The ASD deformation complex and a conjecture of Singer.

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In particular, if $p = 2$ then

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and the eigenvalues of $\star$ are ± 1. Therefore we have a splitting of Λ^2 into the sub-bundles of *self-dual* (SD) and *anti-self-dual* (ASD) two-forms:

$$\Lambda^2 = \Lambda^2_+ \oplus \Lambda^2_-$$

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$$Rm(e^i \wedge e^j) = \frac{1}{2} \sum_{k,\ell} R_{ijk\ell} e^k \wedge e^\ell,$$

where $R_{ijk\ell} = Rm(e_i, e_j, e_k, e_\ell)$. In general, the curvature operator does not preserve $\Lambda^2_\pm$.

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- **Fact:** W respects the splitting of Λ^2 into SD/ASD two-forms, hence we can define

$$W^\pm = W|_{\Lambda^2_\pm}.$$

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• Similarly, we say that (M^4, g) is *self-dual* (SD) if $W^- \equiv 0$. Note that by changing the orientation an ASD manifold becomes SD and vice versa.

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2. The splitting of Λ^2 gives a splitting of the space of harmonic two-forms, hence $H^2_{dR}(M^4;\mathbb{R}) = H^2_+ \oplus H^2_-$, and $\tau(M^4) = \dim H^2_+ - \dim H^2_-$. Also, the Weitzenböck formula for $\omega \in H^2_\pm$ is given by

$$\frac{1}{2}\Delta|\omega|^2 = |\nabla\omega|^2 - 2W^\pm(\omega,\omega) + \frac{1}{3}R|\omega|^2.$$

In particular, if (M,g) is ASD with $R > 0$, then $H^2_+ = 0$ and the intersection form is definite.

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3. Since

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4. If (M, g) is Kähler, then ASD is equivalent to $R = 0$.

The ASD condition, cont.

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• In particular, the ASD property is conformally invariant. This property is crucial in our work since it allows us to study ASD metrics by using a judicious choice of conformal metric.

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4. Poon: There exist ASD metrics on

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• The construction of Floer and Tabues are based on 'gluing' methods, while the work of Poon, Donaldson-Friedman rely in 'twistor' methods (more on these later).

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• The conformal invariance of the Weyl tensor implies that $\mathcal{D}$ is *conformally covariant*: if $\tilde{g} = e^{2u} g$, then

$$\mathcal{D}_{\tilde{g}} h = e^{2u} \mathcal{D}_g (e^{-2u} h).$$

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Question:

When is (M, g) unobstructed?

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where $P = \frac{1}{2}(Ric - \frac{1}{4}Rg)$ is the *Schouten tensor*.

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- The conformal covariance of $\mathcal{D}$ implies that $\mathcal{D}^*$ is conformally covariant: if $\tilde{g} = e^{2u} g$, then

$$\mathcal{D}^*_{\tilde{g}}(Z) = e^{-2u} \mathcal{D}^*_g (e^{-2u} Z).$$

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2. If (M, g) is an oriented Riemannian four-manifold, then the unit sphere bundle $\mathcal{Z}^+(M) \subset \Lambda^2_+(M)$ is a six-dimensional manifold. This manifold has a canonical almost-complex structure, which is integrable iff (M, g) is ASD (Cf. Penrose, Atiyah-Hitchin-Singer). The resulting complex three-manifold is called the *twistor space* associated to (M, g).

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In particular, this identification allows one to use methods of complex geometry to construct examples of ASD manifolds (cf. Poon, Donaldon-Friedman), and to study the deformation of ASD metrics.

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'Direct analysis':

• Itoh: If (M, g) is ASD and Einstein with $R > 0$, then it is unobstructed. Itoh gave an explicit calculation of

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Towards the conjecture

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Theorem

(G-Gover, '22) Assume (M, g) is ASD with $Y(M, [g]) > 0$. If

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then (M, g) is unobstructed.

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(Note: $-\frac{1}{24\pi^2} Y(-\mathbb{CP}^2,[g_{FS}])^2 = -12$, $(2\chi + 3\tau)\left(-\mathbb{CP}^2\right) = 3$)

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After (long) calculations, we find that Z satisfies

$$\frac{1}{2}\Delta|Z|^2 = |\nabla Z|^2 + Z * \nabla(\delta Z) + \frac{1}{2}R|Z|^2,$$

where $*$ denotes tensor product with possible contractions, and

$$(\delta Z)_{ijk} = \nabla^m Z_{mijk}.$$

Key ideas, cont.

It turns out that this formula carries very little information. However, the tractor interpretation also gives us a Weitzenböck formula for δZ:

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Notice the presence of the Schouten tensor. Therefore, in addition to assuming positivity of the Yamabe invariant, we need to add a condition which gives us some control over the Schouten tensor (in a conformally invariant way).

Key ideas, cont.

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Theorem

(Chang-Yang, G-, G-Gover) Suppose (M, g_0) satisfies the assumptions of the theorem. Then there is a conformal metric $g = e^{2u} g_0$ satisfying

$$\Delta_g R \leqslant -6|P|^2 + \frac{2}{3} R^2,$$

where $P = \frac{1}{2}(Ric - \frac{1}{6} Rg)$ is the Schouten tensor and R is the scalar curvature. Moreover, the scalar curvature $R > 0$.

Key ideas, cont.

To find a conformal metric whose scalar curvature satisfies this differential inequality we first prove the existence of a conformal metric whose scalar curvature satisfies

$$\Delta R = \frac{1}{8}R^2 - \frac{3}{2}|E|^2 + Y(M^4, [g])\, R + \mu,$$

where E is trace-free Ricci tensor and μ is a constant.

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where E is trace-free Ricci tensor and μ is a constant. Existence is proved by solving a fourth order variational problem (roughly) of the form

$$\inf_{u \in W^{2,2}} \int \left((\Delta u)^2 + c_1 (\Delta u) |\nabla u|^2 + c_2 |\nabla u|^4 + \cdots \right) dv$$

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This functional is a modified version of a functional that arises in the study of regularized determinants of conformally covariant operators in dimension four. Existence of minimizers (under certain conditions) follows the work of Chang-Yang in the early 90s.

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Thank you.