Trapped submanifolds in de Sitter space

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Trapped submanifolds

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Second fundamental form

It is the symmetric tensor $\mathrm{II} : \mathfrak{X}(\Sigma) \times \mathfrak{X}(\Sigma) \to \mathfrak{X}^{\perp}(\Sigma)$ given by

$$\mathrm{II}(\mathbf{X}, \mathbf{Y}) = -(\overline{\nabla}_{\mathbf{X}} \mathbf{Y})^{\perp}$$

Mean curvature vector field

The mean curvature vector field is defined by

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  - **Future** (past) **trapped** if $\mathbf{H}$ is **timelike and future-pointing** (past-pointing) on Σ.

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  - **Future** (past) **weakly trapped** if $\mathbf{H}$ is **causal and future-pointing** (past-pointing) on Σ.
- The extreme case $\mathbf{H} = 0$ corresponds to a **minimal** submanifold.

- Each normal space $(T_p\Sigma)^\perp$, $p \in \Sigma$, is timelike and two dimensional, and hence admits two future-pointing null directions normal to Σ.

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Shape null operators

The shape operators associated to ξ and η are the symmetric operators $A_\xi, A_\eta : \mathfrak{X}(\Sigma) \to \mathfrak{X}(\Sigma)$ given by

$$\langle A_\xi X, Y \rangle = \langle \mathrm{II}(X, Y), \xi \rangle, \text{ and } \langle A_\eta X, Y \rangle = \langle \mathrm{II}(X, Y), \eta \rangle.$$

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Null mean curvatures

The **null mean curvatures** (or **null expansion scalars**) of Σ are given by

$$\theta_\xi = \frac{1}{n}\mathrm{trace}(A_\xi) \quad \text{and} \quad \theta_\eta = \frac{1}{n}\mathrm{trace}(A_\eta).$$

- Then, we have the expression

$$\mathbf{H} = -\theta_\eta \xi - \theta_\xi \eta$$

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- In particular

$$\langle \mathbf{H}, \mathbf{H} \rangle = -2\theta_\xi \theta_\eta$$

so that

  - Σ is a trapped submanifold if and only if
    i) either both $\theta_\xi < 0$ and $\theta_\eta < 0$ (future trapped),
    ii) or both $\theta_\xi > 0$ and $\theta_\eta > 0$ (past trapped).

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  - Σ is a marginally trapped submanifold if and only if
    - i) either $\theta_\xi = 0$ and $\theta_\eta \neq 0$ (future marginally trapped if $\theta_\eta < 0$ and past marginally trapped if $\theta_\eta > 0$),
    - ii) or $\theta_\xi \neq 0$ and $\theta_\eta = 0$ (future marginally trapped if $\theta_\xi < 0$ and past marginally trapped if $\theta_\xi > 0$).

- Then, we have the expression

$$\mathbf{H} = -\theta_\eta \xi - \theta_\xi \eta$$

for the mean curvature vector field.

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  - ii) or $\theta_\xi \neq 0$ and $\theta_\eta = 0$ (future marginally trapped if $\theta_\xi < 0$ and past marginally trapped if $\theta_\xi > 0$).
- Σ is a weakly trapped submanifold if and only if
  - i) either both $\theta_\xi \leq 0$ and $\theta_\eta \leq 0$ with $\theta_\xi^2 + \theta_\eta^2 > 0$ (future weakly trapped),
  - ii) or both $\theta_\xi \geq 0$ and $\theta_\eta \geq 0$ with $\theta_\xi^2 + \theta_\eta^2 > 0$ (past weakly trapped).

The $(n+2)$-dimensional de Sitter spacetime

- Let $\mathbb{L}^{n+3}$ be the $(n+3)$-dimensional Lorentz-Minkowski space, endowed with the Lorentzian metric

$$\langle , \rangle = -(dx_0)^2 + (dx_1)^2 + \cdots + (dx_{n+2})^2, \qquad x = (x_0, \ldots, x_{n+2})$$

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- The hyperquadric

$$\mathbb{S}_1^{n+2} = \{x \in \mathbb{L}^{n+3} : \langle x, x \rangle = 1\}$$

endowed with the induced metric from $\mathbb{L}^{n+3}$ is the standard model of the **de Sitter space**.

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- Consider on $\mathbb{S}_1^{n+2}$ the **time-orientation** induced by the globally defined timelike vector field $e_0^* \in \mathfrak{X}(\mathbb{S}_1^{n+2})$ given by

$$e_0^*(x) = e_0 - \langle e_0, x \rangle x = e_0 + x_0 x, \qquad e_0 = (1, 0, \ldots, 0),$$

with

$$\langle e_0^*(x), e_0^*(x) \rangle = -1 - \langle e_0, x \rangle^2 \leq -1 < 0.$$

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2. Null hypersurfaces in de Sitter spacetime
   - Marginally trapped submanifolds into the light cone
   - Marginally trapped submanifolds into $\mathcal{J}^-$
3. A uniqueness result for the marginally trapped type equation on compact manifolds

Null hypersurfaces in de Sitter spacetime

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  1. The future component of the **light cone**.

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  1. The future component of the **light cone**.
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- Recall that a null hypersurface into a spacetime M is a smooth codimension one embedded submanifold such that the pull-back of the Lorentzian metric of M is degenerate.

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- Recall that a null hypersurface into a spacetime M is a smooth codimension one embedded submanifold such that the pull-back of the Lorentzian metric of M is degenerate.
- As we will see later, in these cases there always exists a globally defined future-pointing normal null frame $\{\xi, \eta\}$ on Σ.

Marginally trapped submanifolds into the light cone

Light cone of de Sitter spacetime

Fix a point $\mathbf{a} \in \mathbb{S}_1^{n+2}$. The **light cone** in $\mathbb{S}_1^{n+2}$ with **vertex at a** is the subset

$$\Lambda_{\mathbf{a}} = \{x \in \mathbb{S}_1^{n+2} : \langle \mathbf{a}, x \rangle = 1, x \neq \mathbf{a}\}.$$

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The **future** component of $\Lambda_{\mathbf{a}}$ is

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- Define the function $u : \Sigma \to (0, +\infty)$ by $u = -\langle \psi, e_0\rangle$.
- Assume that $\psi(\Sigma)$ is contained into the future connected component of the light cone, $\Lambda_{\mathbf{a}}^+$, that is

$$\langle \psi, \psi\rangle = 1, \quad \langle \mathbf{a}, \psi\rangle = 1 \quad \text{and} \quad u > 0.$$

Normal null frame

In these conditions

$$\xi = \psi - \mathbf{a} \quad \text{and} \quad \eta = -\frac{1 + \|\nabla u\|^2 + u^2}{2u^2}\xi + \frac{1}{u}e_0^\perp$$

are two **future-pointing null normal** vector fields globally defined on Σ with $\langle \xi, \eta \rangle = -1$, where we are denoting

$$e_0 = e_0^\top(p) + e_0^\perp(p) + \langle \psi(p), e_0 \rangle \psi(p), \quad p \in \Sigma.$$

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Null shape operators

The null second forms associated to the global null frame $\{\xi, \eta\}$ are given by

$$A_\xi = I \qquad \text{and} \qquad A_\eta = -\frac{1 + \|\nabla u\|^2 - u^2}{2u^2}I + \frac{1}{u}\nabla^2 u,$$

where $\nabla^2 u$ is the Hessian operator of u.

- In particular, the null expansions are $\theta_\xi = \frac{1}{n}\mathrm{tr}(A_\xi) = 1$ and

$$\theta_\eta = \frac{1}{n}\mathrm{tr}(A_\eta) = \frac{2u\Delta u - n(1 + \|\nabla u\|^2 - u^2)}{2nu^2},$$

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Corollary 1

Let $\psi : \Sigma^n \to \Lambda^+ \subset \mathbb{S}^{n+2}_1$ be a codimension-two spacelike submanifold which is contained in Λ^+_a. The following assertions are equivalent:

- Σ is (necessarily past) **marginally trapped**.
- The positive function $u = -\langle \psi, e_0 \rangle$ satisfies the differential equation

$$\mathbf{2u\Delta u - n(1 + \|\nabla u\|^2 - u^2) = 0} \quad \text{on } \boldsymbol{\Sigma}.$$

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- The positive function $u = -\langle \psi, e_0 \rangle$ satisfies the differential equation

$$\mathbf{2u\Delta u - n(1 + \|\nabla u\|^2 - u^2) = 0} \quad \text{on } \boldsymbol{\Sigma}.$$

- Σ has constant scalar curvature $\mathrm{Scal} = \mathbf{n(n-1)}$.

Example 1

- For each **positive** smooth function $f : \mathbb{S}^n \rightarrow (0, +\infty)$, consider the embedding $\psi_f : \mathbb{S}^n \rightarrow \Lambda^+ \subset \mathbb{S}_1^{n+2}$ given by

$$\psi_f(p) = (f(p), f(p)p, 1).$$

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$$\psi_f(p) = (f(p), f(p)p, 1).$$

- It is not difficult to see that for every $\mathbf{v}, \mathbf{w} \in T_p\mathbb{S}^n$

$$\langle d(\psi_f)_p(\mathbf{v}), d(\psi_f)_p(\mathbf{w}) \rangle = f^2(p) \langle \mathbf{v}, \mathbf{w} \rangle_0,$$

$\langle , \rangle_0$ the standard metric of the round sphere.

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  $\langle , \rangle_0$ the standard metric of the round sphere.
- That is $\psi_f^*(\langle , \rangle) = f^2\langle , \rangle_0$, which means that ψ_f defines a **spacelike immersion** of $\mathbb{S}^n$ into Λ^+ with induced metric **conformal to $\langle , \rangle_0$**.

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- That is $\psi_f^*(\langle , \rangle) = f^2 \langle , \rangle_0$, which means that ψ_f defines a **spacelike immersion** of $\mathbb{S}^n$ into Λ^+ with induced metric **conformal to** $\langle , \rangle_0$.
- Moreover, ψ_f is marginally trapped if and only if f satisfies

$$2f \Delta f - n(1 + \|\nabla f\|^2 - f^2) = 0$$

on $\mathbb{S}^n$ with respect to the **conformal metric** $f^2 \langle , \rangle_0$.

- We will see now that every codimension-two compact spacelike submanifold in Λ^+ is, up to a conformal diffeomorphism, as in Example 1.

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Proposition 1

Let $\psi : \Sigma^n \to \Lambda^+ \subset \mathbb{S}_1^{n+2}$ be a codimension-two compact spacelike submanifold contained in Λ^+.

- Every codimension-two compact spacelike submanifold in Λ^+ is as in Example 1.

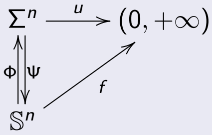
Proposition 1

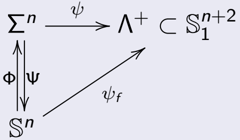
Let $\psi : \Sigma^n \to \Lambda^+ \subset \mathbb{S}_1^{n+2}$ be a codimension-two compact spacelike submanifold contained in Λ^+.
There exists a **conformal diffeomorphism**

$$\Psi : (\Sigma^n, \langle , \rangle) \to (\mathbb{S}^n, \langle , \rangle_0) \text{ such that } \langle , \rangle = u^2 \Psi^*(\langle , \rangle_0),$$

with $u = -\langle \psi, e_0 \rangle = \psi_0 > 0$, and $\psi = \psi_f \circ \Psi$ where $f = u \circ \Psi^{-1}$, and ψ_f is the embedding

$$\psi_f(p) = (f(p), f(p)p, 1)$$





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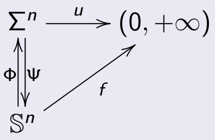

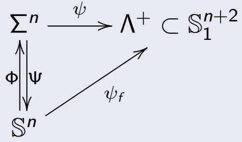

In particular, ψ is an embedding.

Proof:

- Without loss of generality we may assume that the vertex of the light cone is the point $a = (0, \ldots, 0, 1)$, so that

$$\Lambda^+ = \{x \in \mathbb{S}_1^{n+2} : x_{n+2} = 1, x_0 > 0\}.$$

Proof:

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- In particular, this happens when Σ is **compact**.
- But the proof also works under any assumption which implies that the **conformal metric** $\widetilde{\langle , \rangle}$ **is complete**.

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$$\begin{aligned}\psi_f \circ \Psi(p) = &(f(\Psi(p)), f(\Psi(p))\Psi(p), 1)\\ =&(u(p), \psi_1(p), \ldots, \psi_{n+1}(p), 1) = \psi(p).\end{aligned}$$

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- Then, the diagrams

$$\begin{array}{ccc} \Sigma^n & \xrightarrow{\ u\ } & (0, +\infty) \\ \Phi \uparrow\downarrow \Psi & \nearrow_{f} & \\ \mathbb{S}^n & & \end{array} \qquad \begin{array}{ccc} \Sigma^n & \xrightarrow{\ \psi\ } & \Lambda^+ \subset \mathbb{S}_1^{n+2} \\ \Phi \uparrow\downarrow \Psi & \nearrow_{\psi_f} & \\ \mathbb{S}^n & & \end{array}$$

are commutative, and it completes the proof.

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- For every fixed vector $\mathbf{b} \in \mathbb{R}^{n+1}$, let $f_{\mathbf{b}} : \mathbb{S}^n \to (0, +\infty)$ be the function

$$f_{\mathbf{b}}(p) = \frac{1}{\langle p, \mathbf{b} \rangle_0 + \sqrt{1 + \|\mathbf{b}\|_0^2}}$$

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- Equivalently,

$$2f\Delta_0 f + (n-4)\|\nabla^0 f\|_0^2 - nf^2(1 - f^2) = 0 \qquad \text{(EQ2)}$$

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Example 2 (continuation)

- By a direct computation we can see that

$$\|\nabla^0 f_{\mathbf{b}}\|_0^2 = \frac{1}{g^4}\|\nabla^0 g\|_0^2, \quad \text{and} \quad \Delta_0 f_{\mathbf{b}} = -\frac{1}{g^2}\Delta_0 g + \frac{2}{g^3}\|\nabla^0 g\|_0^2,$$

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- Inserting (1) and (2) into (EQ2) we get the validity of (EQ2).

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In particular, Σ is **embedded**.

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- From our previous discussion, the proof of Theorem 1 reduces to find the **positive solutions** of the differential equation

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- In particular, the conformal factor f is the conformal factor of a conformal diffeomorphism of the unit round sphere.
- Recall that, up to orthogonal transformations, every conformal diffeomorphism of $(\mathbb{S}^n, \langle , \rangle_0)$ is given by

$$F_{\mathbf{c}}(p) = \frac{p + (\mu\langle p, \mathbf{c}\rangle_0 + \lambda)\mathbf{c}}{\lambda(1 + \langle p, \mathbf{c}\rangle_0)}$$

for all $p \in \mathbb{S}^n$, where $\mathbf{c} \in \mathbb{B}^{n+1}$, $\mathbb{B}^{n+1}$ the open unit ball in $\mathbb{R}^{n+1}$, and

$$\lambda = (1 - \|\mathbf{c}\|_0^2)^{-1/2} \quad \text{and} \quad \mu = (\lambda - 1)\|\mathbf{c}\|_0^2.$$

- A direct computation shows that the conformal factor f of $F_{\mathbf{c}}$ is given by

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- The steady state space has as **boundary** the null hypersurface L, which represents the **past infinity** of $\mathcal{H}^{n+2}$, usually denoted by $\mathcal{J}^-$.

Marginally trapped submanifolds into $\mathcal{J}^-$

Past infinity of steady state

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- Assume that $\psi(\Sigma)$ is contained in $\mathcal{J}^-$.

Normal null frame

In these conditions

$$\xi = -\mathbf{a} \quad \text{and} \quad \eta = -\frac{1 + \|\nabla u\|^2 + u^2}{2\langle \mathbf{a}, e_0\rangle^2}\xi + \frac{1}{\langle \mathbf{a}, e_0\rangle} e_0^\perp$$

are two **future-pointing null normal** vector fields globally defined on Σ with $\langle \xi, \eta\rangle = -1$.

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Null shape operators

The corresponding null second forms associated to the global null frame $\{\xi, \eta\}$ are given by

$$A_\xi = 0 \qquad \text{and} \qquad A_\eta = \frac{1}{\langle \mathbf{a}, e_0\rangle}(\nabla^2 u + uI).$$

- In particular, the null expansions are $\theta_\xi = \frac{1}{n}\mathrm{tr}(A_\xi) = 0$ and

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$$\mathbf{H} = -\frac{1}{n\langle \mathbf{a}, e_0\rangle}(\Delta u + nu)\xi$$

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Corollary 2

Let $\psi : \Sigma^n \to \mathcal{J}^- \subset \mathbb{S}_1^{n+2}$ be a codimension-two spacelike submanifold which is contained in the **past infinite** of the steady state space. The Σ is **always marginally trapped**, except at points where $\Delta u + nu = 0$ (if any), $u = -\langle \psi, e_0\rangle$, where it is **minimal**.

Example 3

- For each smooth function $f : \mathbb{S}^n \to \mathbb{R}$, consider the embedding $\phi_f : \mathbb{S}^n \to \mathcal{J}^- \subset \mathbb{S}_1^{n+2}$ given by

$$\phi_f(p) = (f(p), p, f(p)).$$

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- It is not difficult to see that for every $\mathbf{v}, \mathbf{w} \in T_p\mathbb{S}^n$

$$\langle d(\phi_f)_p(\mathbf{v}), d(\phi_f)_p(\mathbf{w})\rangle = \langle \mathbf{v}, \mathbf{w}\rangle_0,$$

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- Moreover, ϕ_f is **marginally trapped** except at points (if any) where

$$\Delta_0 f + nf = 0.$$

  on $\mathbb{S}^n$ with respect to the pointwise conformal metric $f^2\langle , \rangle_0$.

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$$\Psi : (\Sigma^n, \langle , \rangle) \to (\mathbb{S}^n, \langle , \rangle_0)$$

such that $\psi = \phi_f \circ \Psi$ where $f = u \circ \Psi^{-1}$ with $u = -\langle \psi, e_0 \rangle = \psi_0$, and ϕ_f is the embedding

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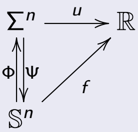

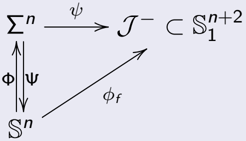

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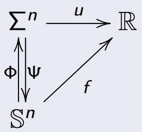

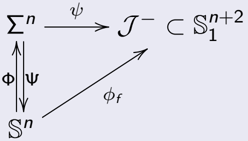

In particular, the immersion ψ is an **embedding** and it is always **marginally trapped** except at points where $\Delta u + nu = 0$ (if any), where it is minimal.

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Proof:

- Let $\psi : \Sigma^n \to \mathcal{J}^- \subset \mathbb{S}_1^{n+2}$ be a codimension-two spacelike submanifold contained in $\mathcal{J}^-$.
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is a **local isometry**.
- Therefore, if we assume Σ to be **complete**, $\mathbb{S}^n$ being simply connected, we conclude that Ψ is in fact a **global isometry**.

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Let $\psi : \Sigma^n \to \mathcal{J}^- \subset \mathbb{S}_1^{n+2}$ be a codimension-two complete spacelike submanifold contained in $\mathcal{J}^-$ and having parallel mean curvature vector.

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such that $\psi = \phi_{\mathbf{b},c} \circ \Psi$, where $\phi_{\mathbf{b},c} : \mathbb{S}^n \to \mathcal{J}^- \subset \mathbb{S}_1^{n+2}$ is the embedding

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Moreover:

(I) Σ is minimal if and only if $c = 0$.

(II) Σ is future marginally trapped if and only if $c < 0$.

(III) Σ is past marginally trapped if and only if $c > 0$.

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- The last assertions follow from (3) since $\mathbf{H} = c\mathbf{a}$, with $\mathbf{a}$ past-pointing.

$\square$

Corollary 4

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for any $\mathbf{b} \in \mathbb{R}^{n+1}$.

Index

A uniqueness result for the marginally trapped type equation on compact manifolds

- Motivated by the geometric meaning of the solutions to the partial differential equation $2u\Delta u - n(1 + \|\nabla u\|^2 - u^2) = 0$, we establish the following intrinsic uniqueness result for this equation.

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Let $(\Sigma, \langle,\rangle)$ be a compact, Riemannian manifold of dimension $n \geq 2$ and Ricci curvature satisfying

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The only positive solution to the partial differential equation

$$2u\Delta u - n(1 + \|\nabla u\|^2 - u^2) = 0 \qquad \text{(MT)}$$

on Σ is the constant function $u \equiv 1$.

Proof:

- Consider the vector field

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- Using this into (4) jointly with (MT) we obtain

$$\begin{aligned}\mathrm{div}(V) =& u^{-(n-1)}\left(\|\nabla^2 u\|^2 - \frac{(\Delta u)^2}{n}\right) \\ &+ u^{-(n-1)}\left(\mathrm{Ric}(\nabla u, \nabla u) - (n-1)\|\nabla u\|^2\right).\end{aligned}$$

- Integrating this and using the divergence theorem we obtain

$$\int_\Sigma u^{-(n-1)} \left(\|\nabla^2 u\|^2 - \frac{(\Delta u)^2}{n} + \mathrm{Ric}(\nabla u, \nabla u) - (n-1)\|\nabla u\|^2 \right) = 0. \tag{5}$$

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- We know from Cauchy-Schwarz inequality that

$$\|\nabla^2 u\|^2 - \frac{(\Delta u)^2}{n} \geq 0,$$

with equality if and only if ∇u is a conformal vector field on Σ.

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$$\int_{\Sigma} u^{-(n-1)} \left(\|\nabla^2 u\|^2 - \frac{(\Delta u)^2}{n} + \mathrm{Ric}(\nabla u, \nabla u) - (n-1)\|\nabla u\|^2 \right) = 0. \tag{5}$$

- We know from Cauchy-Schwarz inequality that

$$\|\nabla^2 u\|^2 - \frac{(\Delta u)^2}{n} \geq 0,$$

with equality if and only if ∇u is a conformal vector field on Σ.

- On the other hand, from $\mathrm{Ric} \geq K$ we also have

$$\mathrm{Ric}(\nabla u, \nabla u) - (n-1)\|\nabla u\|^2 \geq (K - (n-1))\|\nabla u\|^2 \geq 0$$

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- Since $K > (n-1)$, this last equation implies that u is constant and, by (MT) it must be $u \equiv 1$.

$\square$

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That's all !!

Thanks a lot for your attention...