Exponential PDE's in high dimensions

Pierpaolo Esposito
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- H. Brézis, F. Merle, Comm. PDE '91
- Y.Y. Li, I. Shafrir, Indiana Univ. Math. J. '94
- Y.Y. Li, Comm. Math. Phys. '99
- C.C. Chen, C.S. Lin, Comm. Pure Appl. Math. '02 & '03

Quasilinear operators in conformal geometry I

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E-L eqn: $\mathcal{N}(u) + U_g = -k_A \frac{e^{4u}}{\int_M e^{4u} dv_g}$ where

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Difficulty: $\mathcal{N}(u) = (\frac{\gamma_2}{2} + 6\gamma_3)\Delta^2 u - 12\gamma_3 \Delta_4 u + \ldots$ is a quasi-linear operator of mixed orders

- T. Branson, A. Chang, P. Yang, CMP '92
- A. Chang, P. Yang, Ann. Math. '95
- P.E., A. Malchiodi, JDG to appear
- M. Gursky, CMP '99 & '07
- M. Gursky, A. Malchiodi, CMP '12

Simplified problem: retain Δ_4 in $\mathcal{N}$ and consider it in general dimensions $n \geq 2$

Quasilinear operators in conformal geometry II

Simplified problem: retain Δ_4 in $\mathcal{N}$ and consider it in general dimensions $n \geq 2$

Arises also in

$$-\Delta_n u + |\nabla u|^{n-2}\mathrm{Ric}(\nabla u, \nabla u) = \left[|\nabla u|^{n-2}\mathrm{Ric}(\nabla u, \nabla u)\right]_g e^{nu}$$

see

- S. Ma, J. Qing, Calc. Var '21 & Adv. Math. '22

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- X. Ren, J. Wei, J. Differential Equations '95
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Let u_k be solutions of (P) with V_k satisfying (V) and

$$\sup_{k \in \mathbb{N}} \int_\Omega e^{u_k} < +\infty \quad \& \quad \sup_{k \in \mathbb{N}} osc_{\partial\Omega} u_k < +\infty$$

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If $osc_{\partial\Omega} u_k = 0$, (i)-(ii) do hold in $\overline{\Omega}$ with $S \subset \Omega$ in case (ii).

The quasi-linear MF equation

For $\lambda \notin c_n \omega_n \mathbb{N}$ Theorem 1 gives compactness for solutions of

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Theorem 2 (P.E., F. Morlando, JMPA '15)

If $\mathfrak{B}_m(\Omega) = \{\sum_{i=1}^m t_i \delta_{p_i} : \ t_i \geq 0, \ \sum_{i=1}^m t_i = 1, \ p_i \in \Omega\}$ *is non contractible, then* $(P)_\lambda$ *is solvable for* $\lambda \in c_n \omega_n (m, m+1)$

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- W. Ding, J. Jost, J. Li, G. Wang, AIHP '99
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Alternative approaches: via degree (blow-up analysis misses); via perturbative methods (difficult due to nonlinearity of Δ_n)

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$$U_{\lambda,p}(x) = \log \frac{c_n \lambda^n}{(1+\lambda^{\frac{n}{n-1}}|x-p|^{\frac{n}{n-1}})^n} \qquad \lambda > 0,\ p \in \mathbb{R}^n$$

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Theorem 3 (P.E., AIHP '18)

Any solution U of $(P)_\infty$ has the form $U_{\lambda,p}$. In particular $\int_{\mathbb{R}^n} e^U = c_n \omega_n$

The semilinear case $n = 2$

Classification known since a long ago, proved in different ways:

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$\Rightarrow$ enough decay to carry out a simple MP approach

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An alternative approach: via Pohozev identity in

- P.-L. Lions, Appl. Anal. '81
- S. Kesavan, F. Pacella, Appl. Anal. '94
- S. Chanillo, M. Kiessling, Geom. Funct. Anal. '95

Quantization and classification issues

If U is a solution of $(P)_\infty \Rightarrow$ the Kelvin transform $\hat{U}$ satisfies

$$-\Delta_n \hat{U} = \frac{e^{\hat{U}}}{|x|^{2n}} \text{ in } \mathbb{R}^n \setminus \{0\}, \int_{\mathbb{R}^n} \frac{e^{\hat{U}}}{|x|^{2n}} < +\infty$$

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- J. Serrin, Acta Math. '64 and '65
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Mass quantization for singular $n-$Liouville equation:

$$-\Delta_n U = e^U - \gamma \delta_0 \text{ in } \mathbb{R}^n, \quad \int_{\mathbb{R}^n} e^U < +\infty$$

- P. E., Calc. Var. PDE '21 [if $n \geq 2$]
- J. Prajapat, G. Tarantello, Proc. Edinburgh '01 [if $n = 2$]

Interior blow-up

Dropping $\sup_{k\in\mathbb{N}} \text{osc}_{\partial\Omega} u_k < +\infty$, in general concentration masses satisfy $\alpha_p \geq n^n \omega_n$

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In the two-dimensional case $\alpha_p \in 8\pi\mathbb{N}$ is shown in

- Y.Y. Li, I. Shafrir, Indiana Univ. Math. J. '94

based on a Harnack inequality of $\sup + \inf$ type

- I. Shafrir, C.R.A.S. '92

through an isoperimetric argument

The main point comes from the "linear theory": if $-\Delta_n u = f$ in Ω and $B_{2\delta}(x) \subset \Omega$, then

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sup + inf Inequalities

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If $c_1 = (n\omega_n)^{-\frac{1}{n-1}}$, set $\mu_k = e^{-\frac{u_k(x_k)}{n}}$ with $u_k(x_k) = \max_K u_k$:

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Theorem 4 (P.E., M. Lucia, preprint)

Given $K \subset \Omega$ compact and $C_1 < \frac{1}{n-1}$, there exists $C_2 > 0$ so that

$$C_1 \max_K u_k + \inf_\Omega u_k \leq C_2$$

About c_1

The constant c_1 is not explicit but $0 < c_1 \leq (n\omega_n)^{-\frac{1}{n-1}}$, see

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$$= -\frac{1}{2\pi} \int_0^{2\pi} d\theta \int_0^\delta t \log \frac{t}{\delta} f(t\theta) dt = \frac{1}{2\pi} \int_0^{2\pi} d\theta \int_0^\delta \frac{dt}{t} \int_0^t r f(r\theta) dr$$

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General case

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$$\begin{cases} -\Delta_n w_k = V_k e^{u_k} \chi_{B_{R\mu_k}(x_k)} & \text{in } B_\delta(x_k) \\ w_k = 0 & \text{on } \partial B_\delta(x_k) \end{cases}$$

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- since $V_k e^{u_k} \sim V(p) e^{U_{x_k,\mu_k^{-1}}}$ in $B_{R\mu_k}(x_k)$ with $p = \lim_{k \to +\infty} x_k$, further compare w_k from below with the radial case where $c_1 = (n\omega_n)^{-\frac{1}{n-1}}$

Quantization for mass concentration

By $\sup + \inf$ inequalities one gets decay estimates on $h_k e^{u_k}$:

$$V_k e^{u_k} \leq C \frac{\mu_k^\alpha}{|x - x_k|^{n+\alpha}} \qquad \text{in } B_{\frac{d_k}{2}}(x_k) \setminus B_{R\mu_k}(x_k)$$

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Following clusters by clusters, rather standard to show that

Theorem 5 (P.E., M. Lucia, preprint)

$$\alpha_p \in c_n \omega_n \mathbb{N}$$

extending the two-dimensional result in

- Y.Y. Li, I. Shafrir, Indiana Univ. Math. J. '94

Open questions

The decay exponent is in general with $\alpha < \frac{n}{n-1}$. When blow-up is simple, is it possible to reach $\alpha = \frac{n}{n-1}$? Equivalent to

$$V_k e^{u_k} \leq C \frac{\mu_k^{\frac{n}{n-1}}}{|x - x_k|^{\frac{n^2}{n-1}}} \qquad \text{in } B_\delta(p) \setminus B_{R\mu_k}(x_k)$$

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The answer is related to the following fundamental expansion

$$u_k - U_{x_k, \mu_k^{-1}} = O(1) \qquad \text{in } B_\delta(p)$$

and optimal constant $C_1 = \frac{1}{n-1}$ in the sup + inf inequality, see

- D. Bartolucci, C.C. Chen, C.S. Lin, G. Tarantello, Comm. PDE '04
- H. Brézis, Y.Y. Li, I. Shafrir, JFA '93
- Y.Y. Li, Comm. Math. Phys. '99

Thanks for your attention