Extrinsic isoperimetry, estimates for the capacity and parabolicity of submanifolds

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Parabolicity: analytical approach. Liouville's Theorem I

Theorem (Liouville)

Let us suppose that $u : \mathbb{R}^n \to \mathbb{R}$ is harmonic ($\Delta u = 0$) and bounded. Then u is constant.

Remark

- *We say that $\mathbb{R}^n$ satisfies the Liouville's property.*
- *In $\mathbb{R}^2$, if u is subharmonic, ($\Delta u \geq 0$), and bounded then u is constant. Parabolicity.*
- *In $\mathbb{R}^n$, with $n \geq 3$, there are non-constant and bounded subharmonic functions.*

Parabolicity: analytical approach. Liouville's Theorem II

Definition

A non-compact, complete n-dimensional manifold M is *parabolic* if and only if every subharmonic and bounded function defined on it is constant. If such non-constant function exists, then M is *hyperbolic*.

Remark (Question 1)

Let M be a complete Riemannian manifold M. To give a geometric description, (volume growth, curvature assumptions, etc.) for parabolicity/hyperbolicity?

Geometric conditions for parabolicity I

Theorem (L.V. Ahlfors, Com. Math. Helvet., 32, 1935)

Let $M^2_w = [0,\infty) \times_w S^1_1$ be a complete 2-dimensional rotationally symmetric manifold. Then M^2_w is parabolic iff $\int_0^\infty \frac{dr}{Vol(S^w_r)} = \infty$.

Theorem (L. Karp, N. Varopoulos, A. Grigor'yan, 1983)

Let M^n be a complete Riemannian manifold. Then if, for some point $x \in M$, $\int_M \frac{r}{\mathrm{Vol}(B_r(x))} dr = \infty$, M is parabolic.

In particular, if $\mathrm{Vol}(B_r(x)) \leq Cr^2$, ($M$ is of quadratic volume growth), then M is parabolic.

Geometric conditions for parabolicity II

Theorem (J. Milnor, Amer. Math. Monthly 84, 1987)

Let $M^2_w = [0, \infty) \times_w S^1_1$ be a complete 2-dimensional rotationally symmetric manifold. The Gaussian curvature of M^2_w, $K(r)$, is a radial function of the distance to the center of this space.

(A) Let us suppose that $K(r) \geq -\frac{1}{r^2 \log r}$ for r large. Then M^2_w is parabolic.

(B) Let us suppose that there exists $\epsilon > 0$ such that $K(r) \leq -\frac{1+\epsilon}{r^2 \log r}$ for r large. Then M^2_w is hyperbolic.

Geometric conditions for parabolicity III

Theorem (K. Ichihara, Nagoya Math. J. 87, 1982)

(A) Let M^2 be a complete 2-dimensional Riemannian manifold. If $\int_M |K_M| d\sigma < \infty$, then M^2 is parabolic.

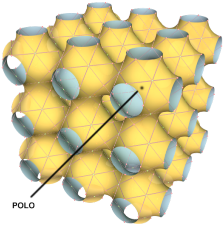
(B) Let M^n be a complete n-dimensional Riemannian manifold. If $K_M \geq K_{M^n_w}$ and $\int_0^\infty \frac{dr}{Vol(S^w_r)} = \infty$, then M is parabolic.

(C) Let M^n be a complete n-dimensional Riemannian manifold. If $K_M \leq K_{M^n_w}$ and $\int_0^\infty \frac{dr}{Vol(S^w_r)} < \infty$, then M is hyperbolic.

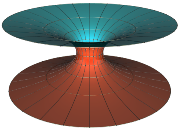
Some examples in $\mathbb{R}^3$ I

Costa's surface, quadratic volume growth, parabolic

P-Schwartz surface, hyperbolic, (universal cover $\mathbb{H}^2$)

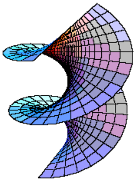


Catenoid, quadratic volume growth, parabolic

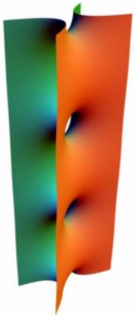


Some examples in $\mathbb{R}^3$ II

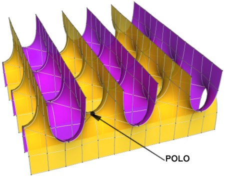
Helicoid, parabolic, (conformally difeomorphic to the plane)



Scherk singly periodic, quadratic volume growth, parabolic

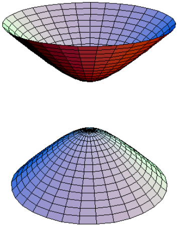


Scherk doubly periodic, hyperbolic, (Thomassen et al)



Some examples in $\mathbb{R}^3$ III

Hyperboloid of two sheets, finite total curvature, parabolic



Hyperbolic paraboloid, finite total curvature, parabolic

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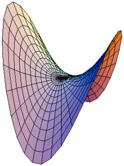


Parabolicity: physical approach. Capacity I

Definition

Let N be a Riemmannian manifold. Let $\Omega \subset N$ be a precompact subset of N, and $K \subseteq \Omega$ a compact subset. Then, the *capacity* of K in Ω is given as the following integral:

$$Cap(K,\Omega) = \int_{\Omega} \|\nabla\phi\|^2 \, d\sigma = \int_{\partial K} \langle \nabla^P \phi, \nu \rangle d\mu$$

where ν is the unit normal to ∂K pointing into $\Omega - K$ and ϕ is the solution of the Laplace equation on $\Omega - K$with Dirichlet boundary values:

$$\begin{cases} \Delta u = 0 \\ u \mid_{\partial\Omega} = 1 \\ u \mid_{\partial K} = 0 \end{cases}$$

Parabolicity: physical approach. Capacity II

Definition

Let N be a complete Riemmannian manifold. Let $\Omega \subset N$ be a precompact subset of N. Let us consider $\{\Omega_i\}_{i=1}^{\infty}$ an exhaustion of N by nested and precompact sets, such that $\Omega \subseteq \Omega_i$ for some i.

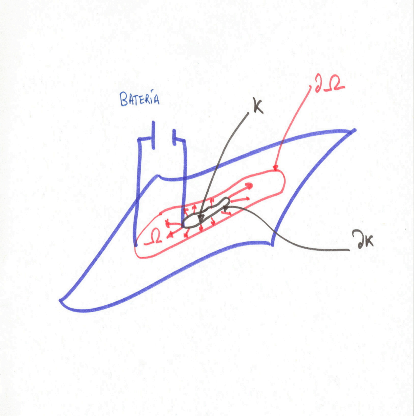
Then, the capacity of Ω in all the manifold, (the *capacity at infinity* denoted as $Cap(\Omega, N)$) is given as the following limit:

$$Cap(\Omega, N) = \lim_{i\to\infty} Cap(\Omega, \Omega_i)$$

Parabolicity: physical approach. Capacity III

Remark

- *This definition is independent of the exhaustion.*
- *$Cap(K, \Omega)$ decreases on expanding of Ω and on shrinking of K.*



Parabolicity: the Kelvin-Nevanlinna-Royden Criterion I

Theorem (Lyons, Sullivan, 1984)

The Riemannian manifold M is hyperbolic iff some of the following equivalent conditions holds:

1. *M admits a non-constant and bounded subharmonic function*
2. *M has positive capacity, i.e., there exists a non-empty precompact $D \subseteq M$ such that $Cap(D, M) > 0$. Thence, M is parabolic iff there exists $D \subseteq M$ such that $Cap(D, M) = 0$.*
3. *The Brownian motion defined on M is transient.*

Submanifolds I

Remark (Question 2)

- *Assume* $\varphi : P^m \longrightarrow N^n$ *is an isometric immersion of a complete non-compact Riemannian m-manifold* P^m *into a complete Riemannian manifold* N^n *with a pole* $o \in N$*.*
- *Do we have something to say about extrinsic curvatures and capacity estimates, (hence, parabolicity of submanifolds)?*

Extrinsic distance I

- Let $\varphi : P^m \longrightarrow N^n$ be an isometric immersion of a complete non-compact Riemannian m-manifold P^m into a complete Riemannian manifold N^n with a pole $o \in N$.
- A pole is a point o such that the exponential map

$$\exp_o : T_o N^n \to N^n$$

is a diffeomorphism.

Extrinsic distance II

- For every $x \in N^n - \{o\}$ we define $r(x) = r_o(x) = \mathrm{dist}_N(o, x)$, and this distance is realized by the length of a unique geodesic from o to x, which is the *radial geodesic from* o.
- We also denote by $r|_P$ or by r the composition $r \circ \varphi : P \to \mathbb{R}_+ \cup \{0\}$. This composition is called the *extrinsic distance function* from o in P^m.
- Let $\varphi : P^m \longrightarrow N^n$ be a C^∞-immersion. Then φ is proper iff $\varphi^{-1}(K)$ is compact in P for all compact K in N. Roughly speaking: when we "go to infinity" in P, then we also "go to infinity" in the ambient manifold N.

Extrinsic distance III

Definition

Given $\varphi : P^m \longrightarrow N^n$ an isometric immersion and N complete Riemannian manifold N^n with a pole $o \in N$.
Define the *extrinsic metric balls* of radius $t > 0$ and center $o \in N$ as

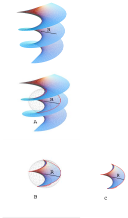
$$D_t(o) = \{x \in P : r(\varphi(x)) < t\} = \{x \in P : \varphi(x) \in B_t^N(o)\} = \varphi^{-1}(B_t^N(o))$$

where $B_t^N(o)$ is the open geodesic t-ball centered at the pole o in N^n. Note that the set $\varphi^{-1}(o)$ can be the empty set.

Extrinsic distance IV

- The extrinsic balls are precompact sets, with smooth boundary ∂D_t, for a dense set of radius t in $\mathbb{R}$ by the Regular Value Theorem and by the Morse-Sard Theorem.

Extrinsic ball in the Helicoid



Extrinsic balls in Costa's surface and the helicoid

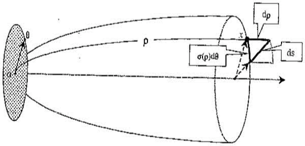
Model spaces I

Definition

A $w-$model M_w^m is a smooth warped product $[0, R[\times_\omega \mathbb{S}_1^{m-1}$, with $w(0) = 0$, $w'(0) = 1$, and $w(r) > 0$ for all $r > 0$. The point $o_w = \pi^{-1}(0)$, where π denotes the projection onto $[0, R[$, is called the *center point* of the model space. If $R = \infty$, then o_w is a pole of M_w^m.

Remark

Mean curvature of geodesic spheres S_r^w is $\eta_\omega(r) = \frac{\omega'(r)}{\omega(r)}$ Sectional curvatures of radial planes are $K(\sigma_x) = -\frac{\omega''(r(x))}{\omega(r(x))}$



Extrinsic isoperimetry I

Theorem (A. Hurtado, S. Markvorsen, V. Palmer, 2009)

*Let N^n be complete with a pole and with $K_N \leq -\frac{w''(r)}{w(r)}$.
Let $\varphi : P^m \longrightarrow N^n$ be C^∞, complete and proper immersion.
Suppose that $-\langle H_P, \nabla^N r\rangle(x) \leq h(r(x))\ \forall x \in P$, (H_P is the mean curvature vector of P). Then*

$$\frac{\mathrm{Vol}(\partial D_r)}{\mathrm{Vol}(D_r)} \geq \frac{\mathrm{Vol}(S_r^W)}{\mathrm{Vol}(B_r^W)} \ \forall\, r > 0$$

and $\frac{\mathrm{Vol}(D_r)}{\mathrm{Vol}(B_r^W)}$ is non-decreasing in $[0, +\infty)$, with B_r^W (resp. S_r^W) the geodesic r-ball, (resp. the geodesic r-sphere), in M_W^m.

Extrinsic isoperimetry II

- Proof: Construction of a comparison model space $M^m_W = [0, R[\times_\omega \mathbb{S}^{m-1}_1$, with $W(r) = \Lambda^{\frac{1}{m-1}}(r)$ satisfying

$$\frac{d}{dr}(\Lambda(r)w(r)) = m\Lambda(r)(w'(r) - h(r)w(r))$$
$$\frac{d}{dr}|_{r=0}\Lambda^{\frac{1}{m-1}}(r) = 1$$

- This new model space M^m_W must satisfy in addition a *balance condition* with respect the bound $h(r)$ for the radial mean curvature of P and the function $w(r)$, namely

$$\frac{\mathrm{Vol}(B^W_r)}{\mathrm{Vol}(S^W_r)}\left(\frac{w'(r)}{w(r)} - h(r)\right) \geq \frac{1}{m}$$

Extrinsic isoperimetry III

- If P is minimal, then $h(r) = 0$, $W(r) = w(r)$ and the balance condition is satisfied when $-\frac{w''(r)}{w(r)} \leq 0$, which includes Cartan-Hadamard manifolds.
- In particular, when $w(r) = w_b(r) = \frac{1}{\sqrt{-b}} \sinh \sqrt{-b} r$, then $M^m_{w_b} = \mathbb{H}^m(b)$, the Hyperbolic space. When $w(r) = w_0(r) = r$, then $M^m_{w_o} = \mathbb{R}^m$, the Euclidean space.

Extrinsic isoperimetry and estimates for the capacity I

Theorem (S. Markvorsen, V. Palmer, 2003)

Let P^m be a complete and minimal submanifold properly immersed in a Cartan-Hadamard manifold N^n with sectional curvatures $K_N \leq b \leq 0$. Then

$$Cap(D_\rho, D_R) \geq Cap(B_\rho^{b,m}, B_R^{b,m})$$

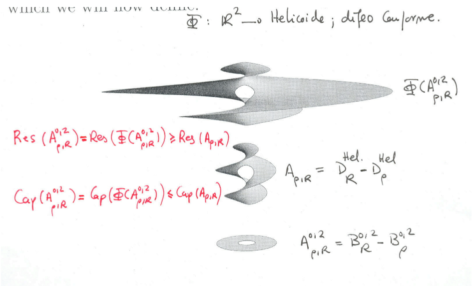
with $B_r^{m,b}$ is the geodesic r-ball in $\mathbb{H}^m(b)$ or $\mathbb{R}^m$. Hence, if $b < 0$,

$$Cap(D_\rho, P) \geq Cap(B_\rho^{b,m}, \mathbb{H}^m)$$

$$\geq \frac{(m-1)\,\mathrm{Vol}(S_1^{0,m-1})}{(\sqrt{-b})^{m-2}(\sinh(\sqrt{-b}\rho))^{1-m})} > 0$$

so P is hyperbolic, for $m \geq 2$.

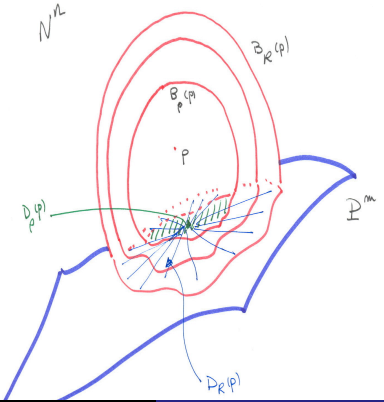
Extrinsic isoperimetry and estimates for the capacity I



Extrinsic isoperimetry and estimates for the capacity: sketch of the Proof I

Key idea of the proof: Our (geometric) approach uses a covering exhaustion of the (sub)-manifold by extrinsic metric balls $\{D_t\}_{t\in\mathbb{R}}$.

Extrinsic isoperimetry and estimates for the capacity: sketch of the Proof II



- The ambient space N^n satisfies $K_N \leq b = -\frac{w_b''}{w_b}$, so, as P is minimal, then $W(r) = w_b(r)$ and $M^m_{w_b}$ will be the Hyperbolic space or the Euclidean space.
- The solution of

$$\begin{cases} \Delta^{M^m_w}\psi = 0 \quad \text{on } [\rho, R] \\ \psi(\rho) = 0 \\ \psi(R) = 1 \end{cases}$$

is (radial) $\psi_{\rho,R}(r)$ where $\psi'_{\rho,R}(r) \geq 0$

- To transplant $\psi_{\rho,R}(r)$ to the annulus in P determined by the extrinsic balls $A_{\rho,R} = D_R(o) \setminus \bar{D}_\rho(o)$.

- Use Greene and Wu's comparison for the Hessian of the distance function r and that $\psi'_{\rho,R}(r) \geq 0$. Then

$$\Delta^P \psi_{\rho,R}(r(x)) \geq 0 = \Delta^P v(x)$$

where $v(x)$ is the solution of the Laplace equation in $A_{\rho,R} = D_R \setminus D_\rho$.

- Applying Maximum Principle,

$$\psi_{\rho,R}(r(x)) - v(x) \leq 0 \ , \ \forall \ x \in A_{\rho,R}$$

and $\psi_{\rho,R}(r(x)) \leq v(x) \ , \forall \ x \in A_{\rho,R}$.

- Then

$$Cap(A_{\rho,R}) = \int_{\partial D_\rho} \|\nabla^P v(x)\| \, d\nu \geq \int_{\partial D_\rho} \|\nabla^P \Psi_{\rho,R}\| \, d\mu$$
$$= \psi'_{\rho,R}(\rho) \int_{\partial D_\rho} \|\nabla^P r\| \, d\mu.$$

- We have $\int_{\partial D_\rho} \|\nabla^P r\| \, d\mu \geq \mathrm{Vol}(S_\rho^{b,m-1})$ using the isoperimetric inequality

$$\frac{\mathrm{Vol}(\partial D_r)}{\mathrm{Vol}(D_r)} \geq \frac{\mathrm{Vol}(S_r^{b,m-1})}{\mathrm{Vol}(B_r^{b,m})}$$

- Hence

$$Cap(A_{\rho,R}) \geq \psi'_{\rho,R}(\rho) \int_{\partial D_\rho} \|\nabla^P r\| \, d\mu$$
$$\geq \psi'_{\rho,R}(\rho) \, \mathrm{Vol}(S_\rho^{b,m-1}) = Cap(B_\rho^{b,m}, B_R^{b,m})$$

A generalization of Ichihara's theorem I

Theorem (S. Markvorsen, V. Palmer, 2005)

N^n complete with a pole o. Suppose $K_{o,N}(\sigma_x) \leq -\frac{\omega''(r)}{\omega(r)}, \forall x$. P^m complete, properly immersed in N with $-\langle H_P, \nabla^N r\rangle(x) \leq h(r(x)) \leq \eta_\omega(r)\ \forall r$. Then, if

$$\int_\rho^\infty \frac{\mathcal{G}^m(r)}{\omega^{m-1}(r)}\, dr < \infty \quad \text{where} \quad \mathcal{G}(r) = \exp(\int_\rho^r h(t)\, dt)$$

we have that P^m es hyperbolic.

$\Rightarrow$

Theorem (Ichihara, 1982)

Let M^n be a complete n-dimensional Riemannian manifold. If $K_M \leq K_{M^n_w}$ and $\int_0^\infty \frac{dr}{Vol(S^w_r)} < \infty$, then M is hyperbolic.

$\Downarrow$

Theorem (Ahlfors, 1934)

Let $M^2_w = [0,\infty) \times_w S^1_1$ be a complete 2-dimensional rotationally symmetric manifold. Then M^2_w is parabolic iff $\int_0^\infty \frac{dr}{Vol(S^w_r)} = \infty$.

$\Downarrow$

Theorem (Markvorsen-Palmer, 2003)

P^m complete, minimal, properly immersed in N^n (Cartan-Hadamard), with $K_{sec} \leq b \leq 0$.

P^m is hyperbolic if, either ($b < 0$ y $m \geq 2$), or ($b = 0$ y $m \geq 3$).

A generalization of Ichihara's theorem: sketch of the Proof I

- Given $w(r)$ and $h(r)$, define $L\psi(r) = \psi''(r) + \psi'(r)\left((m-1)\eta_w(r) - mh(r)\right)$ on radial functions $\psi(r)$
- The solution of

$$\begin{cases} L\psi = 0 \text{ on } [\rho, R] \\ \psi(\rho) = 0 \\ \psi(R) = 1 \end{cases}$$

is $\psi_{\rho,R}(r) = \frac{\int_\rho^r \Lambda(t)\,dt}{\int_\rho^R \Lambda(t)\,dt}$, where $\psi'_{\rho,R}(r) = \Lambda(\rho)\left(\int_\rho^R \Lambda(t)\,dt\right)^{-1}$
and $\Lambda(r) = \frac{\mathcal{G}^m(r)}{\omega^{m-1}(r)}$ with $\mathcal{G}(r) = \exp(\int_\rho^r h(t))\,dt$.

A generalization of Ichihara's theorem: sketch of the Proof II

- It is straightforward to check that

$$0 \leq \psi_{\rho,R}(r) \leq 1 \ \forall r \in [0, R] \text{and } \forall R > 0$$

- To transplant $\psi_{\rho,R}(r)$ to the annulus in P determined by the extrinsic balls $A_{\rho,R} = D_R(o) \setminus \bar{D}_\rho(o)$. Then use Greene and Wu's comparison for the Hessian of the distance function r and that $\psi'_{\rho,R}(r) \geq 0$. Then

$$\Delta^P \psi_{\rho,R}(r(x)) \geq 0 = \Delta^P v(x)$$

where $v(x)$ is the solution of the Laplace equation in $A_{\rho,R} = D_R \setminus D_\rho$. Hence $\psi_{\rho,R}(r(x))$ is subharmonic $\forall R > 0$

A generalization of Ichihara's theorem: sketch of the Proof III

- On the other hand, fixing $x \in P$ such that $r(x) > \rho$, we have that $\{\psi_{\rho,R}(r(x))\}_{R>\rho}$ is decreasing because $\frac{d}{dR}\psi_{\rho,R}(r(x)) = -\Lambda(R)\int_{\rho}^{r(x)}\Lambda(t) \leq 0 \;\forall R$.
- Then, define $\psi_\rho : P \setminus \bar{D}_\rho(o) \longrightarrow \mathbb{R}$ as

$$\psi_\rho(x) := lim_{R\to\infty}\frac{\int_{\rho}^{r(x)}\Lambda(t)\,dt}{\int_{\rho}^{R}\Lambda(t)\,dt}$$

Well defined, non-constant, bounded and subharmonic.

Thank you