsic distance and extrinsic balls II

Definition of extrinsic ball

The extrinsic ball $D_R(o)$ of radius R centered in $o \in M$ is the set of points whose extrinsic distance to o is at most R

$$D_R(o) := \{p \in M\,;\, r_o(p) < R\} \tag{10}$$

Where $r_o(p)$ is the extrinsic distance form o to p.

Volume growth I

With these extrinsic balls we can define the volume comparison quotient

$$\mathcal{Q}_b(R) := \frac{\operatorname{Vol}(D_R)}{\operatorname{Vol}(B_R^{b,n})}, \tag{11}$$

where $B_R^{b,n}$ stands for the geodesic ball of radius R in $\mathbb{K}^n(b)$.

Theorem volume growth (V. Palmer PLMS 1999)

Let $\varphi : M \to N$ be a proper and minimal immersion into a Cartan-Hadamard ambient manifold N ($K_N \leq b \leq 0$), then the volume comparison quotient $\mathcal{Q}_b(R)$ is a non decreasing function on R.

From the previous theorem we can define

Definition

Let $\varphi : M \to N$ be a proper and minimal immersion into a Cartan-Hadamard ambient manifold N ($K_N \leq b \leq 0$). M has **finite volume growth** if and only if

$$\sup_R \mathcal{Q}_b(R) = \lim_{R\to\infty} \mathcal{Q}(R) < \infty.$$

the volume comparison quotient has a finite upper bound.

Relation volume growth - Second fundamental form

Theorem, V Gimeno V. Palmer, JGEA 2013

Let $M^n \to \mathbb{H}^m(b)$ be a proper and complete minimal immersion $n > 2$. Suppose that

$$\|A\| \leq \frac{\delta(r)}{e^{2\sqrt{-b}r}} \quad \text{, such that } \delta \to 0 \text{ when } r \to \infty.$$

Then :

1. M has finite topological type.
2. M has finite volume growth.
3. $$\sup_R \mathcal{Q}_b(R) \leq \mathcal{E}(M) = \text{ ends of } M.$$

Theorem, V. Gimeno V. Palmer, Israel J. of Math., 2013

Let $M^2 \to \mathbb{H}^m(b)$ be a complete minimal immersion, suppose that

$$\int_{M^2} \|A\|^2 dV < \infty$$

then

$$\sup_R \mathcal{Q}_b(R) \leq \frac{1}{4\pi} \int_{M^2} \|A\|^2 dV + \chi(M^2) \quad .$$

Topological Ends

Let M be a non-compact connected manifold. We define an equivalence relation in the set $\mathcal{A} = \{\alpha : [0, \infty) \to M | \alpha \text{ is a proper arc}\}$, by setting $\alpha_1 \sim \alpha_2$ if for every compact set $C \subset M$, α_1, α_2 lie eventually in the same component of $M - C$.

Definition

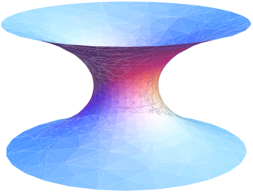
Each equivalence class in $\mathcal{E}(M) = \mathcal{A}/\sim$ is called an *end* of M.

Counting ends I

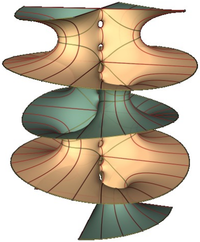
Given an exhaution by compact sets $\{K_i\}$ of the manifold P ($K_i \subset K_{i+1}$ and $\cup_{i\in\mathbb{N}} K_i = P$) , the number of ends $\mathcal{E}(P)$ of P is the supremum of the number of connected components with non compact closure of $P - K_i$. (see Tkachev's paper Manuscripta Math. 82, 1994 and Anderson's I.E.H.S. preprint 1984)

Some examples

1. The number of ends of any compact space is zero.
2. The real line $\mathbb{R}$ has two ends.
3. If $n > 1$, then the Euclidean space $\mathbb{R}^n$ has only one end. This is because $\mathbb{R}^n \setminus F$ has only one unbounded component for any compact set F.
4. The catenoid has two ends



5. the periodic surface of Callahan-Hoffman-Meeks has infinitely many ends



What I know? I

Theorem, V. Gimeno V. Palmer, PAMS 2013

Let $\varphi : M^n \to N$ be a proper complete minimal immersion in a Cartan-Hadamart ambient manifold N $(K_N \leq b \leq 0)$. Suposse that the submanifold has finite volume growth,

$$\sup_R \mathcal{Q}(R) < \infty,$$

then

$$\mathcal{I}_\infty(M) = (n-1)\sqrt{-b}$$

What I know? II

Theorem, V. Gimeno, POTA 2013

Let $\varphi : M^n \to N$ be a proper complete minimal immersion in a Cartan-Hadamart ambient manifold N ($K_N \leq b \leq 0$). Suposse that the submanifold has finite volume growth,

$$\sup_R \mathcal{Q}(R) < \infty,$$

then

$$\lambda^*(M) = \frac{-(n-1)^2 b}{4}$$

Sketch of the proof I

For any $\Phi \in L^2_{1,0}(M) \setminus \{0\}$

$$\lambda^*(M) \leq \frac{\int_M |\nabla \Phi|^2 dV}{\int_M |\Phi|^2 dV}$$

Pick

$$\Phi : M \to \mathbb{R}; \quad \Phi = \phi_R \circ r.$$

$$\phi_R(t) = \begin{cases} \dfrac{sin\left(\frac{2\pi\left(t-\frac{R}{2}\right)}{R}\right)}{\mathrm{Vol}(S_t^b)^{\frac{1}{2}}} & \text{if } t \in [\frac{R}{2}, R] \\ 0 \text{ otherwise.} \end{cases}$$

Sketch of the proof II

By the Rayleigh quotient definition and the coarea formula

$$\lambda^*(M) \leq \frac{\int_M \langle \nabla\Phi, \nabla\Phi\rangle d\mu}{\int_M \Phi^2 d\mu} = \frac{\int_M (\phi')^2 \langle \nabla r_p, \nabla r_p\rangle d\mu}{\int_M \Phi^2 d\mu} \leq \frac{\int_M (\phi')^2 d\mu}{\int_M \Phi^2 d\mu}$$

$$= \frac{\int_0^R \left[\int_{\partial D_s} \frac{(\phi')^2}{|\nabla r|}\right] ds}{\int_0^R \left[\int_{\partial D_s} \frac{\phi^2}{|\nabla r|}\right] ds} = \frac{\int_{\frac{R}{2}}^R (\phi'(s))^2 \left[\int_{\partial D_s} \frac{1}{|\nabla r|}\right] ds}{\int_{\frac{R}{2}}^R \phi^2(s) \left[\int_{\partial D_s} \frac{1}{|\nabla r|}\right] ds} \tag{12}$$

$$= \frac{\int_{\frac{R}{2}}^R (\phi'(s))^2 \, (\mathrm{Vol}(D_s))' \, ds}{\int_{\frac{R}{2}}^R \phi^2(s) \, (\mathrm{Vol}(D_s))' \, ds} \quad .$$

From the definition of $\mathcal{Q}_b$ and taking into account that $\mathcal{Q}$ is a non-decreasing function

$$(\ln \mathcal{Q}_b(s))' = \frac{(\mathrm{Vol}\, D_s)'}{(\mathrm{Vol}\, D_s)} - \frac{\mathrm{Vol}(S_s^b)}{\mathrm{Vol}(B_s^b)} \geq 0 \quad . \tag{13}$$

Sketch of the proof III

So,

$$\mathcal{Q}_b(s)\operatorname{Vol}(S_s^b) \leq (\operatorname{Vol}(D_s))' \leq (\ln \mathcal{Q}_b(s))' \operatorname{Vol}(B_s^b)\mathcal{Q}_b(s)+\mathcal{Q}_b(s)\operatorname{Vol}(S_s^b) \quad . \tag{14}$$

Lemma

There exists an upper bound function $\Lambda : \mathbb{R}^+ \to \mathbb{R}^+$ *to*

$$\frac{\int_0^R (\phi')^2 \operatorname{Vol}(S_s^b) ds}{\int_0^R \phi^2 \operatorname{Vol}(S_s^b) ds} \leq \Lambda(R) \tag{15}$$

such that

$$\lim_{R\to\infty} \Lambda(R) = \frac{-(n-1)^2 b}{4} \tag{16}$$

Denoting now,

$$F(R) := \left(\frac{(m-1)^2}{4} \mathrm{Cot}_b(R/2)^2 + \frac{4\pi^2}{R^2} + \frac{2(m-1)\pi}{R} \mathrm{Cot}_b(R/2) \right)$$

$$\delta(R) := \int_{\frac{R}{2}}^{R} (\ln \mathcal{Q}(s))' \, ds,$$

$$\lambda^*(M) \leq \frac{\mathcal{Q}(R)}{\mathcal{Q}(\frac{R}{2})} \left[\frac{\mathrm{Vol}(B^b_R)}{\mathrm{Vol}(S^b_R)} \frac{4}{R} F(R)\delta(R) + \Lambda(R) \right] \tag{17}$$

Letting R tend to infinity and taking into account that

$$\begin{aligned}
\lim_{R\to\infty} F(R) &= -\frac{(n-1)^2 b}{4} \quad, \\
\lim_{R\to\infty} \delta(R) &= 0 \quad, \\
\lim_{R\to\infty} \frac{\mathrm{Vol}(B^b_R)}{\mathrm{Vol}(S^b_R)} \frac{4}{R} &= \begin{cases} \frac{4}{m-1} \text{ if } b = 0, \\ 0 \text{ if } b < 0. \end{cases} \\
\lim_{R\to\infty} \frac{\mathcal{Q}(R)}{\mathcal{Q}(\frac{R}{2})} &= 1 \quad .
\end{aligned} \tag{18}$$

An improvement? I

Theorem, S Ilias, B. Nelli, M. Soret, Arxiv aug 2013

Let $\varphi : M^n \to N$, N Cartan-Hadamard, if

$$\sup_R \mathcal{Q}_b(R) < \infty$$

then:

- $\mathcal{I}_\infty(M) \leq (m-1)\sqrt{-b}$
- if M is minimal, $\lambda^*(M) = \frac{-(m-1)^2 b}{4}$

They make use of the volume entropy μ_M of M

$$\mu_M := \limsup_{R\to\infty} \left(\frac{\ln(\mathrm{Vol}(D_R))}{R}\right) < \infty.$$

Since

$$\mathrm{Vol}(D_R) \leq \sup_R \mathcal{Q}_b(R)\,\mathrm{Vol}(B_R^b)$$

An improvement? II

,

$$\mu_M = \limsup_{R\to\infty} \left(\frac{\ln(\sup_R \mathcal{Q}_b(R))}{R} + \frac{\ln(\mathrm{Vol}(B_R^b))}{R} \right) < \infty.$$

Therefore,

$$\mu_M := \mu_{\mathbb{H}^n(b)}.$$

Independence on the volume growth

$$¿\ \sup_R \mathcal{Q}_b(R)\ ?$$

We only need its finiteness.

We have seen

$$\sup_R \mathcal{Q}_b(R) \sim \mathcal{E}(M)$$

There exists an other relation?

Outline

Capacity I

Given a compact set $K \subset M$ in a Riemannian manifold M and an open set $\Omega \subset M$ containing K, we call the couple (K, Ω) a *capacitor*. Each capacitor has capacity defined by

$$\mathrm{Cap}(K, \Omega) := \inf_{u} \int_{\Omega \setminus K} \|\nabla u\| d\mu \quad , \tag{19}$$

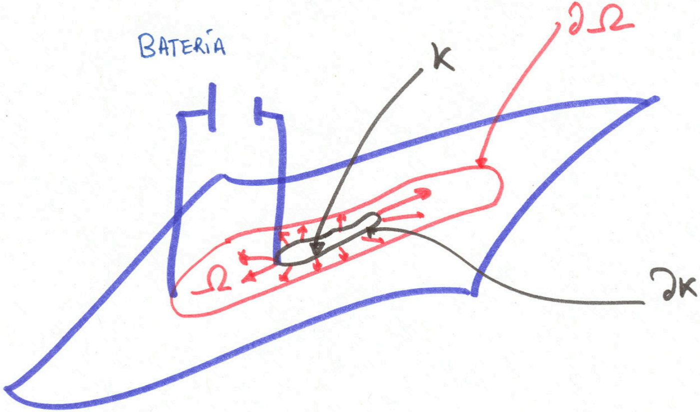
where the inf is taken over all Lipschitz functions u with compact support in Ω such that $u = 1$ on K.
When Ω is precompact, the infimum is attained for the function $u = \Psi$ which is the solution of the following Dirichlet problem in $\Omega \setminus K$:

$$\begin{cases} \Delta\Psi = 0 \\ \Psi|_{\partial K} = 0 \\ \Psi|_{\partial\Omega} = 1 \end{cases} \tag{20}$$

Capacity II

From a physical point of view, the capacity of the capacitor (K, Ω) represents the total electric charge (generated by the electrostatic potential Ψ) flowing into the domain $\Omega \setminus K$ through the interior boundary ∂K. Since the total current stems from a potential difference of 1 between ∂K and $\partial\Omega$, we get from Ohm's Law that the effective resistance of the domain $\Omega \setminus K$ is

$$R_{\text{eff}}(\Omega \setminus K) = \frac{1}{\text{Cap}(K, \Omega)} \quad . \tag{21}$$


BATERÍA
K
∂Ω
Ω
∂K

Capacity of extrinsic annuli

Given an isometric immersion $\varphi : M \to N$, the extrinsic annulus is

$$\mathrm{A}_{\rho,R} := \{x \in M \,|\rho \leq r(x) \leq R\}$$

Theorem, S. Markvorsen V. Palmer, GAFA 2002

Let $\varphi : M^n \to N$ be a proper and minimal immersion into a Cartan-Hadamard ambient manifold with curvatures bounded from above by $K_n \leq b \leq 0$, then

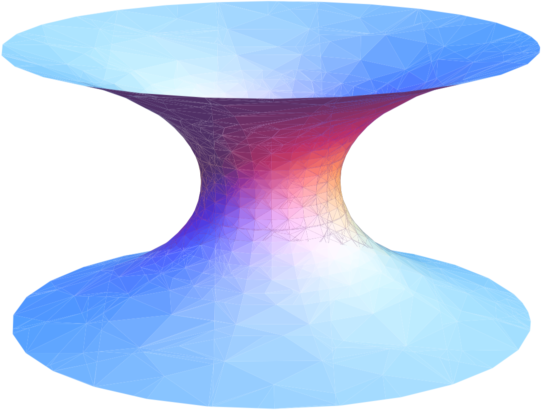
$$\mathrm{Cap}(A_{\rho,R}) \geq \mathrm{Cap}(A_{\rho,R}^{\mathbb{K}_b^n}).$$

Theorem, V. Gimeno S. Markvorsen, in preparation

Let $\varphi : M^n \to N$ be a proper and minimal immersion into a Cartan-Hadamard ambient manifold with curvatures bounded from above by $K_n \leq b \leq 0$, then

$$1 \leq \frac{\operatorname{Cap}(A_{\rho,R})}{\operatorname{Cap}(A^{\mathbb{K}^n_b}_{\rho,R})} \leq \sup_R \mathcal{Q}_b(R).$$

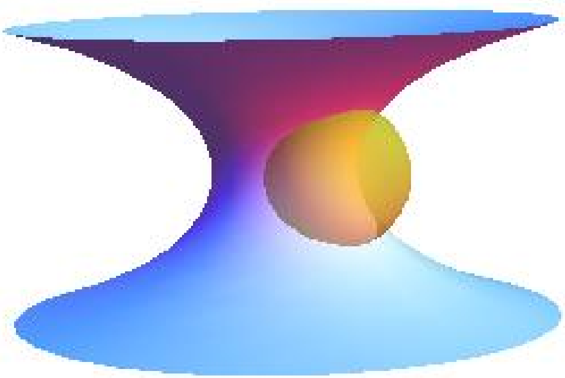
Catenoid

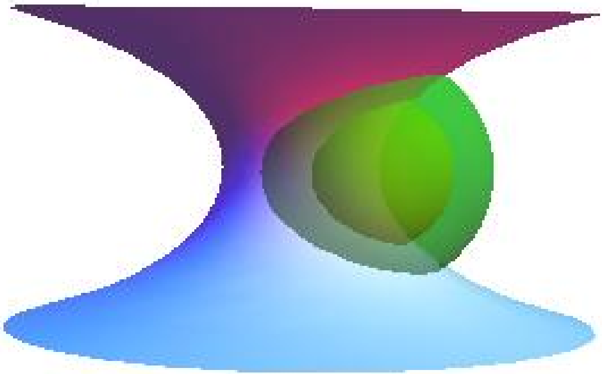


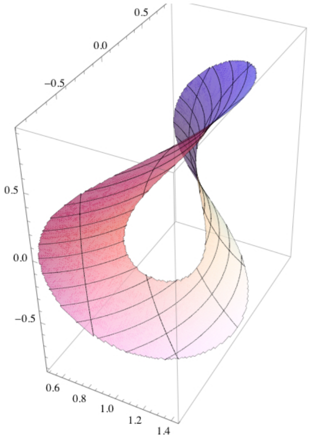
Theorem, Jorge-Meeks

Let M^2 be a minimal surface embedded in $\mathbb{R}^3$ with finite total curvature, then

$$\sup_R \mathcal{Q}(R) = \mathcal{E}(M^2) = \text{number of ends of } M.$$



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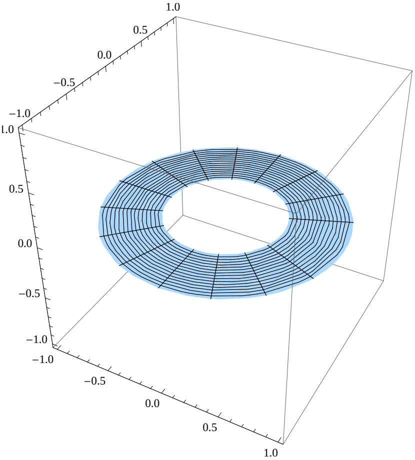
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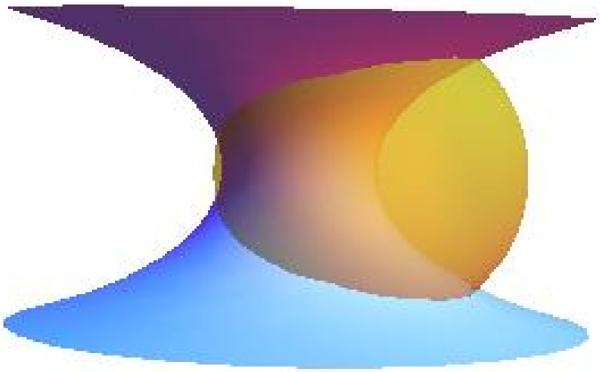

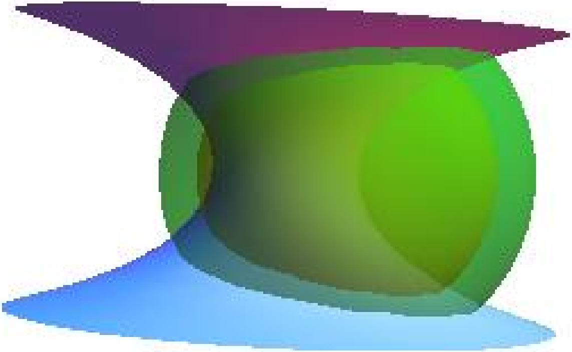

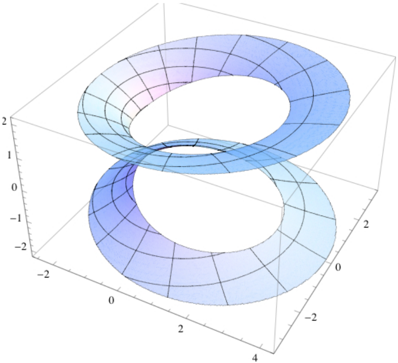
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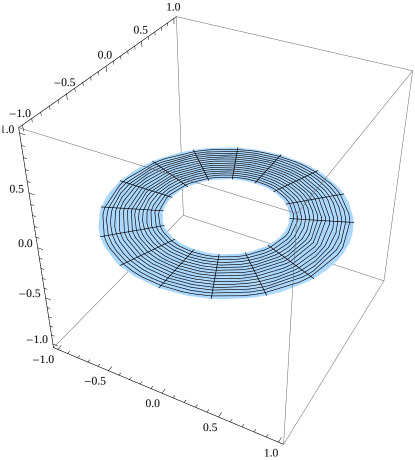
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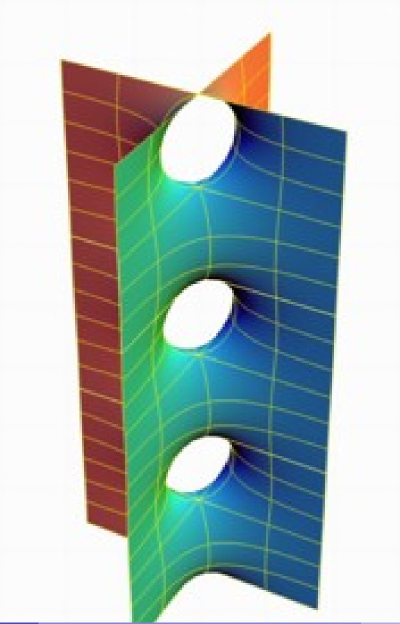
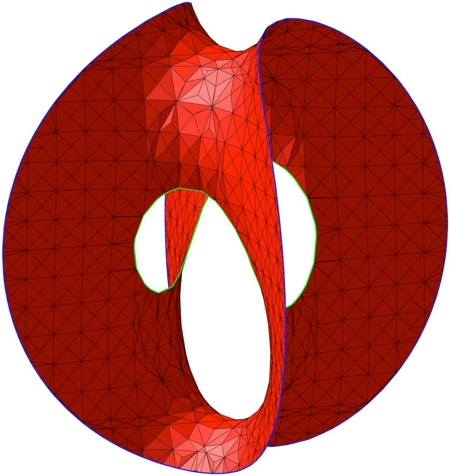
0.0
0.5
1.0

Scherk's singly periodic surface



The Scherk's singly periodic surface has

$$\sup_R \mathcal{Q}(R) = 2.$$



Volume growth number of ends I

Theorem, Anderson + Qing Chen, Manuscripta Math., 1997

Let M be an $n-$dimensional complete properly immersed minimal submanifold in $\mathbb{R}^m$ which satisfies

$$\limsup r\|A\| = 0$$

Then

$$\lim_{R\to\infty} \frac{\mathrm{Vol}(D_R)}{\omega_n R^n} = \mathcal{E}(M) < \infty.$$

Generalizing the ambient manifold

$$\mathbb{R}^m \to \text{ Model space } M^m_w.$$

Volume growth number of ends II

Model space

A $w-$model space M^n_w is a simply connected n-dimensional smooth manifold M^n_w with a point $o_w \in M^n_w$ called the *center point of the model space* such that $M^n_w - \{o_w\}$ is isometric to a smooth warped product with base $B^1 = (0, \Lambda) \subset \mathbb{R}$ (where $0 < \Lambda \leq \infty$), fiber $F^{n-1} = S^{n-1}_1$ (i.e. the unit $(n-1)-$sphere with standard metric), and positive warping function $w : [0, \Lambda) \to \mathbb{R}_+$. Namely:

$$g_{M^n_w} = \pi^*\left(g_{(0,\Lambda)}\right) + (w \circ \pi)^2 \sigma^*\left(g_{S^{n-1}_1}\right) \quad , \tag{22}$$

being $\pi : M^n_w \to (0, \Lambda)$ and $\sigma : M^n_w \to S^{n-1}_1$ the projections onto the factors of the warped product.

Volume growth number of ends III

Examples

$$\mathbb{K}^n_b = M^n_{w_b}.$$

$$w_b(r) = \begin{cases} \frac{1}{\sqrt{b}}\sin(\sqrt{b}r) \text{ if } b > 0 \\ r \text{ if } b = 0 \\ \frac{1}{\sqrt{-b}}\sinh(\sqrt{-b}r) \text{ if } b < 0 \end{cases}$$

Balanced models

Balanced from below:

$$\frac{\mathrm{Vol}(B^w_r)}{\mathrm{Vol}(S^w_r)}\frac{w'(r)}{w(r)} \geq \frac{1}{m}$$

Balanced from above:

$$\frac{\mathrm{Vol}(B^w_r)}{\mathrm{Vol}(S^w_r)}\frac{w'(r)}{w(r)} \leq \frac{1}{m-1}$$

Volume growth number of ends IV

Theorem, V. Gimeno, V. Palmer, JGEA 2013

Let $\varphi : M^n \to M_w^m$ be a proper and complete minimal immersion into a balanced from below model space M_w^m. Suppose that :

- $n > 2$,
- $w'(r) \geq d > 0$.
- $w'(r)w(r)\|A\| \leq \epsilon(r)$ such that $\epsilon \to 0$ when $r \to \infty$.

Then, M has finite topological type and

$$1 \leq \lim_{R\to\infty} \frac{\operatorname{Vol}(D_R)}{\operatorname{Vol}(B_R^w)} \leq \mathcal{E}(M).$$

Volume growth number of ends V

Theorem,Qing Chen, Manuscripta Math., 1997

Let M^n be a complete , proper and $n-$dimensional minimal submanifold of $\mathbb{R}^m$. Suppose that:

$$\sup_{R>0} \frac{\operatorname{Vol}(D_R)}{\omega_n R^n} < \infty.$$

Then

$$\mathcal{E}(M) \leq \sup_{R>0} \frac{\operatorname{Vol}(D_R)}{\omega_n R^n}.$$

Volume growth number of ends VI

Theorem, V. Gimeno and S. Markvorsen, in preparation

Let $\varphi : M^n \to N^m$ be a proper minimal and complete immersion. Where:

- N possesses a pole
- The sectional curvatures K_N of N are bounded by the radial curvatures K_w of a balanced from below model space M^n_w

$$K_N(p) \leq K_{M^n_w}(r(p)) = -\frac{w''}{w}(r(p)) \quad .$$

- $w' > 0$ and there exist R_0 such that $K_{M^n_w}(R) \leq 0$ for any $R > R_0$
- $$\limsup_{t\to\infty}\left(\frac{\int_0^t w(s)^{m-1}ds}{t^m/m}\right) = C_w < \infty \quad .$$

Then, if M has finite $w-$volume growth,

$$\mathcal{E}(P) \leq 2^m C_w \lim_{t\to\infty}\frac{\mathrm{Vol}(D_t)}{\mathrm{Vol}(B^w_t)}. \tag{23}$$

Thanks!!