A PDE proof of the Riemannian Penrose inequality

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joint work with
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(M, g) complete noncompact 3-Riemannian manifold

- LINEAR potential theory: $\Delta u = 0$
  - $\mathbb{R}^3 \rightsquigarrow$ Willmore inequality [A.-Mazzieri, 2016]

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  - $\mathrm{Ric} \geq 0 \rightsquigarrow$ generalised Willmore inequality [A.-Fogagnolo-Mazzieri, 2018]

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$$0 \leq m_{\mathrm{ADM}}(M, g)$$

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$$\sqrt{\frac{|\partial M|}{16\pi}} \leq m_{\mathrm{ADM}}(M, g) \qquad \leftarrow \qquad \frac{1}{2} \mathrm{C}_p(\partial M)^{\frac{1}{3-p}} \leq m_{\mathrm{ADM}}(M, g)$$

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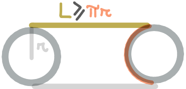

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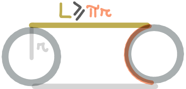

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summary:

- nonlinear potential in (M, g), with $\partial M \neq \emptyset$
- monotonicity result
- RPI
- sketches of the proof of RPI and of the monotonicity

(M, g) complete, noncompact, 3-Riem. mfld, ∂M smooth and compact

$$\begin{cases} \Delta_p u \underset{1}{=} 0 & \text{in } M \\ u = 0 & \text{on } \partial M \\ u \to 1 & \text{at } \infty \end{cases} \tag{P}$$

- range of u
- regularity
- critical points

[1] $\Delta_p u := \operatorname{div}(|\mathrm{D}u|^{p-2}\mathrm{D}u) = 0 \iff \int_\Omega \langle |\mathrm{D}u|^{p-2}\mathrm{D}u, \mathrm{D}\psi \rangle \mathrm{d}\mu = 0$, for every $\psi \in \mathrm{C}_c^\infty(M)$

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- quantity $\mathcal{E}_p(t) := 4\pi t - \dfrac{t^{\frac{2}{p-1}}}{c_p} \displaystyle\int_{\{u=\alpha_p(t)\}} |\mathrm{D}u|\, \mathrm{H}\, \mathrm{d}\sigma + \dfrac{t^{\frac{5-p}{p-1}}}{c_p^2} \displaystyle\int_{\{u=\alpha_p(t)\}} |\mathrm{D}u|^2\, \mathrm{d}\sigma$

$$c_p \underset{2}{=} \left(\frac{p-1}{3-p}\right) \mathrm{C}_p^{\frac{1}{p-1}} \qquad \alpha_p(t) = 1 - \left(\frac{t_p}{t}\right)^{\frac{3-p}{p-1}}$$

defined for every $t \geq t_p := \big((p-1)/(3-p)\, c_p\big)^{(p-1)/(3-p)}$ such that $\alpha_p(t)$ is regular

- if $p = 2$, then $c_2 = 1$ and

$$\mathcal{E}_2(t) = \mathcal{E}(t) := 4\pi t - t^2 \int_{\{u=1-\frac{1}{t}\}} |\mathrm{D}u|\, \mathrm{H}\, \mathrm{d}\sigma + t^3 \int_{\{u=1-\frac{1}{t}\}} |\mathrm{D}u|^2\, \mathrm{d}\sigma$$

[2] $\mathrm{C}_p := \inf \left\{ \left(\frac{p-1}{3-p}\right)^{p-1} \frac{1}{4\pi} \int_M |\mathrm{D}w|^p \mathrm{d}\mu : w \in \mathrm{C}_c^\infty(M),\ w \equiv 1 \text{ on } \partial M \right\}$

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Theorem [A.-Mantegazza-Mazzieri-Oronzio, 2022]:

$\mathrm{R} \geq 0$ & ∂M connected & $H_2(M, \partial M; \mathbb{Z}) = \{0\}$ & $\exists$ sln u to pb (P)

$\Downarrow$

$$\left.\begin{array}{r} s \leq t \\ \alpha_p(s), \alpha_p(t) \text{ reg. values} \end{array}\right\} \implies \mathcal{E}_p(s) \leq \mathcal{E}_p(t)$$

comment: $H_2(M;\mathbb{Z}) = \{0\} \quad \Rightarrow$ every regular level set of u is connected

Indeed:

- suppose $\{u = \tau\}$ regular $= \Sigma' \sqcup \Sigma''$
- $H_2(M;\mathbb{Z}) = \{0\} \quad \Rightarrow \quad \begin{cases} \Sigma' = \partial\Omega' \\ \Sigma'' = \partial\Omega'' \end{cases}$
- Strong maximum principle

$\Rightarrow \;\; o \in \Omega' \cap \Omega'' \;\; \Rightarrow \;\; \Omega' \subset\subset \Omega'' \;\; \Rightarrow \;\; o \in \Omega' \subset\subset \Omega'' \;\; \Rightarrow \;\; u \equiv const.$ in $\overline{\Omega}'' \setminus \Omega'$: imp.!

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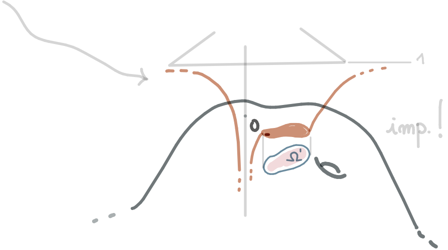


TIME SYMMETRIC CASE & CONNECTED HORIZON

$\Lambda = 0$ **Theorem [Huisken-Ilmanen, 2001]**: (M, g) complete noncompact 3-mfld.

(i) $\mathrm{R} \geq 0$

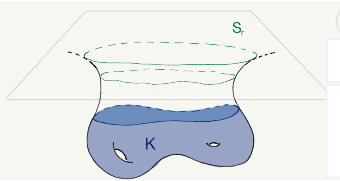
(ii) (M, g) is A. F.

(iii) ∂M outermost minimal

$\Longrightarrow$

- $m_{\mathrm{ADM}}(M, g) \geq \sqrt{\dfrac{|\partial M|}{16\pi}}$
- "=" holds iff

$$(M, g) \cong \left([2m, +\infty) \times \mathbb{S}^2, \frac{\mathrm{d}r \otimes \mathrm{d}r}{1 - \frac{2m}{r}} + r^2 g_{\mathbb{S}^2}\right)$$



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- $M \setminus K$ diffeo $\mathbb{R}^3 \setminus \overline{B}$ • $g_{ij}(x) = \delta_{ij} + O_2(|x|^{-\tau})$, $\tau > 1/2$

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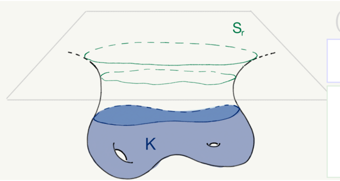

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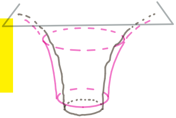

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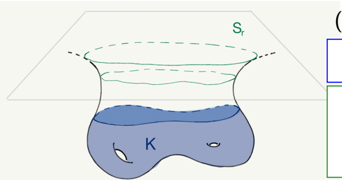

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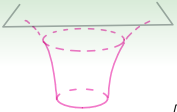
- $M \setminus K$ diffeo $\mathbb{R}^3 \setminus \overline{B}$ • $g_{ij}(x) = \delta_{ij} + O_2(|x|^{-\tau})$, $\tau > 1/2$

$$m_{\mathrm{ADM}}(M,g) = \lim_{r \to +\infty} \frac{1}{16\pi} \int_{S_r} \sum_{i,k=1}^{n} \left(\frac{\partial g_{ik}}{\partial x^k} - \frac{\partial g_{kk}}{\partial x^i} \right) \nu^i \mathrm{d}\sigma$$

Geometry of the spatial Schwarzschild metric:



$$M = \big[2m, +\infty\big) \times \mathbb{S}^2, \qquad g = \frac{dr \otimes dr}{1 - \frac{2m}{r}} + r^2 g_{\mathbb{S}^2}, \qquad u = \sqrt{1 - \frac{2m}{r}},$$

static s.lu

- the Ricci curvature of (M, g) is given by

$$\mathrm{Ric} = -\frac{2m}{r^3} \frac{dr \otimes dr}{\big(1 - \frac{2m}{r}\big)} + \frac{m}{r} g_{\mathbb{S}^2};$$

- the scalar curvature of (M, g) is

$$\mathrm{R} = 0;$$

- the coordinate spheres S_r are totally umbilic surfaces with

$$\mathrm{h}_{ij} = \frac{1}{r}\sqrt{1 - \frac{2m}{r}}\, g_{ij}$$

and constant mean curvature

$$\mathrm{H} = \frac{2}{r}\sqrt{1 - \frac{2m}{r}}$$

simplistic physical facts on m

Newtonian theory of gravity

- $V : \mathbb{R}^3 \to \mathbb{R}$ gravitational potential
- $\rho : \mathbb{R}^3 \to \mathbb{R}$ mass density $\qquad \begin{cases} \Delta_V = 4\pi\rho, & \text{in } \mathbb{R}^3 \\ V \to 0 & \text{at } \infty \end{cases}$

$$\Rightarrow \quad \int_{\mathbb{R}^3} \rho \, \mathrm{d}\mu = \frac{1}{4\pi} \lim_{r \to +\infty} \int_{S_r} \frac{\partial V}{\partial r} \mathrm{d}\sigma = \frac{1}{4\pi} \lim_{r \to +\infty} \int_{S_r} \left(\frac{m}{r^2} + O(r^{-1}) \right) \mathrm{d}\sigma = m$$

General relativity

- (M, g) complete 3D mfld = time-snapshot of an isolated gravitational system
- $\rho : \mathbb{R}^3 \to \mathbb{R}$ mass density $\qquad \begin{cases} \mathrm{R}_g = 16\pi\rho, & \text{in } M \\ (M, g) \ \text{A.F.} \end{cases}$

~~$\int_M \mathrm{R}_g \, d\mu = m$~~

$$m := \frac{1}{16\pi} \int_M \mathrm{d}\mathrm{R}_{\bar{g}}[g - \bar{g}] \mathrm{d}\mu = \frac{1}{16\pi} \lim_{r \to +\infty} \int_{S_r} \left\langle -\bar{\mathrm{D}} \bar{\mathrm{tr}} g + \overline{\mathrm{div}} g, \bar{\nu} \right\rangle \mathrm{d}\sigma = \frac{1}{16\pi} \lim_{r \to +\infty} \int_{S_r} \left(\frac{\partial g_{ij}}{\partial x^j} - \frac{\partial g_{jj}}{\partial x^i} \right) \nu^i \mathrm{d}\sigma$$

TIME SYMMETRIC CASE & CONNECTED HORIZON

$\Lambda = 0$ **Theorem [Huisken-Ilmanen, 2001]**: (M,g) complete noncompact 3-mfld.

(i) $\mathrm{R} \geq 0$
(ii) (M,g) is A. F.
(iii) ∂M outermost minimal

$\Longrightarrow$

- $m_{\mathrm{ADM}}(M,g) \geq \sqrt{\frac{|\partial M|}{16\pi}}$
- "=" holds iff

$$(M,g) \cong \left([r_m,+\infty)\times\mathbb{S}^2, \frac{\mathrm{d}r\otimes\mathrm{d}r}{1-\frac{2m}{r}} + r^2 g_{\mathbb{S}^2}\right)$$

$\Lambda < 0$ **Theorem [Lee-Neves, 2015]**: (M,g) complete noncompact 3-mfld.

(i) $\mathrm{R} \geq -6$
(ii) (M,g) is A. H.
(iii) ∂M outermost minimal $\cong \partial M_\infty$
(iv) $m(M,g) \leq 0$

$\Longrightarrow$

- $m(M,g) \geq \frac{1}{\gamma}\left[1 - \mathrm{g}(\partial M) + \frac{|\partial M|}{4\pi}\right]\sqrt{\frac{|\partial M|}{16\pi}}$

$$\gamma := \left(\max\{1, \mathrm{g}(\partial M) - 1\}\right)^{3/2}$$

- "=" holds iff

$$(M,g) \cong \left([r_{m,k},+\infty)\times\Sigma_k, \frac{\mathrm{d}r\otimes\mathrm{d}r}{k+r^2-\frac{2m}{r}} + r^2 g_{\Sigma_k}\right)$$

with $k = -1$

A

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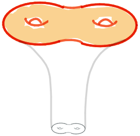

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$\Lambda = -3$, $m(M, g) \geq 0$,

Theorem [Ambrozio, 2015]: (M, g) complete noncompact 3-mfld.

(i) $\mathrm{R} \geq -6$
(ii) (M, g) is C^3-close to the Schwarzschild anti-de Sitter of mass $m > 0$
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$\Longrightarrow$

- $m(M, g) \geq \left[1 + \frac{|\partial M|}{4\pi}\right] \sqrt{\frac{|\partial M|}{16\pi}}$
- "=" holds iff $(M, g) \cong \left([r_m, +\infty) \times \mathbb{S}^2, \frac{\mathrm{d}r \otimes \mathrm{d}r}{1 + r^2 - \frac{2m}{r}} + r^2 g_{\mathbb{S}^2}\right)$

Theorem [Chodosh, 2016]: Asymptotically hyperbolic RPI for (M, g) Schwarzschild anti-de Sitter at infinity,..., and $m \geq 0$

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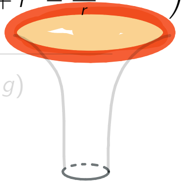

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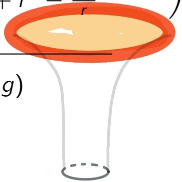

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sketch of our proof of RPI: $\qquad \Sigma_t := \{u = \alpha_p(t)\}$

- $$\mathcal{E}_p(t) := 4\pi t - t^{\frac{2}{p-1}} c_p^{-1} \int_{\Sigma_t} |\mathrm{D}u|\, \mathrm{H}\, \mathrm{d}\sigma + t^{\frac{5-p}{p-1}} c_p^{-2} \int_{\Sigma_t} |\mathrm{D}u|^2\, \mathrm{d}\sigma$$
- $s \leq t \quad \Longrightarrow \quad \mathcal{E}_p(s) \leq \mathcal{E}_p(t)$
- ∂M is minimal $\Rightarrow \mathcal{E}_p(t_p) = 4\pi t_p + \dfrac{t^{\frac{5-p}{p-1}}}{c_p^2} \displaystyle\int_{\partial M} |\mathrm{D}u|^2\, \mathrm{d}\sigma \geq 4\pi t_p$
- (M, g) is C^2_1-A.F. $\quad \Rightarrow \quad = 1 - \left(\dfrac{p-1}{3-p}\right) \dfrac{c_p}{|x|^{\frac{3-p}{p-1}}} + o_2(|x|^{-\frac{3-p}{p-1}})$

$$\Rightarrow \quad \lim_{t \to +\infty} \mathcal{E}_p(t) \leq 8\pi\, m_{\mathrm{ADM}}(M, g)$$

$$\Longrightarrow \quad m_{\mathrm{ADM}}(M, g) \quad \geq \quad \frac{1}{2} \mathrm{C}_p(\partial M)^{\frac{1}{3-p}} \quad \xrightarrow[p \to 1^+]{} \quad \sqrt{\frac{|\partial M|}{16\pi}}$$

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(M, g) complete, noncompact, 3-Riem. mfld, ∂M smooth and compact

$$\begin{cases} \Delta_p u = 0 & \text{in } M \\ u = 0 & \text{on } \partial M \\ u \to 1 & \text{at } \infty \end{cases}$$

$$\mathcal{E}_p(t) := 4\pi t - \frac{t^{\frac{2}{p-1}}}{c_p} \int_{\{u=\alpha_p(t)\}} |\mathrm{D}u|\, \mathrm{H}\, \mathrm{d}\sigma + \frac{t^{\frac{5-p}{p-1}}}{c_p^2} \int_{\{u=\alpha_p(t)\}} |\mathrm{D}u|^2\, \mathrm{d}\sigma$$

Theorem [A.-Mantegazza-Mazzieri-Oronzio, 2022]:

$\mathrm{R} \geq 0$ & ∂M connected & $H_2(M, \partial M; \mathbb{Z}) = \{0\}$ & $\exists$ sln u to pb (P)

$\Downarrow$

$$\left.\begin{array}{r} s \leq t \\ \alpha_p(s), \alpha_p(t) \text{ reg. values} \end{array}\right\} \implies \mathcal{E}_p(s) \leq \mathcal{E}_p(t)$$

idea of the proof:

- $\{\text{critical values}\} = \emptyset$

$$\mathcal{E}_p'(t) = 4\pi - \int_{\{u=\alpha_p(t)\}} \frac{\mathrm{R}^{\Sigma_t}}{2} \mathrm{d}\sigma$$

$$+ \int_{\{u=\alpha_p(t)\}} \left\{ \frac{\left|\mathrm{D}^{\Sigma_t} |\mathrm{D}u|\right|^2}{|\mathrm{D}u|} + |\mathring{\mathrm{h}}|^2 + \frac{\mathrm{R}}{2} + \left(\frac{5-p}{p-1}\right) \left[\frac{\mathrm{H}}{2} - \left(\frac{p-1}{3-p}\right) \frac{|\mathrm{D}u|}{1-u} \right]^2 \right\} \mathrm{d}\sigma$$

{critical values} $= \emptyset$:

$$
\begin{aligned}
\mathcal{E}_2'(\tau) &= \frac{\mathrm{d}}{\mathrm{dt}} \left(4\pi t \; - \; t^2 \int\limits_{\{u = 1 - \frac{1}{t}\}} |\mathrm{D}u| \, \mathrm{H} \, \mathrm{d}\sigma \; + \; t^3 \int\limits_{\{u = 1 - \frac{1}{t}\}} |\mathrm{D}u|^2 \, \mathrm{d}\sigma \right) \\
&= 4\pi \\
&\quad -2t \int\limits_{\{u = 1 - \frac{1}{t}\}} |\mathrm{D}u|^2 \mathrm{H} \mathrm{d}\sigma + \int\limits_{\{u = 1 - \frac{1}{t}\}} \left\{ \frac{\left| \mathrm{D}^{\Sigma_t} |\mathrm{D}u| \right|^2}{|\mathrm{D}u|} + |\mathring{\mathrm{h}}|^2 + \frac{\mathrm{R}}{2} - \frac{\mathrm{R}^{\Sigma_t}}{2} + \frac{3}{4}\mathrm{H}^2 \right\} \mathrm{d}\sigma \\
&\quad +3t^2 \int\limits_{\{u = 1 - \frac{1}{t}\}} |\mathrm{D}u|^2 \mathrm{d}\sigma - t \int\limits_{\{u = 1 - \frac{1}{t}\}} |\mathrm{D}u| \mathrm{H} \mathrm{d}\sigma \\
&= \underbrace{4\pi - \int\limits_{\{u = 1 - \frac{1}{t}\}} \frac{\mathrm{R}^{\Sigma_t}}{2} \mathrm{d}\sigma}_{\text{Gauss-Bonnet term}} + \int\limits_{\{u = 1 - \frac{1}{t}\}} \left\{ \frac{\left| \mathrm{D}^{\Sigma_t} |\mathrm{D}u| \right|^2}{|\mathrm{D}u|} + |\mathring{\mathrm{h}}|^2 + \frac{\mathrm{R}}{2} + 3\left(\frac{\mathrm{H}}{2} - \frac{|\mathrm{D}u|}{1-u} \right)^2 \right\} \mathrm{d}\sigma
\end{aligned}
$$

$\Rightarrow \quad \mathcal{E}_2'(t) \geq 0, \quad$ for very $t \geq 1$

{critical values} = Ø :

$$\mathcal{E}_2'(\tau) = \frac{\mathrm{d}}{\mathrm{dt}}\left(4\pi t - t^2 \int_{\{u=1-\frac{1}{t}\}} |\mathrm{D}u|\, \mathrm{H}\, \mathrm{d}\sigma + t^3 \int_{\{u=1-\frac{1}{t}\}} |\mathrm{D}u|^2\, \mathrm{d}\sigma\right)$$

$\{u = 1 - \frac{1}{t}\}$!!

$$= 4\pi$$

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$$= \underbrace{4\pi - \int_{\{u=1-\frac{1}{t}\}} \frac{\mathrm{R}^{\Sigma_t}}{2}\mathrm{d}\sigma}_{\text{Gauss-Bonnet term}} + \int_{\{u=1-\frac{1}{t}\}} \left\{\frac{\left|\mathrm{D}^{\Sigma_t}|\mathrm{D}u|\right|^2}{|\mathrm{D}u|} + |\mathring{\mathrm{h}}|^2 + \frac{\mathrm{R}}{2} + 3\left(\frac{\mathrm{H}}{2} - \frac{|\mathrm{D}u|}{1-u}\right)^2\right\}\mathrm{d}\sigma$$

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$$= 4\pi$$

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$$= 4\pi$$

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- $|$ {critical values} $| = 0$

$|\{\text{critical values}\}| = 0$

- $$[1,+\infty) \ni t \mapsto \mathcal{E}_2(t) \;:=\; 4\pi t \;-\; t^2 \int_{\{u=1-\frac{1}{t}\}} |\mathrm{D}u|\, \mathrm{H}\, \mathrm{d}\sigma \;+\; t^3 \int_{\{u=1-\frac{1}{t}\}} |\mathrm{D}u|^2\, \mathrm{d}\sigma$$

$$\Rightarrow \qquad \mathcal{E}_2(t) = 4\pi t + \int_{\{u=1-\frac{1}{t}\}} \left\langle Y, \frac{\mathrm{D}u}{|\mathrm{D}u|} \right\rangle \mathrm{d}\sigma, \qquad Y := \frac{\mathrm{D}|\mathrm{D}u|}{(1-u)^2} + \frac{|\mathrm{D}u|}{(1-u)^3}\mathrm{D}u$$

- outside $\mathrm{Crit}(u)$:

$$\mathrm{div}(Y) = \frac{|\mathrm{D}u|}{(1-u)^2}\left[\frac{3|\mathrm{D}u|^2}{(1-u)^2} + \frac{|\mathrm{D}\mathrm{D}u|^2 - |\,\mathrm{D}|\mathrm{D}u|\,|^2}{|\mathrm{D}u|^2}\right] + \frac{|\mathrm{D}u|}{(1-u)^2}\left[\frac{3\langle \mathrm{D}|\mathrm{D}u|, \mathrm{D}u\rangle}{(1-u)\,|\mathrm{D}u|} + \frac{\mathrm{Ric}(\mathrm{D}u,\mathrm{D}u)}{|\mathrm{D}u|^2}\right]$$

$$= \frac{|\mathrm{D}u|}{(1-u)^2}\left\{-\frac{\mathrm{R}^\Sigma}{2} + \underbrace{(\ldots)}_{\geq 0}\right\}$$

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$$\geq 4\pi(t-s) - 2\pi \int_{1-1/s}^{1-1/t} \frac{\chi(\{u=\tau\})}{(1-\tau)^2}\mathrm{d}\tau \geq 4\pi(t-s) - 4\pi \int_{1-1/s}^{1-1/t} \frac{\mathrm{d}\tau}{(1-\tau)^2} = 0$$

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$$\mathrm{div}(Y) = \\ = \frac{|\mathrm{D}u|}{(1-u)^2} \left\{ -\frac{\mathrm{R}^\Sigma}{2} + \underbrace{(\ldots)}_{\geq 0} \right\}$$

- so if $\{1 - 1/s < u < 1 - 1/t\} \cap \mathrm{Crit}(u) = \emptyset$, then

$$\begin{aligned} \mathcal{E}(t) - \mathcal{E}(s) &= 4\pi(t-s) + \int\limits_{\{1-1/s<u<1-1/t\}} \mathrm{div}\, Y \mathrm{d}\mu \\ &= 4\pi(t-s) + \int_{1-1/s}^{1-1/t} \frac{\mathrm{d}\tau}{(1-\tau)^2} \int\limits_{\{u=\tau\}} \left\{ -\frac{\mathrm{R}^\Sigma}{2} + \underbrace{(\ldots)}_{\geq 0} \right\} \mathrm{d}\sigma \\ &\geq 4\pi(t-s) - 2\pi \int_{1-1/s}^{1-1/t} \frac{\chi(\{u=\tau\})}{(1-\tau)^2} \mathrm{d}\tau \geq 4\pi(t-s) - 4\pi \int_{1-1/s}^{1-1/t} \frac{\mathrm{d}\tau}{(1-\tau)^2} = 0 \end{aligned}$$

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- now if $\{1 - 1/s < u < 1 - 1/t\} \cap \mathrm{Crit}(u) \neq \emptyset$, then we define $Y_k := \eta_k\left(\frac{|\mathrm{D}u|}{1-u}\right) Y$

$$\Rightarrow \qquad \mathrm{div}(Y_k) = \eta_k P + \eta_k D + \eta_k'\left(\frac{|\mathrm{D}u|}{1-u}\right)\left|\mathrm{D}|\mathrm{D}u| + \frac{|\mathrm{D}u|}{1-u}\mathrm{D}u\right|^2$$

$$\Rightarrow \qquad \mathcal{E}(t) - \mathcal{E}(s) = 4\pi(t-s) + \int_{\{1-1/s<u<1-1/t\}} \mathrm{div}\, Y_k \,\mathrm{d}\mu$$

$$\geq 4\pi(t-s) + \int_{\{1-1/s<u<1-1/t\}} \{\eta_k P + \eta_k D\}\,\mathrm{d}\mu$$

$$\xrightarrow[k\to+\infty]{} 4\pi(t-s) + \int_{\{1-1/s<u<1-1/t\}} \mathbb{1}_{M\setminus \mathrm{Crit}(u)} \mathrm{div}\, Y \,\mathrm{d}\mu \geq 0$$

$|\{\text{critical values}\}| = 0$

$$\mathcal{E}_2(t) = 4\pi t + \int_{\{u=1-\frac{1}{t}\}} \left\langle Y, \frac{\mathrm{D}u}{|\mathrm{D}u|} \right\rangle \mathrm{d}\sigma, \qquad Y := \frac{\mathrm{D}|\mathrm{D}u|}{(1-u)^2} + \frac{|\mathrm{D}u|}{(1-u)^3}\mathrm{D}u$$

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$$\mathcal{E}_2(t) = 4\pi t + \int_{\{u=1-\frac{1}{t}\}} \left\langle Y, \frac{\mathrm{D}u}{|\mathrm{D}u|} \right\rangle \mathrm{d}\sigma, \qquad Y := \frac{\mathrm{D}|\mathrm{D}u|}{(1-u)^2} + \frac{|\mathrm{D}u|}{(1-u)^3}\mathrm{D}u$$

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idea of the proof:

- $\{\text{critical values}\} = \emptyset$

$$\mathcal{E}_p'(t) = 4\pi - \int_{\{u=\alpha_p(t)\}} \frac{\mathrm{R}^{\Sigma_t}}{2} \mathrm{d}\sigma$$

$$+ \int_{\{u=\alpha_p(t)\}} \left\{ \frac{\left|\mathrm{D}^{\Sigma_t} |\mathrm{D}u|\right|^2}{|\mathrm{D}u|} + |\mathring{\mathrm{h}}|^2 + \frac{\mathrm{R}}{2} + \left(\frac{5-p}{p-1}\right) \left[\frac{\mathrm{H}}{2} - \left(\frac{p-1}{3-p}\right) \frac{|\mathrm{D}u|}{1-u} \right]^2 \right\} \mathrm{d}\sigma$$

- $|\{\text{critical values}\}| = 0$
- $|\{\text{critical values}\}| > 0$
  - for $0 < T < 1$, $M_T := \{0 \leq u \leq T\}$, $|\cdot| := \sqrt{|\cdot|^2 + \varepsilon}$

$$\begin{cases} \mathrm{div}\left(|\mathrm{D}v|_\varepsilon^{p-2} \mathrm{D}v\right) = 0 & \text{in } M_T \\ v = 0 & \text{on } \partial M \\ v = T & \text{on } \{u_p = T\} \end{cases}$$

  - the solution u^ε is smooth
  - $$\mathcal{E}_p^\varepsilon(t) := 4\pi t - \frac{t^{\frac{2}{p-1}}}{c_{p,\varepsilon}} \int_{\{u=\alpha_p^\varepsilon(t)\}} |\mathrm{D}u|\, \mathrm{H}\, \mathrm{d}\sigma + \frac{t^{\frac{5-p}{p-1}}}{c_{p,\varepsilon}^2} \int_{\{u=\alpha_p^\varepsilon(t)\}} |\mathrm{D}u|^2\, \mathrm{d}\sigma$$

idea of the proof:

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  - $\mathcal{E}_p^\varepsilon(t) \to \mathcal{E}_p(t)$, as $\varepsilon \to 0^+$
  - suppose by contradiction that $s < t$ and $\mathcal{E}_p(t) < \mathcal{E}_p(s)$, then

$$0 > -\delta \ge \mathcal{E}_p^\varepsilon(t) - \mathcal{E}_p^\varepsilon(s) \ge -\frac{\varepsilon}{6}\left(\frac{p+1}{p-1}\right)^2 \int_{\{\alpha_p^\varepsilon(s) < u^\varepsilon < \alpha_p^\varepsilon(t)\}} \frac{c_{p,\varepsilon}^{\frac{p-1}{3-p}} |\mathrm{D}u^\varepsilon|^2}{\left[\frac{3-p}{p-1}\left(1-u^\varepsilon\right)\right]^{\frac{p-1}{3-p}+3}}\,\mathrm{d}\mu$$

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