Lucio Boccardo (Sapienza Università di Roma - Istituto Lombardo)
The impact of terms of order 1 in Dirichlet problemswith discontinuous coefficients
Granada, 24.4.2025

1

The study of the linear boundary value problem

$$(1)\qquad \begin{cases} -\mathrm{div}(M(x)Du) + u = -\mathrm{div}(u\,E(x)) + f(x), & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega. \end{cases}$$

and of the dual problem

$$(2)\qquad \begin{cases} -\mathrm{div}(M(x)D\psi) + \psi = E(x)\cdot D\psi + g(x), & \text{in } \Omega, \\ \psi = 0, & \text{on } \partial\Omega. \end{cases}$$

starts with G. Stampacchia; then we recall the results of [b-bumi] (see also [b-jde], [b-ucm]) concerning the existence of weak or distributional solutions.

Here Ω is a bounded, open subset of $\mathbb{R}^N$ with $\underline{N \geq 3}$, $M(x)$ is a measurable matrix such that

$$(3)\qquad \alpha|\xi|^2 \leq M(x)\xi\cdot\xi, \qquad |M(x)| \leq \beta,\ \forall\, \xi \in \mathbb{R}^N,$$

with $\alpha, \beta > 0$,

$$(4)\qquad E(x) \in (L^r(\Omega))^N, \quad r \geq N \quad (\text{or } r = 2)$$

$$(5)\qquad f(x) \in L^m(\Omega),\ m \geq 1.$$

The first difficulty in the study of the boundary value problem (1) is the loss of coercivity (explicit ex.) of the operator -div$\big(M(x)Du - u\,E(x)\big)$.

Nevertheless, a **formal** use of $\frac{u}{1+|u|}$, or $\frac{1}{h}\,T_h(u)$ $(h \to 0)$ as test function in the weak formulation of (1) gives these meager, but basic estimates (see [b-bumi] and [b-jde])

$$\begin{cases} \dfrac{\alpha}{2}\displaystyle\int_\Omega |D\log(1+|u|)|^2 \leq \dfrac{1}{2\alpha}\displaystyle\int_\Omega |E|^2 + \displaystyle\int_\Omega |f| \\ \|u\|_1 \leq \|f\|_1 \end{cases}$$

Our approach does not need

- $\operatorname{div}(E(x)) = 0$
- $\|E\|_{L^N}$ "small"

Here, I do not recall several contributions.

- Properties of u with respect to m (defined in (5)), if $r = N$ (r defined in (4)):

$$\begin{cases} m = 1 \\ 1 < m < \frac{2N}{N+2} \\ \frac{2N}{N+2} \leq m < \frac{N}{2} \\ m = \frac{N}{2} \\ m > \frac{N}{2} \end{cases} \tag{6}$$

- $r \neq N$

In a recent paper (see [bbc-jde]) the existence of weak solutions in $W_0^{1,2}(\Omega)$ for boundary value problems of the type

$$(7)\qquad \begin{cases} -\mathrm{div}(M(x)Du) + u = -\mathrm{div}(h(u)\,E(x)) + f(x), & \text{in } \Omega,\\ u = 0, & \text{on } \partial\Omega, \end{cases}$$

is studied, with

$$\lim_{t\to\infty} \frac{h(t)}{t} = \infty,$$

that is with a loss of coercivity more important. Two main cases are

$$(8)\qquad \begin{cases} h(u) = u\,|u|^{\theta},\ 0<\theta<1/N\\ E \in (L^r(\Omega))^N,\ r = \frac{N}{1-\theta N}, \end{cases} \qquad \begin{cases} h(u) = u\,\log(1+|u|),\\ E \in (L^N \log^N(\Omega))^N. \end{cases}$$

The loss of coercivity observed above it is even **stronger** (w.r.t. (1)) in the last boundary value problem (7), with $h(u)$ defined in (8).

A possible development is the study of the above problems in presence of singular coefficients like

$$(9)\qquad \begin{cases} -\mathrm{div}(M(x)Du) + a(x)\,u = -\mathrm{div}(h(u)\,E(x)) + f(x), & \text{in } \Omega,\\ u = 0, & \text{on } \partial\Omega, \end{cases}$$

with $0 \le a(x) \in L^1(\Omega)$, $f(x) \in L^1(\Omega)$. There are two direction to widen the study:

- existence of "entropy" solutions (following the trail of [b-jde]) for elliptic systems of the type

$$\begin{cases} -\mathrm{div}(M(x)Du) + u = -\mathrm{div}(u\,M(x)D\psi) + f(x), & \text{in } \Omega, \\ -\mathrm{div}(M(x)D\psi) + \psi = u^\theta & \text{in } \Omega, \\ u = \psi = 0, & \text{on } \partial\Omega. \end{cases} \quad \begin{cases} f(x) \geq 0 \\ 0 < \theta \leq 1 \end{cases}$$

- the use of the regularizing effect of the **Q-condition**

  $$|f(x)| \leq Q\,a(x) \in \underline{L^1(\Omega)}, \quad Q > 0,$$

  introduced in [ab].

2. N=2

Work in progress

3. No time for

"drift"(that is (2)), nonlinear principal part, parabolic, …