Regular(ized) varieties and Płonka sums of algebras

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A variety[1] of algebras is called *regular* when it does satisfy regular identities only, where an identity $\sigma \approx \tau$ (in a fixed algebraic language) is regular provided that exactly the same variables actually occur in both the terms σ and τ. Examples of regular varieties include semigroups, (commutative) monoids and semilattices. A variety is irregular if it is not regular. In particular, a variety $\mathcal{V}$ is *strongly irregular* if it possess a term-definable (binary) operation $f(x, y)$ such that $\mathcal{V} \models f(x, y) \approx x$. Intuitively, f behaves in $\mathcal{V}$ as a projection in the first component. Strongly irregular varieties are very common, as they include, for instance, groups, rings and any variety having a lattice (or group) reduct.

Given a variety $\mathcal{V}$, one can consider its *regularization* $R(\mathcal{V})$, namely the variety satisfying only the regular identities holding in $\mathcal{V}$. In case $\mathcal{V}$ is strongly irregular, then the elements of $R(\mathcal{V})$ admits a well-behaved structure theory: any algebra in $R(\mathcal{V})$ can be represented as a Płonka sum over a semilattice direct system of algebras in $\mathcal{V}$. The construction was introduced by the Polish algebraist J. Płonka [4, 5, 6, 7].

In this seminar, I will provide the details of the Płonka representation theorem for regularized varieties and show some of its applications, such as the description of the lattice of the subvarieties of $R(\mathcal{V})$ [2], of the subdirectly irreducible members [3] and of the equational basis. Finally, I will explain how the theory of Płonka sums has been applied in algebraic logic [1].

References

[1] S. Bonzio, F. Paoli, and M. Pra Baldi. *Logics of Variable Inclusion*. Springer, Trends in Logic, 2022.

[2] J. Dudek and E. Graczyinska. The lattice of varieties of algebras. *Bull. Acad. Polon. Sci. Ser. Sci. Math*, 29:337–340, 1981.

[3] H. Lakser, R. Padmanabhan, and C. R. Platt. Subdirect decomposition of Płonka sums. *Duke Mathematical Journal*, 39:485–488, 1972.

[4] J. Płonka. On a method of construction of abstract algebras. *Fundamenta Mathematicae*, 61(2):183–189, 1967.

[5] J. Płonka. On distributive quasilattices. *Fundamenta Mathematicae*, 60:191–200, 1967.

[6] J. Płonka. On the sum of a direct system of universal algebras with nullary polynomials. *Algebra Universalis*, 19(2):197–207, 1984.

[7] J. Płonka and A. Romanowska. Semilattice sums. In A. Romanowska and J. D. H. Smith, editors, *Universal Algebra and Quasigroup Theory*, pages 123–158. Heldermann, 1992.

[1] I am using the term with its usual meaning in Universal Algebra.