Cubic minimal cones and Jordan algebras

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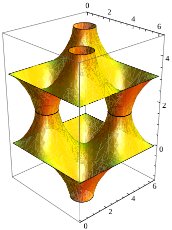


Entire solutions

Bernstein's theorem, 1916

Any solution $u(x_1, x_2)$ of the minimal surface equation (=MSE)

$$H[u] := \Delta u + \Delta_1 u$$
$$= (1 + |Du|^2)\Delta u - \frac{1}{2} Du \cdot (D|Du|^2) = 0.$$

defined in the **whole** $\mathbb{R}^2$ is an affine function: $u(x) = \mathbf{x} \cdot \mathbf{a} + b$.

Here $H(u)$ stands for the mean curvature operator and

$$\Delta_p u := |Du|^2 \Delta u + \frac{p-2}{2} Du \cdot D|Du|^2$$

is the p-Laplace operator.

Entire solutions

The “Bernstein property” (= B.P.) for $n \geq 3$

- W.H. Fleming (1962) and E. De Giorgi (1965): to prove the B.P. for solutions in $\mathbb{R}^n$ is sufficed to check that *no non-trivial minimal cones existed in* $\mathbb{R}^n$.

- There are no minimal cones in $\mathbb{R}^n$ for $n \leq 7$: Fleming $n = 3$ (1962), F.J. Almgren $n = 4$ (1966), and J. Simons $n \leq 7$ (1968).

- In 1969, E. Bombieri, E. De Giorgi and E. Giusti found the first non-affine entire solution of the minimal surface equation

$$(1 + |Du|^2)\Delta u - \frac{1}{2} Du \cdot (D|Du|^2) = 0, \qquad x \in \mathbb{R}^8.$$

  The construction heavily depends on certain properties of the quadratic minimal (Clifford–Simons) cones over $\mathbb{S}^3 \times \mathbb{S}^3$, namely

$$\{(x, y) \in \mathbb{R}^4 \times \mathbb{R}^4 : |x|^2 - |y|^2 = 0\}.$$

- More examples in $\mathbb{R}^n$, $n \geq 8$, were found by L. Simon (1989).

- There is no explicit representation available for the constructed examples.

Some important questions

- Why 8?
- Is it possible to provide any explicit entire (non-affine) solution of MSE?
- Are there any polynomial solutions of MSE?

Some more motivations

Doubly and triply periodic examples in the Minkowski space-times $\mathbb{R}^{1,2}$ and $\mathbb{R}^{1,3}$ due to V. Sergienko and V.Tk. (2001), and J. Hoppe (1995), resp.

A fourfold periodic minimal hypersurface in $\mathbb{R}^4$, V.Tk. (2008)

Let $s(x):\mathbb{R}\to[-1,1]$ be the Jacobi sinus of modulus $\sqrt{-1}$, i.e. $s'^2(t)=1-s^4(t)$, $s(t+\omega)=-s(t)$, where $\omega=\frac{\Gamma(\frac{1}{4})^2}{2\sqrt{2\pi}}$. Then

$$M=\{x=(x_1,x_2,x_3,x_4)\in\mathbb{R}^4:\ s(x_1)s(x_2)-s(x_3)s(x_4)=0\},$$

is an embedded minimal 4-periodic minimal hypersurface in $\mathbb{R}^4$ with isolated Clifford cone type singularities at the vertices of a periodic lattice Λ:

$$x_1x_2-x_3x_4=0 \qquad \text{or} \qquad x_1^2+x_2^2=x_3^2+x_4^2.$$

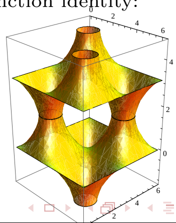
Let $\Gamma\subset O(4)$ be the stabilizer of Λ. Then M is Γ-invariant.

Remark. The proof uses the following Jacobi sinus function identity:

$S(Ax)=Q(x)S(x+h\omega), \quad h=\frac{1}{2}(1,1,1,1)\in\mathbb{R}^4$

where $Q(x)\neq 0$ is bounded in $\mathbb{R}^4$ and

$S(x)=s(x_1)s(x_2)-s(x_3)s(x_4)$.



... and a few more motivations

- embedded minimal hypersurfaces in $\mathbb{R}^n$ with conic singularities;
- minimal hypersurfaces in the spheres;
- searching for explicit entire solutions of MSE;

...

Eigencubics

Suppose u is a homogeneous polynomial, $\deg u = k \geq 2$ then the cone $u^{-1}(0)$ is minimal if and only **if u divides $\Delta_1 u$**, i.e.

$$\Delta_1 u = |Du|^2 \Delta u - \frac{1}{2} Du \cdot (D|Du|^2) \equiv 0 \mod u \tag{1}$$

W. Hsiang, J. Diff. Geom., 1(1967)

Problem 1. How does one classify irreducible minimal cubic forms in n variables, $n \geq 4$, with respect to the natural action of $O(n)$)? Or we may ask a weaker question, namely, whether there are always irreducible minimal cubic forms in n variables for all $n \geq 4$.

[...] For example, it is very difficult to classify irreducible cubic forms in n variables such that

$$\Delta_1 u = |Du|^2 \Delta u - \frac{1}{2} Du \cdot (D|Du|^2) = \lambda |x|^2 u(x).$$

Problem 2. For a given dimension n, $n \geq 4$, are there irreducible homogeneous polynomials in n real variables of *arbitrary high degree*, which give minimal cones of codimension one in $\mathbb{R}^n$? Or, if the degree is bounded, how does one express the bound in terms of n?

Problem 3. Are there any closed minimal submanifolds of codimension one in $\mathbb{S}^m$ which are not algebraic? Or, if possible, show that every closed minimal submanifold of codimension one in $\mathbb{S}^m$ is algebraic.

Definition

A polynomial solution of $\Delta_1 u \equiv 0 \mod u$ is called an **eigenfunction**.

An eigenfunction of $\deg u = 3$ is called an **eigencubic**.

A solution of $\Delta_1 u(x) = \lambda |x|^2 \cdot u$ is called a **radial** eigencubic.

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A solution of $\Delta_1 u(x) = \lambda |x|^2 \cdot u$ is called a **radial** eigencubic.

- any *linear* function is an eigenfunction of Δ_1;
- the only *quadratic* eigenfunctions are $u(x) = (q-1)(x_1^2 + \ldots + x_p^2) - (p-1)(x_{p+1}^2 + \ldots + x_n^2), \quad p+q=n.$
- in degree $k \geq 3$ the main difficulty is the absence of any normal form;
- some eigenfunctions of $\deg u = 3, 4, 6$ sporadically distributed in $\mathbb{R}^n$ were found in 1960s-1970s.

In what follows, we always suppose that $\deg u = 3$.

Remark. Observe that two cubic forms u_1 and u_2 produces two congruent cones in $\mathbb{R}^n$ if and only if they are **congruent**, i.e.

$$u_2(x) = C \cdot u_1(Ox), \qquad O \in O(n).$$

Hsiang's trick

Let $\mathfrak{G}'(k,\mathbb{R})$ be the real vector space of quadratic forms of k real variables with **trace zero**:

$$\begin{array}{ccccc} \mathfrak{G}'(k,\mathbb{R}) \cong & \mathrm{Herm}'_k(\mathbb{R}) & \cong & \mathbb{R}^N, & \\ & \uparrow & & \uparrow & \\ & O(k) & \hookrightarrow & O(N) & \end{array} \qquad \text{where } N := \frac{(k-1)(k+2)}{2}$$

- $\mathfrak{G}'(k,\mathbb{R})$ is invariant under action of $O(k)$ as substitutions
- $\det(x+\lambda\mathbf{1}) = \lambda^k + b_2(x)\lambda^{k-2} + \ldots + b_k(x)$ is a basic $O(k)$-invariant:

$$\mathbb{R}[x_1,\ldots,x_N]^{O(k)} = \mathbb{R}[b_2,\ldots,b_k]$$

- it is well-known that Δ_1 is $O(N)$-invariant; in particular, if u is an invariant polynomial with respect to $O(k)\hookrightarrow O(N)$, then so also is $\Delta_1 u$, therefore

$$\Delta_1 : \mathbb{R}[b_2,\ldots,b_k] \to \mathbb{R}[b_2,\ldots,b_k];$$

- in view of $\deg \Delta_1 b_3 = 5$ one has

$$\Delta_1 b_3 = c_1 \cdot b_2 b_3 + c_2 \cdot b_5,$$

and it follows that for $k \le 4$

$$\Delta_1 b_3 = c_1 b_2 \cdot b_3,$$

i.e. b_3 is an eigenfunction!

Hsiang's trick: $k = 3$

For $k = 3$ one also has $\deg \Delta b_3 = 1$ and $\deg |Db_3|^2 = 4$, hence

$$|D\, b_3|^2 = c_3 b_2^2 = c_3 |x|^4 \quad \text{and} \quad \Delta b_3 = 0.$$

É. Cartan (1938)

The only cubic polynomial solutions of

$$|Du(x)|^2 = 9|x|^4, \qquad \Delta u(x) = 0 \tag{2}$$

are

$$u_d(x) := \frac{3\sqrt{3}}{2} \det \begin{pmatrix} x_2 - \frac{1}{\sqrt{3}} x_1 & \bar{z}_1 & \bar{z}_2 \\ z_1 & -x_2 - \frac{1}{\sqrt{3}} x_1 & \bar{z}_3 \\ z_2 & z_3 & \frac{2}{\sqrt{3}} x_1 \end{pmatrix}, \quad x \in \mathbb{R}^{3d+2}, \tag{3}$$

where $z_k \in \mathbb{R}^d \cong \mathbb{F}_d$ is the real division algebra of dimension $d \in \{1, 2, 4, 8\}$.

It follows that b_3 is proportional to u_1.

Remark. Alternatively, the homogeneity of u_d and (2) implies

$$\Delta_1 u_d = -\frac{1}{2} D u_d \cdot D |Du_d|^2 = -54 |x|^2 u_d.$$

Thus, **all** u_d are eigencubics in $\mathbb{R}^5, \mathbb{R}^8, \mathbb{R}^{14}, \mathbb{R}^{26}$.

Hsiang's trick: $k = 4$

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For $k = 4$ one obtains the **Hsiang eigencubic** in $\mathbb{R}^9$ given by

$$h_1(x) = b_3(X) \sim \operatorname{tr} X^3,$$

where

$$X = \begin{pmatrix} x_1 - \frac{x_2}{\sqrt{3}} - \frac{x_3}{\sqrt{6}} & \bar{z}_4 & \bar{z}_5 & \bar{z}_6 \\ z_4 & -x_1 - \frac{x_2}{\sqrt{3}} - \frac{x_3}{\sqrt{6}} & \bar{z}_7 & \bar{z}_8 \\ z_5 & z_7 & \frac{2x_2}{\sqrt{3}} - \frac{x_3}{\sqrt{6}} & \bar{z}_9 \\ z_6 & z_8 & z_9 & \frac{\sqrt{2}x_3}{\sqrt{3}} \end{pmatrix}, \quad z_i \in \mathbb{R}$$

Remarks.

1. In fact, one still has a similar result for h_d for $z_i \in \mathbb{F}_d$ and $d = 1, 2, 4$, but not for $d = 8$.
2. By the Allison-Faulkner (1980) 'extracting Jordan algebras theorem':

$$h_1(x) \cong \det \begin{pmatrix} x'_1 & x'_2 & x'_3 \\ x'_4 & x'_5 & x'_6 \\ x'_7 & x'_8 & x'_9 \end{pmatrix}$$

Clifford eigencubics

Clifford eigencubics

Start with the Lawson cubic (1970)

$$u = \operatorname{Im} v^2 w = 2x_1x_2x_3 + x_4(x_1^2 - x_2^2), \quad \text{where} \quad v = x_1 + x_2 i,\ w = x_3 + x_4 i$$

$$= x_3 \cdot v^\top \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} v + x_4 \cdot v^\top \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} v$$

In general, given a **Clifford system**, i.e. a family $\mathcal{A} = \{A_i\}_{0\le i\le q}$ of self-adjoint endomorphisms of $\mathbb{R}^{2m}$ s.t.

$$A_iA_j + A_jA_i = 2\delta_{ij} \cdot \mathbf{1}_{\mathbb{R}^{2m}},$$

we define

$$u_{\mathcal{A}} = v^\top A_w v := v^\top (\sum_{i=0}^{q} A_i w_i) v, \qquad x = (v, w) \in \mathbb{R}^{2m} \times \mathbb{R}^{q+1}.$$

Remark. An obstruction to the existence of a Clifford system is $\delta(q)|m$, where

q	1	2	3	4	5	6	7	8	...	q
$\delta(q)$	1	2	4	4	8	8	8	8	...	$16\delta(q-8)$

Clifford eigencubics

Classification of Clifford eigencubics, [Tka10b]

- $u_{\mathcal{A}}$ is an eigencubic: $\Delta_1 u_{\mathcal{A}} = -8|x|^2 u$;
- $u_{\mathcal{A}}$ and $u_{\mathcal{B}}$ are congruent iff the Clifford systems $\mathcal{A}$ and $\mathcal{B}$ are geometrically equivalent;
- the pair (q, m) (called the **index**) is an inner invariant of $u_{\mathcal{A}}$;
- the number of congruence classes of Clifford eigencubics of index (q, m) is 1 if $q \not\equiv 0 \pmod 4$, and $\lfloor m/2\delta(q) \rfloor + 1$ if $q \equiv 0 \pmod 4$;
- the following trace formula holds:

$$\operatorname{tr}(D^2 u)^3 = -24(q-1)u.$$

Practically, the index is restored from u by

$$q = 1 + \frac{|x|^2 \cdot \operatorname{tr}(D^2 u)^3}{\Delta_1 u}, \qquad m = n - q - 1.$$

Radial eigencubics

In summary: all known **irreducible** eigencubics are radial eigencubics, i.e.

$$\Delta_1 u(x) = \lambda |x|^2 \cdot u \qquad (4)$$

Remark. There exist, however, non-radial *reducible* eigencubics, e.g.,

$$u = x_5(2x_1^2 + 2x_2^2 - x_3^2 - x_4^2 - x_5^2),$$

which satisfies

$$\Delta_1 u = Q \cdot u,$$

with

$$Q(x) = -12(2x_1^2 + 2x_2^2 + x_3^2 + x_4^2 + 3x_5^2) \neq \lambda |x|^2.$$

Problem. Do there exist *irreducible* eigencubics?

Radial eigencubics

Definition

A radial eigencubic u is called **Clifford** eigencubic if it is congruent to some $u_{\mathcal{A}}$. Otherwise, u is called **exceptional**.

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Propostion

The Cartan eigencubic u_d is exceptional.

Proof. Indeed, suppose in some orthogonal coordinates,

$$u_d(x) = C \cdot v^{\top} A_w v, \quad C \in \mathbb{R}, \ x = (v, w) \in \mathbb{R}^n.$$

Then $|Du_d|^2$ is at most quadratic in w-variables. On the other hand, we know that $|Du_d|^2 = 9|x|^4$. A contradiction follows because both the squared norm of the gradient and $|x|^4$ are orthogonal invariants.

Classification of radial eigencubics, I

Theorem A [Tka10c]

(a) Any radial eigencubic u is **harmonic**.

(b) There exists $\mathbb{R}^n = V_\xi \oplus V_\eta \oplus V_\zeta \oplus \mathbb{R}e_n$ such that

$$u \cong x_n^3 + \frac{3}{2}x_n(-2\xi^2 - \eta^2 + \zeta^2) + \Psi_{030} + \Psi_{111} + \Psi_{102} + \Psi_{012},$$

where $\Psi_{ijk} \in \xi^i \otimes \eta^j \otimes \zeta^k$.

(c) $n = 3n_1 + 2n_2 - 1$, where $\dim V_\xi = n_1$ and $\dim V_\eta = n_2$;

(d) **The trace formula**: $\mathrm{tr}(D^2u)^3 = 3(n_1 - 1)\lambda \cdot u$.

(e) **A hidden Jordan algebra structure**: $|D\Psi_{030}|^2 = 9|\eta|^4$.

(f) **A hidden Clifford algebra structure**: $(n_2 + n_1 - 1) \mid \delta(n_1 - 1)$.

Classification of radial eigencubics, I

Definition

It follows from (c) and (d) that the pair $(n_1, n_2) \in \mathbb{Z}^+ \times \mathbb{Z}^+$ is an inner invariant of u, and is called the **type** of u.

Remark 1. The type of u is recovered by invariant formulae

$$n_1 = \frac{|x|^2 \cdot \mathrm{tr}(D^2u)^3}{3\Delta_1 u} + 1, \qquad n_2 = \frac{n+1-3n_1}{2}.$$

Remark 2. A Clifford eigencubic $u_{\mathcal{A}}$ has type

$$(n_1, n_2) = (q, m+1-q).$$

Remark 3. The only type is insufficient for determining whether a given radial eigencubic is exceptional or Clifford. For instance, there are radial eigencubics of both kinds having the same type $(n_1, n_2) = (1, 5)$.

A short excursion into Jordan algebras

A short excursion into Jordan algebras

P. Jordan (1932): a program to discover a new algebraic setting for quantum mechanics by capture intrinsic algebraic properties of Hermitian matrices.

A Jordan (J.) algebra

J is vector space with a bilinear **commutative** product $\bullet : J \times J \to J$ satisfying the **the Jordan identity**

$$x^2 \bullet (x \bullet y) = x \bullet (x^2 \bullet y)$$

The algebra J is **formally real** if additionally $x^2 + y^2 = 0$ implies $x = y = 0$.

For any $x \in J$, the subalgebra $J(x)$ generated by x is associative. The **rank** of J is $\max\{\dim J(x) : x \in J\}$ and the **minimum polynomial** of x is

$$m_x(\lambda) = \lambda^r - \sigma_1(x)\lambda^{r-1} + \ldots + (-1)^r \sigma_r(x) \qquad \text{such that } m_x(x) = 0.$$

$\sigma_1(x) = \operatorname{Tr} x$ = the **generic trace** of x,
$\sigma_n(x) = N(x)$ = the **generic norm** (or generic determinant) of x.

Example 1. An associative algebra becomes a J. algebra with $x \bullet y = \frac{1}{2}(xy + yx)$.

Example 2. The Jordan algebra of $n \times n$ matrices over $\mathbb{R}$: $\operatorname{rank} x = n$, $\operatorname{Tr} x = \operatorname{tr} x$, $N(x) = \det x$.

Formally real Jordan algebras

Classification of formally real Jordan algebras

P. Jordan, J. von Neumann, E. Wigner, *On an algebraic generalization of the quantum mechanical formalism*, *Annals of Math.*, **1934**

Any (finite-dimensional) formally real J. algebra is a direct sum of the simple ones:

- the algebra $\mathfrak{h}_n(\mathbb{F}_1)$ of symmetric matrices over the reals;
- the algebra $\mathfrak{h}_n(\mathbb{F}_2)$ of Hermitian matrices over the complexes;
- the algebra $\mathfrak{h}_n(\mathbb{F}_4)$ of Hermitian matrices over the quaternions;
- the spin factors $\mathfrak{J}(\mathbb{R}^{n+1})$ with $(x_0, x) \bullet (y_0, y) = (x_0 y_0 + \langle x, y\rangle; \, x_0 y + y_0 x)$;
- $\mathfrak{h}_3(\mathbb{F}_8)$, the Albert exceptional algebra.

In particular, the only possible formally real J. algebras J with $\mathrm{rank}(J) = 3$ are:

a Jordan algebra	the norm of a trace free element
$J = \mathfrak{h}_3(\mathbb{F}_d), \ d = 1, 2, 4, 8$	$\sqrt{2}N(x) = u_d(x)$
$J = \mathbb{R} \oplus \mathfrak{J}(\mathbb{R}^{n+1})$	$\sqrt{2}N(x) = 4x_n^3 - 3x_n\|x\|^2$
$J = \mathbb{F}_1^3 = \mathbb{R} \oplus \mathbb{R} \oplus \mathbb{R}$	$\sqrt{2}N(x) = x_2^3 - 3x_2 x_1^2$

According [Tka10a], the second column is exactly the only cubic solutions to

$$|Du(x)|^2 = 9|x|^4.$$

Does there exist any explicit correspondence between cubic solutions of the latter equation and formally real cubic Jordan algebras?

Cubic eiconals and cubic Jordan algebras

Theorem , N. Nadirashvili, S. Vlăduţ and V.T. [NTV12b]

Let u be a cubic solution of

$$Q(Du) = 9Q(x)^2$$

on a quadratic space (V, Q). Then the multiplication

$$(x_0, x) \bullet (y_0, y) = (x_0 y_0 + Q(x, y),\ x_0 y + y_0 x + \frac{1}{3\sqrt{2}} \operatorname{hess} u(x, y)$$

turn $J_u := \mathbb{R} \oplus \mathbb{R}^n$ into a cubic Jordan algebra with the generic norm

$$N_u(x_0, x) = x_0^3 - \frac{3x_0 Q(x)}{2} + \frac{u(x)}{\sqrt{2}}.$$

Conversely, if (J, N) is a cubic Jordan algebra then

$$u(x) = \sqrt{2} N(x), \qquad x \in \{x \in J_u : \operatorname{Tr} x = 0\}.$$

Two solutions u_1 and u_2 are orthogonally equivalent if and only if the associated Jordan algebras J_{u_1} and J_{u_2} are isomorphic.

Classification of radial eigencubics, II

Theorem B, [Tka10c], [Tka12]

Let u be a radial eigencubic and let $\mathbb{R}^n = V_\xi \oplus V_\eta \oplus V_\zeta \oplus \mathbb{R}e_n$ be the associated orthogonal decomposition. Then there exists a natural Jordan algebra structure on $J := \mathbb{R}e_n \oplus V_\eta$ such that

$$u(x)|_J = \mathrm{Tr}(x^3), \quad x \in J$$

where Tr is the generic trace on J. Moreover, the following conditions are equivalent:

- u is an **exceptional** radial eigencubic;
- $\Psi_{030} \equiv u|_{V_\eta}$ is **reducible**;
- Ψ_{030} is **harmonic** and $n_2 \neq 2$;
- J is a **simple** Jordan algebra;
- $\mathrm{tr}(D^2u)^2 = \mathrm{const} \cdot |x|^2$ and $n_2 \in \{0, 5, 8, 14, 26\}$.

In particular, a radial eigencubic is exceptional iff $\mathrm{tr}(D^2u)^2 = c|x|^2$ and $n_2 \in \{0, 5, 8, 14, 26\}$.

n_1	1	2	3	5	9	0	1	2	4	0	1	2	5	9	0	1	3	1	3	7
n_2	0	0	0	0	0	5	5	5	5	8	8	8	8	8	14	14	14	26	26	26
n	2	5	8	14	26	9	12	15	21	15	18	21	30	42	27	30	36	54	60	72
								?				?	?	?			?	?	?	?

Radial eigencubics vs isoparametric hypersurfaces

Theorem C, [Tka10c]

Let u be a radial eigencubic in $\mathbb{R}^n$ of type (n_1, n_2) and $n_2 \neq 0$. Then u is congruent to the cubic form

$$u = (|\xi|^2 - |\eta|^2)x_n + a(\xi,\zeta) + b(\eta,\zeta) + c(\xi,\eta,\zeta), \tag{5}$$

where

$$x = (\xi, \eta, \zeta, x_n) \in \mathbb{R}^m \times \mathbb{R}^m \times \mathbb{R}^{n-2m-1} \times \mathbb{R}^1$$

and $a \in \xi \otimes \zeta^2$, $b \in \eta \otimes \zeta^2$, $c \in \xi \otimes \eta \otimes \zeta$. Moreover, the **quartic** polynomials

$$h_0(\xi,\eta) := (|\xi|^2 + |\eta|^2)^2 - 2|D_\zeta c|^2 \in \mathrm{Isop}(n_1 - 1, m - n_1),$$
$$h_1(\xi,\eta) := -|\xi|^4 + 6|\xi|^2|\eta|^2 - |\eta|^4 - 2|D_\zeta c|^2 \in \mathrm{Isop}(n_1, m - n_1 - 1).$$

If f, in addition, is an **exceptional** eigencubic then m is uniquely defined by $m = d + n_1 + 1$, where $n_2 = 3d + 2$, $d \in \{1, 2, 4, 8\}$.

Remark. Combining the latter correspondence with the recent characterization of isoparametric hypersurfaces with 4 principal curvatures (by T. Cecil, Q.S.Chi, G. Jensen, S. Immerwoll), one obtains an obstruction to the existence of some exceptional families of radial eigencubics.

Radial eigencubics revisited

A unified construction [Tka12]

Let W be a simple rank 3 Jordan algebra over $\mathbb{F}=\mathbb{R}$ or $\mathbb{F}=\mathbb{C}$, and let $J\subset W$ be a simple (Jordan) subalgebra of W. Let $\mathbb{R}^n = W\ominus J$ be the Euclidean vector space equipped with the inner product $\langle x,y\rangle = \operatorname{Tr} x\bar{y}$. Then

$$u(x) := \operatorname{Re}\operatorname{Tr} x^3, \qquad x\in\mathbb{R}^n,$$

is an **exceptional** radial eigencubic.

Example 5. Consider $J=\mathfrak{h}_3(\mathbb{F}_1)\subset\mathfrak{h}_3(\mathbb{F}_8)=W$. Then $W\ominus J\cong\mathbb{R}^{21}$ is the vector space of matrices

$$x := \begin{pmatrix} 0 & z_3 & z_2 \\ -z_3 & 0 & z_1 \\ -z_2 & -z_1 & 0 \end{pmatrix}, \quad \bar{z}_i = -z_i \in \mathbb{F}_d = \mathbb{O},$$

hence $u := \operatorname{Re}\operatorname{Tr} x^3$ is an exceptional eigencubic in $\mathbb{R}^{21}$.

Singular solutions of fully-nonlinear PDE's

Singular solutions of fully-nonlinear PDE's

- Evans, Crandall, Lions, Jensen, Ishii: If $\Omega \subset \mathbb{R}^n$ is bounded with C^1-boundary, ϕ continuous on $\partial\Omega$, F uniformly elliptic operator then the Dirichlet problem

$$\begin{aligned} F(D^2u) &= 0, \quad \text{in} \quad \Omega \\ u &= \phi \quad \text{on} \quad \partial\Omega \end{aligned}$$

  has a unique viscosity solution $u \in C(\Omega)$;
- Krylov, Safonov, Trudinger, Caffarelli, early 80's: the solution is always $C^{1,\varepsilon}$
- Nirenberg, 50's: if $n = 2$ then u is classical (C^2) solution
- Nadirashvili, Vlăduţ, 2007: if $n = 12$ then there are solutions which are not C^2

In 2005–2011, N. Nadirashvili and S. Vlăduţ constructed (uniformly elliptic) Hessian equations with $F(S)$ being smooth, homogeneous, depending only on the eigenvalues of S, and such that they have singular $C^{1,\delta}$-solutions.

- $$u(x) = \frac{\operatorname{Re} z_1 z_2 z_3}{|x|}, \qquad x = (z_1, z_2, z_3),$$
  where $z_i \in \mathbb{F}_d$, $d = 4, 8$ (quaternions, octonions), is a viscosity solution of a fully nonlinear uniformly elliptic equation. In fact,
  $$u(x) = \frac{N(x)}{|x|}, \qquad x \in \mathfrak{h}_3(\mathbb{F}_d) \ominus \mathbb{R}^3$$
  is an exceptional eigencubic.
- N.N., S.V., V.T. [NTV12a]:
  $$u(x) = \frac{N(x)}{|x|}, \qquad x \in \mathfrak{h}_3(\mathbb{R}) \ominus \mathbb{R}^1$$
  is a non-classical solution in $\mathbb{R}^5$.

Thank you!

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