Canonical variational completion of differential equations

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Abstract

Given a non-variational system of differential equations, the simplest way of turning it into a variational one is by adding a correction term. In the paper, we propose a way of obtaining such a correction term, based on the so-called Vainberg-Tonti Lagrangian, and present several applications in general relativity and classical mechanics.

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1 Introduction

For a given non–variational system of differential equations, there are multiple ways of transforming it into a variational one - among these, variational multipliers (or variational integrating factors), [1], are maybe the most well known. Another possibility is to simply add a correction term.

In the paper, we consider systems of ordinary or partial differential equations - represented by *source forms,* or *source tensors,* similar to Euler-Lagrange systems for extremals of integral variational functionals in the calculus of variations. We propose a way of obtaining such a correction term - which we call a variational completion, as follows. Any ordinary or partial differential system can be expressed as the vanishing of some source form ε on sections of an appropriate jet bundle. Further, to this source form, one can naturally attach a Lagrangian λ_ε, called the Vainberg-Tonti Lagrangian of ε, [7]; this Lagrangian has the property that the difference

$$\tau := E(\lambda_\varepsilon) - \varepsilon \tag{1}$$

between its Euler-Lagrange form $E(\lambda_\varepsilon)$ and ε offers a measure of the non-variationality of ε. Using τ in (1) as the correction term, the system $\varepsilon + \tau = 0$ becomes variational.

The method appears to have several interesting applications. We present here three of them.

1) *Einstein tensor obtained from variational completion of the Ricci tensor.* Historically, the first variant of gravitational field equations proposed by Einstein was:

$$R_{ij} = \frac{8\pi\kappa}{c^4} T_{ij}, \tag{2}$$

where: R_{ij} is the Ricci tensor of a 4-dimensional Lorentzian manifold (X, g), T_{ij} is the energy-momentum tensor, while κ and c are constants, [8]. This variant correctly predicted some physical facts, but failed to fulfil another request: local energy-momentum conservation. This led Einstein to adding in the left hand side the "correction term" $-\frac{1}{2}Rg_{ij}$ (by a reasoning based on Bianchi identities), thus leading to the nowadays famous:

$$R_{ij} - \frac{1}{2}Rg_{ij} = \frac{8\pi\kappa}{c^4} T_{ij}. \tag{3}$$

The variational deduction of (3), due to Hilbert, relies on a heuristic argument - simplicity. Hilbert chose to construct the action for the left hand side using the "simplest scalar" (i.e., simplest differential invariant) which can be constructed from the metric tensor and its derivatives alone. Happily, the Euler-Lagrange expression ensuing from this simplest scalar - which is the scalar curvature R - coincides with the left hand side of (3).

There is, still, another way of finding this correction term. Equation (2) is not variational. Actually, the term which fails to be variational is R_{ij}; in the paper, we prove that the Hilbert Lagrangian is (up to multiplication by a non-essential constant), nothing else than the Vainberg-Tonti Lagrangian corresponding to R_{ij}. Accordingly, the correction term $-\frac{1}{2}Rg_{ij}$ can be obtained from R_{ij} as a canonical variational completion.

2) *Energy-momentum tensors.* In special relativity, energy-momentum tensors are obtained by adding to the Noether current corresponding to the invariance of the matter Lagrangian to space-time translations a symmetrization term. The way of obtaining the symmetrization term is subject to an old debate, [2], [3]. The canonical variational completion method offers the possibility of recovering the expression of a full, symmetric energy-momentum tensor from just one of its terms - e.g., from a non-symmetrized Noether current. In particular, the energy-momentum tensor of the electromagnetic field can be obtained this way.

3) In *classical mechanics*, equations of *damped small oscillations* are known to be non-variational. Without aiming to give a general physical interpretation of the obtained correction term, we determine the canonical variational completion of these equations.

In Sections 2 and 3, we briefly present some known notions and results to be used in the following.

2 Differential forms on jet bundles

The mathematical background for a modern formulation of both field theory and mechanics are fibered manifolds and their jet bundles.

Consider a fibered manifold Y of dimension $m+n$, with n-dimensional base X and projection $\pi : Y \to X$. Fibered charts (V, ψ), $\psi = (x^i, y^\sigma)$ on Y induce the fibered charts (V^r, ψ^r), $\psi^r = (x^i, y^\sigma, y^\sigma_{j_1}, ..., y^\sigma_{j_1 j_2 ... j_r})$ on the r-jet prolongation $J^r Y$ of Y and (U, ϕ), $\phi = (x^i)$ on X. The manifold $J^r Y$ can be regarded as a fibered manifold in multiple ways, by means of the projections:

$$\pi^{r,s} : J^r Y \to J^s Y, \quad (x^i, y^\sigma, y^\sigma_{j_1}, ..., y^\sigma_{j_1 j_2 ... j_r}) \mapsto (x^i, y^\sigma, y^\sigma_{j_1}, ..., y^\sigma_{j_1 j_2 ... j_s}),$$

where $r > s$, $J^0 Y := Y$ and:

$$\pi^r : J^r Y \to X.$$

The set of $\mathcal{C}^\infty$-smooth sections $\gamma : X \to Y$, locally expressed by some functions $(x^i) \mapsto \gamma(x^i) = (x^i, y^\sigma(x^i))$ is denoted by $\Gamma(Y)$. Given a section $\gamma \in \Gamma(Y)$, its prolongation to $J^r Y$ is: $J^r\gamma : (x^i) \mapsto J^r\gamma(x^i) = (x^i, y^\sigma(x), y^\sigma_{,j}(x), ..., y^\sigma_{,j_1 j_2 ... j_r}(x))$, where the symbol $_{,j}$ denotes partial differentiation with respect to x^j.

In field theoretical applications, the coordinates x^i play the role of space-time coordinates, while y^σ are "field" coordinates (to be accurate, real fields are encoded in *sections* $y^\sigma = y^\sigma(x^i)$). The case of mechanics is characterized by $\dim X = 1$; in this case, the coordinates on $J^r Y$ are usually denoted by $(t, q^\sigma, \dot{q}^\sigma, \ddot{q}^\sigma, ..., q^{(r)})$ and are interpreted as: time, generalized coordinates, generalized velocity etc.

By $\Omega^r_k W$, we denote the set of k-forms of order r over an open set $W \subset Y$, i.e., the set of k-forms over the r-th prolongation $J^r W \subset J^r Y$. In particular, $\mathcal{F}(W) := \Omega^r_0 W$ is the set of real-valued smooth functions over $J^r W$.

The subset of $\Omega^r_k W$ consisting of k-forms:

$$\rho = \frac{1}{k!} a_{i_1 i_2 ... i_k} dx^{i_1} \wedge dx^{i_2} \wedge ... \wedge dx^{i_k}, \tag{4}$$

(where $a_{i_1 i_2 ... i_k}$, $k \leq n$, are smooth functions of the coordinates $x^i, y^\sigma, y^\sigma_{j_1}, ..., y^\sigma_{j_1 j_2 ... j_r}$) is called the set of ($\pi^r$-)*horizontal k-forms* of order r; similarly, one can speak about $\pi^{r,s}$-horizontal forms of order r as forms generated by exterior products of the differentials $dx^i, dy^\sigma, ..., dy^\sigma_{j_1 ... j_s}$.

Examples of π^r-horizontal forms are volume forms and Lagrangians.

For $X = \mathbb{R}^n$, the *Euclidean volume form* is:

$$\omega_0 = dx^1 \wedge dx^2 \wedge ... \wedge dx^n. \tag{5}$$

On pseudo-Riemannian manifolds (X, g_{ij}), a coordinate-invariant volume form is locally given by: $dV = \sqrt{|g|}\omega_0$, where $g := \det(g_{ij})$.

A *Lagrangian* of order r is defined as a π^r-horizontal n-form of order r :

$$\lambda = \mathcal{L}\omega_0, \qquad \mathcal{L} = \mathcal{L}(x^i, y^\sigma, ..., y^\sigma_{i_1...i_r}). \tag{6}$$

A form $\theta \in \Omega^r_k Y$ is a *contact form* if it is annihilated by all jets $J^r\gamma$ of sections $\gamma \in \Gamma(Y)$. Important examples are the *basic contact 1-forms* on J^rY defined on a coordinate neighborhood by:

$$\omega^\sigma = dy^\sigma - y^\sigma{}_j dx^j, \quad \omega^\sigma{}_{i_1} = dy^\sigma{}_{i_1} - y^\sigma{}_{i_1 j} dx^j, ... \tag{7}$$
$$\omega^\sigma{}_{i_1 i_2 ... i_{r-1}} = dy^\sigma{}_{i_1 i_2 ... i_{r-1}} - y^\sigma{}_{i_1 i_2 ... i_{r-1} j} dx^j.$$

A differential form is called p-contact if it is generated by p-th exterior powers of contact forms.

3 Source forms and variationality conditions

A *source form* of order r on a fibered manifold Y, [5], is a $\pi^{r,0}$-horizontal, 1-contact $(n+1)$-form on J^rY. In local coordinates, any source form is expressed as:

$$\varepsilon = \varepsilon_\sigma \omega^\sigma \wedge \omega_0, \quad \varepsilon_\sigma = \varepsilon_\sigma(x^i, y^\sigma, y^\sigma{}_i, ..., y^\sigma{}_{j_1...j_r}). \tag{8}$$

The set of source forms of order at most r over Y is closed under addition and under multiplication with functions $f \in \mathcal{F}(J^rY)$.

The most notable example of a source form is the *Euler Lagrange form* $E(\lambda)$ of a Lagrangian $\lambda = \mathcal{L}(x^i, y^\sigma, ..., y^\sigma_{i_1...i_r})\omega_0 \in \Omega^r_n(Y)$:

$$E(\lambda) := E_\sigma \omega^\sigma \wedge \omega_0,$$
$$E_\sigma = \frac{\partial \mathcal{L}}{\partial y^\sigma} - d_{k_1} \frac{\partial \mathcal{L}}{\partial y^\sigma{}_{k_1}} + ... + (-1)^r d_{k_1}...d_{k_r} \frac{\partial \mathcal{L}}{\partial y^\sigma{}_{k_1...k_r}}.$$

A section $\gamma : X \to Y$ is critical for the Lagrangian λ if and only if the $E(\lambda)$ is annihilated by the r-jet of γ, i.e., $E_\sigma(\lambda) \circ J^r\gamma = 0$, $\sigma = 1, ..., m$.

A source form ε is called:

a) *locally variational* if around any point of the fibered manifold Y, there exists a local fibered chart (V, ψ) and a Lagrangian λ on some jet prolongation V^r ($r \in \mathbb{N}$) of V, such that, on V^r, $\varepsilon = E(\lambda)$;

b) *globally variational* if there exists a Lagrangian λ on the whole manifold Y such that $\varepsilon = E(\lambda)$.

Local variationality of a source form $\varepsilon = \varepsilon_\sigma \omega^\sigma \wedge \omega_0$ of order r is equivalent to a generalization of classical Helmholtz conditions, [6]:

$$H_{\sigma\nu}{}^{j_1...j_k}(\varepsilon) = 0, \quad k = 0, ..., r, \tag{9}$$

where:

$$H_{\sigma\nu}{}^{j_1...j_k}(\varepsilon) = \frac{\partial \varepsilon_\sigma}{\partial y^\nu{}_{j_1...j_k}} - (-1)^k \frac{\partial \varepsilon_\nu}{\partial y^\sigma{}_{j_1...j_k}} - \tag{10}$$
$$- \sum_{l=k+1}^{r} (-1)^l \binom{l}{k} d_{i_{k+1}} d_{i_{k+2}} ... d_{i_l} \frac{\partial \varepsilon_\nu}{\partial y^\sigma{}_{j_1...j_k i_{k+1}...i_l}}$$

locally describe the *Helmholtz form* $H_\varepsilon = \frac{1}{2}\sum_{k=0}^{r} H_{\sigma\nu}{}^{j_1...j_k}(\varepsilon)\omega^\nu{}_{j_1...j_k} \wedge \omega^\sigma \wedge \omega_0$.

4 Canonical variational completion

By *variational completion* of a given source form ε on Y, we will mean any source form τ on Y with the property that $\varepsilon + \tau$ is variational. Of course, one can speak about local and about global variational completions.

In the following, we will only study local variational completions.

Clearly, every source form has infinitely many variational completions: indeed, any Lagrangian λ induces the completion $\tau := E(\lambda) - \varepsilon$. Thus, the question is how to choose the Lagrangian λ in a meaningful way. In the following, we will try to give an answer to this question.

Given an arbitrary source form $\varepsilon = \varepsilon_\sigma \omega^\sigma \wedge \omega_0 \in \Omega^r_{n+1}Y$ of order r, a local Lagrangian attached to ε is the *Vainberg-Tonti Lagrangian* $\lambda_\varepsilon = \mathcal{L}_\varepsilon \omega_0$, [7], [4], defined by:

$$\mathcal{L}_\varepsilon(x^i, y^\sigma, ..., y^\sigma{}_{j_1...j_s}) = y^\sigma \int_0^1 \varepsilon_\sigma(x^i, uy^\sigma, ..., uy^\sigma{}_{j_1...j_s})du. \tag{11}$$

The Euler-Lagrange form $E(\lambda_\varepsilon) = E_\nu \omega^\nu \wedge \omega_0$ of the Vainberg-Tonti Lagrangian λ_ε is given, [7], by:

$$E_\nu = \varepsilon_\nu - \int_0^1 u\{y^\sigma(H_{\nu\sigma} \circ \chi_u) + y^\sigma{}_j(H_{\nu\sigma}{}^j \circ \chi_u) + ... + y^\sigma{}_{j_1...j_r}(H_{\nu\sigma}{}^{j_1...j_r} \circ \chi_u)\}du,$$

where $\chi_u : J^{2r}Y \to J^{2r}Y$ denotes the homothety $(x^i, y^\sigma, y^\sigma{}_j, ..., y^\sigma{}_{j_1...j_{2r}}) \mapsto (x^i, uy^\sigma, uy^\sigma{}_j, ..., uy^\sigma{}_{j_1...j_{2r}})$ and the coefficients $H_{\sigma\nu}{}^{j_1...j_k}$ are as in (9).

From (9), it follows that the coefficients $H_{\sigma\nu}{}^{j_1...j_k}$ above have the meaning of "obstructions from variationality" of the source form ε. In particular, if the source form ε is variational, then $E(\lambda_\varepsilon) = \varepsilon$.

It thus appears as natural

Definition 1 *The canonical variational completion of a source form $\varepsilon \in \Omega^r_{n+1}(Y)$, is the source form $\tau(\varepsilon)$ given by the difference between the Euler-Lagrange form of the Vainberg–Tonti Lagrangian of ε and ε itself:*

$$\tau(\varepsilon) = E(\lambda_\varepsilon) - \varepsilon. \tag{12}$$

The local coefficients τ_ν of the canonical variational completion $\tau(\varepsilon) = \tau_\nu \omega^\nu \wedge \omega_0$ can be directly expressed in terms of the coefficients $H_{\nu\sigma}{}^{j_1...j_k}$:

$$\tau_\nu = -\int_0^1 u\{y^\sigma(H_{\nu\sigma}\circ\chi_u) + y^\sigma{}_j(H_{\nu\sigma}{}^j\circ\chi_u) + ... + y^\sigma{}_{j_1...j_r}(H_{\nu\sigma}{}^{j_1...j_r}\circ\chi_u)\}du.$$

Remark. Generally speaking, the Vainberg-Tonti Lagrangian and, accordingly, the canonical variational completion of a source form of order r, are of order $2r$. Still, under certain conditions, [4] (which are fulfilled by a large number of equations in physics), the Vainberg-Tonti Lagrangian is actually equivalent to a Lagrangian of order r.

5 Source forms in general relativity

Consider a Lorentzian manifold (X, g_{ij}) of dimension 4, with local charts (U, ϕ), $\phi = (x^i)_{i=\overline{0,3}}$ and Levi-Civita connection ∇. We denote by R_{ij} the Ricci tensor of ∇ and by $R = g^{ij}R_{ij}$, the scalar curvature. We assume in the following that measurement units are chosen in such a way that $c = 1$. Indices of tensors will be lowered or raised by means of the metric g_{ij} and its inverse g^{ij}.

Einstein field equations (3) arise by varying with respect to the metric tensor the Lagrangian $\lambda = \lambda_g + \lambda_m$, where:

i) $\lambda_g = -\dfrac{1}{16\pi\kappa}R\sqrt{|g|}\omega_0$ (with $\omega_0 = dx^0 \wedge dx^1 \wedge dx^2 \wedge dx^3$) is the Hilbert Lagrangian;

ii) the *matter Lagrangian* $\lambda_m = L_m\sqrt{|g|}\omega_0,$ is given by a differential invariant $L_m = L_m(g_{ij}, g_{ij,h}, ...; y^\sigma, y^\sigma{}_j, ..., y^\sigma{}_{j_1...j_r})$ depending on the metric tensor components and their derivatives up to a certain order $s \in \mathbb{N}$ and on the r-jet of a field y^σ. Typically, in classical general relativity, $s = 0$.

In the case of vacuum Einstein equations

$$R_{ij} - \frac{1}{2}Rg_{ij} = 0, \tag{13}$$

the "field components" to be varied are the metric tensor components g_{ij} (or, more commonly, the inverse metric components g^{ij}), hence the fibered manifold Y is the bundle of metrics $Met(X)$, defined as the set of symmetric nondegenerate tensors of type (0,2) on X. Since both R_{ij} and R are of second order in g_{ij}, the space we have to work on is the second order jet bundle $J^2Met(X)$.

We denote the local charts on $Met(X)$ by (V, ψ), with $\psi = (x^i, g_{jk})$ and the induced fibered chart on $J^2Met(X)$, by (V^2, ψ^2), with $\psi^2 = (x^i, g_{jk}; g_{jk,i}; g_{jk,il})$. We will also use the notations:

$$\omega_{jk} = dg_{jk} - g_{jk,i}dx^i; \;\; \omega_{jk,l} = dg_{jk,l} - g_{jk,li}dx^i$$

for the basic contact forms on $J^2Met(X)$. The Riemann tensor, the Ricci tensor and the Ricci scalar thus become objects on $J^2Met(X)$.

5.1 Canonical variational completion of the Ricci tensor

We will prove in the following that vacuum Einstein equations (13) can be obtained by means of the canonical variational completion of the source form with components R_{ij}.

Take the following source form on $J^2Met(X)$:

$$\varepsilon := \alpha R^{ij}\sqrt{|g|}\omega_{ij} \wedge \omega_0, \tag{14}$$

where α is a (momentarily) arbitrary constant. Its components $\varepsilon^{ij} = \varepsilon^{ij}(g_{kl}; g_{kl,i}, g_{kl,ij})$ are given by

$$\varepsilon^{ij} = \alpha R^{ij}\sqrt{|g|}.$$

The Vainberg-Tonti Lagrangian $\lambda_\varepsilon = \mathcal{L}_\varepsilon \omega_0$ is defined as:

$$\mathcal{L}_\varepsilon = g_{ij}\int\limits_0^1 \varepsilon^{ij}(ug_{kl}; ug_{kl,i}; ug_{kl,ij})du.$$

Let us study the behavior of the integrand with respect to homotheties $\chi_u : (g_{kl}; g_{kl,i}; g_{kl,ij}) \mapsto (ug_{kl}; ug_{kl,i}; ug_{kl,ij})$. These homotheties induce the transformation $g^{kl} \mapsto u^{-1}g^{kl}$ of the inverse metric tensor components. The Christoffel symbols

$$\Gamma^i{}_{jk} = \frac{1}{2}g^{ih}(g_{hj,k} + g_{hk,j} - g_{jk,h})$$

are invariant to χ_u and hence the curvature tensor components $R_j{}^i{}_{kl} = \Gamma^{\ i}_{\ jk,l} - \Gamma^i{}_{jl,k} + \Gamma^h{}_{jk}\Gamma^i{}_{hl} - \Gamma^h{}_{jl}\Gamma^i{}_{hk}$ are also invariant. The Ricci tensor $R_{jk} = R_j{}^i{}_{ki}$ is obtained just by a summation process from $R_j{}^i{}_{kl}$, which means that it is also insensitive to χ_u. That is, $R^{ij} = g^{ih}g^{jl}R_{hl}$ will acquire a u^{-2}.

It remains to compute the contribution of χ_u to the factor $\sqrt{|g|}$. Each line of the matrix (g_{jk}) is multiplied by u, that is, $g = \det(g_{ij})$ will acquire a factor of u^4 and finally,

$$\sqrt{|g \circ \chi_u|} = u^2\sqrt{|g|}.$$

Substituting into the expression of $\mathcal{L}_\varepsilon$, we get this way,

$$\mathcal{L}_\varepsilon = g_{ij}\int\limits_0^1 u^0 \alpha R^{ij}\sqrt{|g|}du = \alpha g_{ij}R^{ij}\sqrt{|g|}\int\limits_0^1 u^0 du = \alpha R\sqrt{|g|}.$$

Thus, if we choose

$$\alpha := \frac{-1}{16\pi\kappa},$$

the Vainberg-Tonti Lagrangian $\lambda_\varepsilon = \mathcal{L}_\varepsilon\omega_0$ becomes the Hilbert Lagrangian λ_g :

$$\lambda_\varepsilon = \lambda_g. \tag{15}$$

We know, however, that the Euler-Lagrange expressions of $R\sqrt{|g|}$ with respect to g_{ij} are given by (minus) the contravariant components of the Einstein tensor. In differential form writing, this is:

$$E(\lambda_\varepsilon) = \frac{1}{16\pi\kappa}(R^{ij} - \frac{1}{2}Rg^{ij})\sqrt{|g|}\omega_{ij} \wedge \omega_0$$

hence, we find the variational completion $\tau = E(\lambda_\varepsilon) - \varepsilon$ as

$$\tau = \frac{1}{16\pi\kappa}(2R^{ij} - \frac{1}{2}Rg^{ij})\sqrt{|g|}\omega_{ij} \wedge \omega_0.$$

Remark. The factor α in (14) is actually unessential, the variationally completed equation

$$E(\lambda_\varepsilon) = 0$$

is still the correct vacuum Einstein equation, regardless of its value.

5.2 Energy-momentum tensors

Having one term of an energy-momentum tensor, the canonical variational completion method offers a way of recovering its full expression. We will apply this method in the case when the known piece is a (non-symmetrized) Noether current.

In the case of Einstein equations with matter (3), we will have to work on a fibered product $Y \times_X Met(X)$ over X (where Y is a fibered manifold with base X) with coordinate charts (V, ψ), $\psi = (x^i, y^\sigma, g_{jk})$. In this case, one can speak separately about variations with respect to y^σ and to g_{jk} and accordingly, about Y-variationality and $Met(X)$-variationality, Y- and $Met(X)$-variational completions.

Consider a first order Lagrangian λ_m on $Y \times_X Met(X)$; we suppose in addition that λ_m does not depend on x^j and on the derivatives $g_{ij,k}$. Thus, $\lambda_m = L_m\sqrt{|g|}\omega_0$, where

$$L_m = L_m(y^\sigma, y^\sigma{}_j, g_{ij}).$$

In classical relativity theory, there are two major ways of defining energy-momentum tensors, corresponding to two different contexts:

1) The *canonical energy-momentum tensor*, corresponding to special relativity (where $X = \mathbb{R}^4$ and the metric tensor is fixed as $\eta_{ij} = diag(1, -1, -1, -1)$). A Lagrangian $\lambda_m = L_m\omega_0$, which is invariant to the group of space-time translations $\tilde{x}^i = x^i + a^i$, $a^i = const.$, gives rise to a system of conserved Noether currents, called the canonical energy-momentum tensor (invariance to space-time translations amounts to the above assumption that L_m does not explicitly depend on x^i). These Noether currents are given by, [8]:

$$\tilde{T}^{ij} = \eta^{ik}(y^\sigma{}_{,k}\frac{\partial L_m}{\partial y^\sigma{}_{,j}} - \delta^j_k L_m). \tag{16}$$

The canonical energy-momentum tensor $\tilde{T}^{ij}$ is, generally, not symmetric - which is inconvenient, since symmetry is required on physical grounds (angular momentum conservation). This is usually solved by adding a divergence-free term, thus obtaining a tensor $\overset{can}{T}{}^{ij}$ which is symmetric and still conserved, i.e., $\overset{can}{T}{}^{ij}{}_{,j} = 0$. There are multiple possibilities of choosing the symmetrization term, [2].

2) In general relativity (where (X, g_{ij}) is an arbitrary Lorentzian manifold), energy-momentum tensors (*Hilbert, or metric energy-momentum tensors*) $\overset{met}{T}{}^{ij}$ are defined by means of functional derivatives of the matter Lagrangian $\lambda_m = \mathcal{L}_m \omega_0$, $\mathcal{L}_m = L_m \sqrt{|g|}$, with respect to g_{ij}:

$$-\frac{1}{2}\mathcal{T}^{ij} := \frac{\delta \mathcal{L}_m}{\delta g_{ij}}; \quad \overset{met}{T}{}^{ij} := \frac{1}{\sqrt{|g|}}\mathcal{T}^{ij}. \tag{17}$$

Here, $L_m = L_m(y^\sigma, y^\sigma{}_j, y^\sigma{}_j; g_{ij})$ is a differential invariant (a "*scalar*"), hence the Lagrangian λ_m is invariant to (transformations on $J^r Y$ induced by) arbitrary diffeomorphisms on X. As a result, $\overset{met}{T}{}^{ij}$ obeys on-shell the *covariant conservation law* $\overset{met}{T}{}^{ij}{}_{;j} = 0$ and also, has gauge invariance properties, [2]. Moreover, $\overset{met}{T}{}^{ij}$ is, by construction, symmetric.

The two procedures of defining the energy-momentum tensor are fundamentally different and obviously require a thorough geometric analysis. Just as a first remark, they generally do not even make sense at the same time: in special relativity, where the metric is fixed, it makes no sense to speak about variations of a Lagrangian with respect to the metric. On the other hand, in general relativity, where X is an arbitrary manifold, space-time translations $\tilde{x}^i = x^i + a^i$, $a^i = const.$, cannot be defined geometrically. However, there is a realm (see, e.g., [10]) where both procedures can be applied, namely, when:

$$X = \mathbb{R}^4, \quad g_{ij} - arbitrary \tag{18}$$

(actually, in [10], it is pointed out the particular case of *weak metrics* – in which the author studies the equivalence between the two definitions. Still, for our purposes, we do not need the assumption that the metric is weak).

For a special-relativistic Lagrangian

$$\lambda_m = L_m \omega_0, \quad L_m = L_m(y^\sigma, y^\sigma{}_i, g_{ij} = \eta_{ij}), \tag{19}$$

the canonical variational completion offers a recipe of symmetrization of the Noether current $\tilde{T}^{ij}$. We will do this in three steps:

Step 1. We leave for the moment the special relativistic context and formally allow g^{ij} to vary. Abiding by the principle of general covariance, [8], a straightforward generalization of (16) to the new context is given by the tensor density:

$$\tilde{\mathcal{T}}^{ij} = g^{ik}(y^\sigma{}_{;k}\frac{\partial L_m}{\partial y^\sigma{}_{;j}} - \delta^j_k L_m)\sqrt{|g|}, \tag{20}$$

where the semicolon $_{;k}$ denotes (formal) covariant differentiation with respect to $\partial/\partial x^k$.

Note: In the above, y^σ are tensors of some unspecified rank (the upper position of the index is chosen just for convenience; y^σ can very well be components of, e.g., a scalar, a covector field or of a tensor of type (0,2)).

Step 2. Taking into account (17), we consider the source form $\varepsilon = \alpha\tilde{\mathcal{T}}^{ij}\omega_{ij}\wedge\omega_0$ on $J^1(Y\times_X Met(X))$, with components $\varepsilon^{ij} = \alpha\tilde{\mathcal{T}}^{ij}(y^\sigma, y^\sigma{}_j, g_{kh})$, where $\alpha \in \mathbb{R}$ is a constant. Its $Met(X)$-Vainberg-Tonti Lagrangian $\lambda_\varepsilon := \mathcal{L}_\varepsilon\omega_0$ is:

$$\mathcal{L}_\varepsilon = \alpha g_{ij}\int_0^1 (\tilde{\mathcal{T}}^{ij}\circ\chi_u)du,$$

where $\chi_u(y^\sigma, y^\sigma{}_i, g_{ij}) := (y^\sigma, y^\sigma{}_i, ug_{ij})$ only affects the metric components. Substituting $\tilde{\mathcal{T}}^{ij}$ from (20) and taking into account that χ_u leaves Christoffel symbols invariant and that $\delta^i_i = \dim(X) = 4$, we have:

$$\mathcal{L}_\varepsilon = \alpha\int_0^1 u(y^\sigma{}_{;i}\frac{\partial(L_m\circ\chi_u)}{\partial y^\sigma{}_{;i}} - 4L_m\circ\chi_u)\sqrt{|g|}du. \tag{21}$$

Further, we calculate the Hilbert energy-momentum tensor of λ_ε as:

$$-\frac{1}{2}\overset{met}{T}{}^{ij} := \frac{1}{\sqrt{|g|}}\frac{\delta\mathcal{L}_\varepsilon}{\delta g_{ij}}. \tag{22}$$

Step 3. Finally, particularize in (22) g_{ij} as η_{ij} and define

$$\overset{met}{T}{}^{ij} :|_{g_{ij}=\eta_{ij}} =: T^{ij}.$$

This way, $\overset{met}{T}{}^{ij}$ is defined up to multiplication by the constant α. This constant can then be adjusted, for instance, in such a way that the obtained symmetrization term

$$\tau^{ij} := T^{ij} - \tilde{T}^{ij} \tag{23}$$

is independent from $\tilde{T}^{ij}$ (it does not contain any multiple of $\tilde{T}^{ij}$).

The covariant conservation law of $\overset{met}{T}{}^{ij}$ (obtained as a consequence of the fact that $\overset{met}{T}{}^{ij}$ is a Hilbert energy-momentum tensor) now transforms into the usual conservation law: $T^{ij}{}_{,j} = 0$. Thus, the obtained energy-momentum tensor T_{ij} is, as required, both symmetric and conserved. Moreover, the symmetrzation term τ^{ij} offers a measure of the non-$Met(X)$-variationality of $\tilde{T}^{ij}$.

Example: *energy-momentum tensor of the electromagnetic field.*
The electromagnetic field is described by the potential 1-form $A = A_i dx^i$ on X and by the 2-form $F := dA = \dfrac{1}{2}F_{ij}dx^i\wedge dx^j$.

In the special relativistic case $g_{ij} = \eta_{ij}$, we have $F_{ij} = A_{j,i} - A_{i,j}$, or, in terms of the contravariant components A^i : $F_{ij} = \eta_{jk} A^k{}_{,i} - \eta_{ik} A^k{}_{,j}$. The Lagrangian of the electromagnetic field is $\lambda_f = L_f \omega_0$ with

$$L_f = -\frac{1}{16\pi} F_{ij} F^{ij}; \tag{24}$$

Translational invariance of λ_f leads to the Noether current, [8]:

$$\tilde{T}^{ij} = -\frac{1}{4\pi} \eta^{ih} \frac{\partial A^l}{\partial x^h} F^j{}_l + \frac{1}{16\pi} \eta^{ij} F_{kl} F^{kl}. \tag{25}$$

The curved space generalization of $\tilde{T}^{ij}$ in (25) is the tensor density:

$$\tilde{\mathcal{T}}^{ij} = \left(-\frac{1}{4\pi} g^{ih} A^l{}_{;h} F^j{}_l + \frac{1}{16\pi} g^{ij} F_{kl} F^{kl}\right)\sqrt{|g|} \tag{26}$$

where, this time:

$$F_{ij} = g_{jk} A^k{}_{;i} - g_{ik} A^k{}_{;j}. \tag{27}$$

Further, we calculate the Vainberg-Tonti Lagrangian of the source form

$$\varepsilon = \alpha \tilde{\mathcal{T}}^{ij} \omega_{ij} \wedge \omega_0,$$

where $\tilde{\mathcal{T}}^{ij} = \tilde{\mathcal{T}}^{ij}(A^k; A^k_{,l}; g_{kl}; g_{kl,h})$. We prefer to use A^k rather than $A_k := g_{kl} A^l$ as the field variables for a reason which will become transparent below. This way, χ_u acts as follows:

$$g_{ij} \circ \chi_u = u g_{ij}, \quad g^{ij} \circ \chi_u = u^{-1} g^{ij},$$

while χ_u does not affect the field variables $y^\sigma = A^k$. Again, the Christoffel symbols $\Gamma^i{}_{jk}$ are invariant to χ_u. Expressing F_{ij} as in (27), we can now determine the effect of χ_u on each term of $\tilde{\mathcal{T}}^{ij}$:

$$A^l{}_{;i} \circ \chi_u = A^l{}_{;i}; \quad F_{jl} \circ \chi_u = u F_{jl}; \quad F^{kl} \circ \chi_u = u^{-1} F^{kl}, \quad \sqrt{|g|} \circ \chi_u = u^2 \sqrt{|g|}.$$

All in all, we have:

$$\tilde{\mathcal{T}}^{ij} \circ \chi_u = u \tilde{\mathcal{T}}^{ij}$$

and hence, the Vainberg-Tonti Lagrangian $\lambda_\varepsilon = \mathcal{L}_\varepsilon \omega_0$ is given by:

$$\mathcal{L}_\varepsilon = g_{ij} \int_0^1 u \alpha \tilde{\mathcal{T}}^{ij} du = \frac{\alpha}{2} g_{ij} \tilde{\mathcal{T}}^{ij},$$

that is,

$$\mathcal{L}_\varepsilon = \alpha\left(-\frac{1}{8\pi} A^{l;k} F_{kl} + \frac{1}{8\pi} F_{kl} F^{kl}\right)\sqrt{|g|}. \tag{28}$$

Taking into account that $F_{kl} = -F_{lk}$, the term $A^{l;k}F_{kl}$ in the above can be re-expressed as: $A^{l;k}F_{kl} = \frac{1}{2}(A^{l;k} - A^{k;l})F_{kl} = \frac{1}{2}F^{kl}F_{kl}$; substituting into (28), we finally obtain the $Met(X)$-Vainberg-Tonti Lagrangian of (26) as:

$$\lambda_\varepsilon = \frac{\alpha}{16\pi}F^{kl}F_{kl}\sqrt{|g|}\omega_0 = -\alpha\lambda_f. \tag{29}$$

But, variation of λ_f with respect to g^{ij} is well-known, [8]; namely, we will get for $\lambda_\varepsilon = -\alpha\lambda_f$ the Hilbert energy-momentum tensor

$$\overset{met}{T}{}^{ij}(\alpha) = -\alpha(-\frac{1}{4\pi}F^{il}F^{j}_{\ l} + \frac{1}{16\pi}g^{ij}F_{kl}F^{kl}).$$

Particularizing now $g^{ij} = \eta^{ij}$, we get the symmetrized energy-momentum tensor:

$$T^{ij} := \overset{met}{T}{}^{ij} = -\alpha(\tilde{T}^{ij} + \frac{1}{4\pi}A^{i}_{\ ,l}F^{jl}).$$

Taking $\alpha := -1$ (which provides $\overset{met}{T}{}^{ij} = \tilde{T}^{ij} + independent\_term$), we obtain $\lambda_\varepsilon = \lambda_f$ and the symmetrization term:

$$\tau^{ij} = \frac{1}{4\pi}A^{i}_{\ ,l}F^{jl},$$

or, in covariant writing,

$$\tau_{ij} = \frac{1}{4\pi}A_{i,l}F_{j}^{\ l}$$

as the correction term. This is the classical symmetrization term, [8], yet, obtained here by a completely different reasoning.

Remarks.

1) If, for a given (symmetrized or not) energy-momentum tensor $\tilde{T}^{ij}$, the Lagrangian λ_m is not known, a Lagrangian can be constructed as the $Met(X)$-Vainberg-Tonti Lagrangian (21); if a Lagrangian λ_m is already known, the above gives an alternative construction.

1) If the given matter Lagrangian density $\mathcal{L}_m$ is *homogeneous* both in the metric components and in the derivatives $y^{\sigma}_{\ ;i}$ (the homogeneity degrees need not coincide) then, applying Euler's theorem in (21), we see that the Vainberg-Tonti Lagrangian density $\mathcal{L}_\varepsilon$ in (21) actually coincides, up to multiplication by some constant, with the matter Lagrangian density $\mathcal{L}_m$. In this case, we can always choose α such that $\mathcal{L}_\varepsilon = \mathcal{L}_m$. In this case, the symmetrization term coincides with the one in [3], yet, it is found just by considerations of variationality.

2) If we had worked with the potential 1-form components A_i (instead of the vector field components A^i) as our field variables, we would have had $F_{ij} = A_{j;i} - A_{i;j}$ - invariant to χ_u and by a similar reasoning to the above, we would have got $\tilde{\mathcal{T}}^{ij}(ug_{kl}) = u^{-1}\tilde{\mathcal{T}}^{ij}(g_{kl})$ and, consequently, to $\mathcal{L}_\varepsilon = (g^{ij}\tilde{\mathcal{T}}_{ij})\int_0^1 u^{-1}du$. But, since the latter integral does not have a finite value, we could not have

calculated $\mathcal{L}_\varepsilon$ this way. Hence, it appears that, at least in this case, the 4-potential *vector field* components A^i are a more advantageous choice for our dynamical variables.

6 An example in first order mechanics

Take $Y = \mathbb{R} \times \mathbb{R}^n$, with local coordinates (t, q^σ); on the second jet prolongation J^2Y, we denote the induced local coordinates by $(t, q^\sigma, \dot{q}^\sigma, \ddot{q}^\sigma)$.

Consider the second order source form

$$\varepsilon = \varepsilon_\sigma \omega^\sigma \wedge dt,$$

$$\varepsilon_\sigma = m_{\sigma\nu}\ddot{q}^\nu + k_{\sigma\nu}q^\nu + \frac{\partial F}{\partial \dot{q}^\sigma}, \tag{30}$$

where:

- $m_{\sigma\nu}$, $k_{\sigma\nu}$ are constant and symmetric;
- $F = F(\dot{q}^\sigma)$ is homogeneous of some degree $p \geq 1$ in $\dot{q}^\sigma$.

The ODE system $\varepsilon_\sigma = 0$ is generally non-variational. Let us determine its canonical variational completion. The Vainberg-Tonti Lagrangian attached to ε is $\lambda_\varepsilon = \mathcal{L}_\varepsilon dt$, with

$$\mathcal{L}_\varepsilon = q^\sigma \int_0^1 \varepsilon_\sigma(t, uq^\nu, u\dot{q}^\nu, u\ddot{q}^\nu)du = q^\sigma \int_0^1 (m_{\sigma\nu}u\ddot{q}^\nu + k_{\sigma\nu}uq^\nu + \frac{\partial F}{\partial \dot{q}^\sigma}(u\dot{q}^\nu))du.$$

Taking into account the homogeneity degree of F, this is:

$$\begin{aligned}
\mathcal{L}_\varepsilon &= q^\sigma \int_0^1 [u(m_{\sigma\nu}\ddot{q}^\nu + k_{\sigma\nu}q^\nu) + u^{p-1}\frac{\partial F}{\partial \dot{q}^\sigma}]du = \\
&= \frac{1}{2}(m_{\sigma\nu}\ddot{q}^\nu q^\sigma + k_{\sigma\nu}q^\sigma q^\nu) + \frac{1}{p}q^\sigma \frac{\partial F}{\partial \dot{q}^\sigma}.
\end{aligned}$$

The term $\frac{1}{2}m_{\sigma\nu}\ddot{q}^\nu q^\sigma$ differs by a total derivative $d_t(\frac{1}{2}m_{\sigma\nu}\dot{q}^\nu q^\sigma)$ from $-\frac{1}{2}m_{\sigma\nu}\dot{q}^\nu\dot{q}^\sigma$, hence the two expressions are dynamically equivalent. We will thus prefer to take the latter, which is of lower order and thus, we obtain the following Lagrangian function, which is equivalent to the Vainberg-Tonti one:

$$\mathcal{L} = \frac{1}{2}(-m_{\sigma\nu}\dot{q}^\nu\dot{q}^\sigma + k_{\sigma\nu}q^\sigma q^\nu) + \frac{1}{p}q^\sigma \frac{\partial F}{\partial \dot{q}^\sigma}. \tag{31}$$

Let us determine the Euler-Lagrange form of $\mathcal{L}$. We have, on one hand:

$$\frac{\partial \mathcal{L}}{\partial q^\rho} = k_{\sigma\rho}q^\sigma + \frac{1}{p}\frac{\partial F}{\partial \dot{q}^\rho}$$

and, on the other hand,

$$\begin{aligned}
\frac{\partial \mathcal{L}}{\partial \dot{q}^\rho} &= -m_{\sigma\rho}\dot{q}^\rho + \frac{1}{p}\frac{\partial^2 F}{\partial \dot{q}^\sigma \partial \dot{q}^\rho} q^\sigma, \\
d_t\left(\frac{\partial \mathcal{L}}{\partial \dot{q}^\rho}\right) &= -m_{\sigma\rho}\ddot{q}^\rho + \frac{1}{p}\frac{\partial^3 F}{\partial \dot{q}^\sigma \partial \dot{q}^\rho \partial \dot{q}^\nu}\ddot{q}^\nu q^\sigma + \frac{1}{p}\frac{\partial^2 F}{\partial \dot{q}^\sigma \partial \dot{q}^\rho}\dot{q}^\sigma;
\end{aligned}$$

taking again into account that $\frac{\partial F}{\partial \dot{q}^\rho}$ is homogeneous of degree $p-1$, the latter term is: $\frac{1}{p}\frac{\partial^2 F}{\partial \dot{q}^\sigma \partial \dot{q}^\rho}\dot{q}^\sigma = \frac{p-1}{p}\frac{\partial F}{\partial \dot{q}^\rho}$ and, finally,

$$E_\rho(\mathcal{L}) = (m_{\sigma\rho}\ddot{q}^\sigma + k_{\sigma\rho}q^\sigma) + \frac{2-p}{p}\frac{\partial F}{\partial \dot{q}^\rho} - \frac{1}{p}\frac{\partial^3 F}{\partial \dot{q}^\sigma \partial \dot{q}^\rho \partial \dot{q}^\nu}\ddot{q}^\nu q^\sigma.$$

We find the variational completion $\tau = \tau_\rho(t, q, \dot{q})\omega^\rho \wedge dt$ as:

$$\tau_\rho = 2\left(\frac{1}{p} - 1\right)\frac{\partial F}{\partial \dot{q}^\rho} - \frac{1}{p}\frac{\partial^3 F}{\partial \dot{q}^\sigma \partial \dot{q}^\rho \partial \dot{q}^\nu}\ddot{q}^\nu q^\sigma. \tag{32}$$

Particular cases:

1) If $F = 0$, the system $\varepsilon_\sigma = 0$ is equivalent to

$$m_{\sigma\nu}\ddot{q}^\nu + k_{\sigma\nu}q^\nu = 0.$$

These equations characterize free small oscillations with multiple degrees of freedom, [8]. They are known to be variational; their Lagrangian function $\mathcal{L} = \frac{1}{2}(-m_{\sigma\nu}\dot{q}^\nu\dot{q}^\sigma + k_{\sigma\nu}q^\sigma q^\nu)$ coincides (as expected), with (31).

2) If $p = 2$ and F is quadratic in $\dot{q}$:

$$F = \frac{1}{2}\alpha_{\sigma\nu}\dot{q}^\sigma\dot{q}^\nu,$$

(where $\alpha_{\sigma\nu} = \alpha_{\nu\sigma} \in \mathbb{R}$) the ODE system $\varepsilon_\sigma = 0$ characterizes, [9], Section 25, linearly damped oscillations. In this case, the function F is called the *Rayleigh dissipation function* and is interpreted as the rate of energy dissipation in the system. In (30), the last term (with a minus in front) $-\frac{\partial F}{\partial \dot{q}^\sigma} = -a_{\sigma\nu}\dot{q}^\nu$ is interpreted as a friction force. In this case, the canonical variational completion (32) is given by

$$\tau_\rho = -\frac{\partial F}{\partial \dot{q}^\rho}$$

and the variationally completed equations are:

$$m_{\rho\nu}\ddot{q}^\nu + k_{\rho\nu}q^\nu = 0, \tag{33}$$

which are precisely the equations of "undamped" oscillations. That is, the friction force $\dfrac{\partial F}{\partial \dot{q}^\rho}$ has, in this case, the meaning of obstruction from variationality of the equations.

Remark. In other cases (e.g., when $-\dfrac{\partial F}{\partial \dot{q}^\rho}$ is quadratic or cubic in $\dot{q}^\sigma$), the variationally completed equations will not coincide anymore with the equations (33) of undamped oscillations.

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