# Some linear and non-linear (conformal) Dirichlet-Neumann maps

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Blitz, **G-.**, Kopinski, Waldron , in progress

Related work:
**G-.** and L. Peterson: Conformal boundary operators, $\cdots$ Dirichlet-Neumann, *Pacific J.M.*, (2021)
Branson, **G-.** Conformally invariant non-local operators, *Pacific J.M.* (2001)

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# Conformal geometry and the Yamabe operator

Recall that on a Riemannian manifold $(M^d, g)$ the Yamabe operator

$$P_2 : \mathcal{E} \to \mathcal{E} \quad \text{defined by} \quad P_2 f := (\Delta^g - \frac{d-2}{4(d-1)} \mathrm{Sc}^g) f$$

is conformally covariant: If $\hat{g} = \Omega^2 g$ then

$$P_2^{\hat{g}} \circ \Omega^{1-\frac{d}{2}} = \Omega^{-1-\frac{d}{2}} \circ P_2^g .$$

Thus it descends to a well defined invariant operator

$$P_2 : \mathcal{E}[1 - \frac{d}{2}] \to \mathcal{E}[-1 - \frac{d}{2}]$$

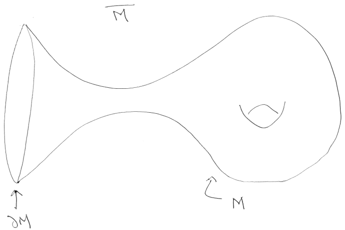
on conformal manifolds $(M, \boldsymbol{c})$ where the (conformal) density bundles $\mathcal{E}[w]$ are defined by

$$\mathcal{E}[w] := ((\Lambda^d TM)^2)^{\frac{w}{2n}}$$

and $\boldsymbol{c}$ is the conformal equivalence class $\boldsymbol{c} = [g]$ of metrics $\hat{g} \sim g$.

# A simple conformal Dirichlet-to-Neumann

Consider a **conformal manifold with boundary** $(\overline{M}^d, \boldsymbol{c})$:




A conformally invariant **Dirichlet-to-Robin** operator (on $\partial M$):

**1.** From a density $f$ (weight $1-\frac{d}{2}$) on the boundary $\partial M$ solve the Dirichlet problem $P_2\tilde{f} = 0$ with $\tilde{f}|_{\partial M} = f$.

**2.** Then form $\delta_1 \tilde{f}|_{\partial M}$, where

$$\delta_1 \stackrel{g}{:=} n^a\nabla_a - (1-\frac{d}{2})H^g, \qquad H^g \text{ is the mean curvature of } \partial M$$

is the **conformally invariant** Cherrier-Robin operator. Thus

$$\boxed{\text{DtoR} : f \mapsto \delta_1\tilde{f}|_{\partial M}} \quad \text{is \textbf{conf. invt} with symbol } \sqrt{-\Delta_{\partial M}}$$

**Question:** $\exists$ higher order analogues of DtoR?

**1.** On densities $\tilde{f}$ (of weight $k - \frac{n}{2}$) the GJMS operators $P_{2k}\tilde{f} = (\Delta^k \tilde{f} + \text{lower order})\tilde{f}$ are conformally invariant.

**2. Problem:** We seek **conformally invariant higher order analogues** $\delta_m$ of $\delta_1$. These differential operators should be defined along $\partial M$, and be canonically determined by the embedding $\partial M \hookrightarrow \overline{M}$ and the conformal structure. For each $m \in \mathbb{N}$ we want these to have order and **transverse order** $m$. A solution $\rightsquigarrow$

Then[1] we get a conformal odd powers of $\sqrt{-\Delta_{\partial M}}$ by

generalised Dirichlet data of $\tilde{f} \mapsto \delta_\ell \tilde{f}|_{\partial M}$ where $P_k \tilde{f} = 0$

where $P_k \tilde{f} = 0$, and the transverse order $\ell$ of $\delta_\ell$ is suitable.

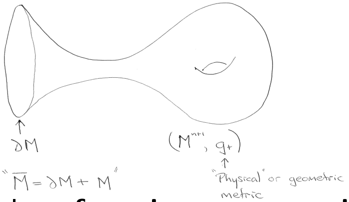
What is the link to scattering – on conformally compact manifolds?

[1] Ignoring lots of fine print!! : Good elliptic problem, assume unique soln, etc

# Poincaré-Einstein and conformally compact manifolds

Henceforth in these talks, **conformal compactification** of a Riemannian manifold $(M^{d=n+1}, g_+)$ is a manifold $\overline{M}$ with boundary $\partial M$ s.t.:

- $\exists\ \overline{g}$ on $\overline{M}$, with
- $g_+ = r^{-2}\overline{g}$, where $r$ a defining function for $\partial M$.

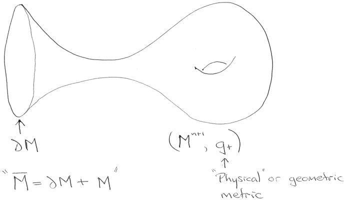



$\Rightarrow$ canonical **conformal structure on boundary**: $(\partial M, [\overline{g}|_{\partial M}])$

- Called here **Poincaré-Einstein** metric if also $g_+$ Einstein – then negative Einstein (in Riemannian signature as here).

d=n+1

Is there a well-defined (non-linear) DN map $[g_{\partial M}] \mapsto ??$




Suppose on $M$ one wants to solve $(\Delta^g + s(n-s)\frac{2J^g}{d})f = 0$, i.e.

$$\left(\Delta^g - 2w(n+w)\frac{J^g}{d}\right)f = 0 \qquad s = n + w \text{ is the } \textbf{spect. param.}$$

where $\Delta^g$ is the metric **Laplacian** $g^{ab}\nabla_a\nabla_b$ for the conformally compact metric $g = g_+ = r^{-2}\overline{g}$ that is singular at $\partial M$. Then, in the scale of the metric $\overline{g}$, ("Dirichlet") solns look like

$$f = F + r^{n+2w}G + (\text{potential log terms if } n + 2w \in \mathbb{Z}_{>0})$$

1. When can we obtain the scattering map $F|_{\partial M} \mapsto G|_{\partial M}$ as a natural and invariant DN map? Maybe if $n + 2w \in \mathbb{Z}_{>0}$. Difficulty is that $G$ shows up exactly where the formal soln is "ambiguous".

On a conformal manifold $(\overline{M}, \boldsymbol{c})$ there is a conformally invariant **tractor bundle** $\mathcal{T}$ and **connection** $\nabla^{\mathcal{T}}$.

$$\mathcal{T} = \mathcal{E}[1] \oplus T^*M[1] \oplus \mathcal{E}[-1], \qquad X_A : \mathcal{E}[-1] \hookrightarrow \mathcal{T} =: \mathcal{T}_A$$

Given $g \in \boldsymbol{c}$

$$\mathcal{T} \stackrel{g}{=} \mathcal{E}[1] \oplus T^*M[1] \oplus \mathcal{E}[-1], \qquad \mathcal{E}[1] := ((\Lambda^n TM)^2)^{\frac{1}{2n}}$$

$$\nabla^{\mathcal{T}}_a(\sigma, \mu_b, \rho) = (\nabla_a \sigma - \mu_a,\ \nabla_a \mu_b + P_{ab}\sigma + \boldsymbol{g}_{ab}\rho,\ \nabla_a \rho - P_{ab}\mu^b),$$

and $\nabla^{\mathcal{T}}$ preserves a tractor metric $h$

$$\mathcal{T} \ni V = (\sigma, \mu_b, \rho) \mapsto 2\sigma\rho + \mu_b\mu^b = h(V, V).$$

There is also a second order Thomas operator:

$$\Gamma(\mathcal{E}[w]) \in f \mapsto D_A f \stackrel{g}{=} \begin{pmatrix} (n+2w-2)wf \\ (n+2w-2)\nabla_a f \\ -(\Delta f + w\mathsf{J} f) \end{pmatrix}$$

where J is a number times $\mathrm{Sc}(g)$.

# The normal tractor

For **Riemannian hypersurfaces** $\Sigma \hookrightarrow (M, g)$ the **co-normal field** $n_b$ plays a fundamental role. Using that $T\Sigma \cong n^\perp \subset TM|_\Sigma$:

second fundamental form $\quad L_{ab} = \nabla^T_a n_b,$ mean c. $\quad H^g = \dfrac{1}{n-1} \bar{g}^{ab} L_{ab},$

$$\text{Gauss formula} \quad \overline{\nabla}_a v^b = \nabla^T_a v^b + n^b L_{ac} v^c \quad v^b \in \Gamma(T\Sigma).$$

For **conformal hypersurfaces** $\Sigma \hookrightarrow (M, \boldsymbol{c})$ (e.g. $\partial M$) the **normal tractor** $N_B$ is the analogue:

$\boxed{N_A \overset{g \in \boldsymbol{c}}{=} (0,\ n_a, -H^g)}$ then (**Thm:**) $\boxed{\mathcal{T}_\Sigma = N^\perp = \Pi(\mathcal{T}) \subset \mathcal{T}\big|_\Sigma.}$

$$\text{Tractor shape form} \quad L_{aB} = \nabla^T_a N_B = (0,\ \mathring{L}_{ab}, -\frac{1}{d-2} \overline{\nabla}^b \mathring{L}_{ab})$$

Gauss formula $\quad \overline{\nabla}_a V^B = \nabla^T_a V^B + N^B L_{aC} V^C + S_a{}^B{}_C V^C \quad V^B \in \Gamma(\mathcal{T}\Sigma).$

Here $S_{aBC} = \mathbb{X}^c_{BC} \mathcal{F}_{ac}$ where (with $\mathring{L}$ :=trace-free second f. form)

$\mathcal{F}_{ab} = \frac{1}{n-3}\Big( \mathring{L}^2_{ab} + W_{acbd} n^c n^d - \frac{|\mathring{L}|^2}{2(n-2)} \bar{\boldsymbol{g}}_{ab} \Big)$ is the **Fialkow tensor**.

# Boundary operators: a naïve construction

In **Riemannian geometry** the Neumann operator is $n^a\nabla_a$. Higher transverse order transverse boundary operators similarly given: $n^a n^b \nabla_a \nabla_b$ etc.

The tools above allow an immediate analogue. This combines the normal tractor just mentioned and the (second) order Thomas-D. Recall $\delta_1 \stackrel{g}{:=} n^a\nabla_a - wH^g$, is the conformal Cherrier-Robin operator. This is recovered by

$$(d + 2w - 2)\delta_1 = N^A D_A.$$

More generally:

**Lemma**

$$\delta_{j+1} := N^{A_1} N^{A_2} \cdots N^{A_j} \delta_1 D_{A_1} D_{A_2} \cdots D_{A_j} \qquad (1)$$

*constructs a family of natural conformally invariant hypersurface operators* $\delta_K : \mathcal{T}^\Phi[w] \to \mathcal{T}^\Phi[w-K]$ *along* $\Sigma$.

# Hidden problems, hidden treasures

It would **appear** from the formula

$$\delta_K := N^{A_1} N^{A_2} \cdots N^{A_{K-1}} \delta_1 D_{A_1} D_{A_2} \cdots D_{A_{K-1}} \quad \text{along} \quad \Sigma$$

that the operator has "high" transverse order and is always at least of transverse order 1. But e.g.: (where $n = \dim(\Sigma)$ etc)

$$\delta_2 f = -(\bar{\Delta} - \frac{n-2}{4(n-1)} \bar{\mathrm{Sc}}) f + \frac{n-2}{4(n-1)} \mathring{L}^{ab} \mathring{L}_{ab} f \;, \quad \text{for } f \in \mathcal{E}\left[1 - \tfrac{n}{2}\right] .$$

This is the **intrinsic to** $\Sigma$ **Yamabe operator** of $(\Sigma, \boldsymbol{c}_\Sigma)$ (plus the conformal invariant $\mathring{L}^{ab} \mathring{L}_{ab}$). So:

**at this weight** $\delta_2$ has transverse order 0.

At the **interior Yamabe weight** $1 - \frac{d}{2}$ we have instead

$$\delta_2 = -(\Delta - \frac{d-2}{4(d-1)} \mathrm{Sc}) \quad \text{along} \quad \Sigma.$$

– i.e. the **interior Yamabe operator**.

# Bad weights

The above could be viewed as treasures, **BUT**, for $\delta_3$

$$\boxed{\delta_3 = 0 \quad \text{at weight} \quad w = 2 - \tfrac{d}{2}\,,}$$

i.e. at **interior Paneitz weight**.

Leading order behaviour: a straightforward induction proves:

**Proposition**

*Let $w \in \mathbb{R}$ and $K \in \mathbb{Z}_{>0}$ be given, and suppose that $\delta_K$ acts on $\mathcal{T}^{\Phi}[w]$. Then along $\Sigma$,*

$$\delta_K = \Big[\prod_{i=1}^{K-1}(d + 2w - K - i)\Big](\nabla_n)^K + ltots\,.$$

So the set of **"Bad weights"** (where max. transverse order not reached) are as follows: $E(\delta_1) = \emptyset$, and for any $K \in \mathbb{Z}_{\geq 2}$,

$$E(\delta_K) = \left\{\frac{2K-1-d}{2}, \frac{2K-2-d}{2}, \cdots, \frac{K+1-d}{2}\right\}\,. \qquad (2)$$

# Digging deeper – conformally flat case

On **conformally flat manifolds** we **can improve** the operators and eliminate every second bad weight:

**Theorem ($\delta^0_k$ – for conformally flat M)**

*Let $K \in \mathbb{Z}_{>0}$ and $\Sigma$ in a conformally flat manifold. There is a family of natural conformally invariant differential operators along $\Sigma$, $\boxed{\delta^0_K : \mathcal{T}^\Phi[w] \to \mathcal{T}^\Phi[w-K]\ ,}$ determined by*

$$\Big[\prod_{j=1}^{\lfloor\frac{K-1}{2}\rfloor}(d+2w-2K+2j)\Big]\delta^0_K = \delta_K\ , \tag{3}$$

*and polynomial continuation in $w$. The universal symbolic formula for $\delta^0_K$ is polynomial in $w$ and $d$.*

So: $E(\delta^0_K) = \left\{\frac{2K-1-d}{2}, \frac{2K-1-d}{2} - 1, \cdots, \frac{2K-1-d}{2} - \left\lfloor\frac{K-2}{2}\right\rfloor\right\}$ .

In particular $\boxed{d \text{ even} \Rightarrow 0 \notin E(\delta^0_K) = \text{the bad weights.}}$

# How the Theorem for $\delta^0_K$ (c. flat M) works

The proof of the above uses e.g. that for $f \in \mathcal{T}^\Phi[2-\frac{d}{2}]$

$$D_A \circ D_B f = (0, 0, \cdots, 0, P_4 f) \quad \text{where} \quad P_4 = \text{Paneitz op.}$$

So at other weights $w \in \mathbb{R}$, $f \in \mathcal{T}^\Phi[w]$ we can deduce (using polynomial in $w$ nature of the $D$ operators)

$$D_A \circ D_B f = ((d+2w-4)*, (d+2w-4)*, \cdots, (d+2w-4)*, \Delta^2 f + lots)$$

While for $(M, \boldsymbol{c})$ **conformally flat** and $f \in \mathcal{T}^\Phi[3-\frac{d}{2}]$ we have

$$D_A \circ D_B \circ D_C f = (0, 0, \cdots, 0, -P_6 f),$$

and so for $w \in \mathbb{R}$, $f \in \mathcal{E}[w]$

$$D_A \circ D_B \circ D_C f = ((d+2w-6)*, (d+2w-6)*, \cdots, (d+2w-6)*, \star).$$

We then show that **these factors survive in the formulae for** $\delta_K$.

**Curved case ? :** $(M, \boldsymbol{c})$ **conformally curved** then $f \in \mathcal{T}^\Phi[3-\frac{n}{2}]$ gives

$$D_A \circ D_B \circ D_C f = (0, 0, \text{mess}, \cdots, \text{mess}, -\Delta^3 f + \text{mess} f)$$

# Recovering the curved case

In fact the "mess" terms have been understood from earlier work. For example for $f \in \mathcal{E}[3 - \frac{n}{2}]$

$$P_{ABC} f := D_A \circ D_B \circ D_C f - \frac{2}{n-4} X_A W_B{}^F{}_C{}^E D_F D_E f = (0, \cdots, 0, -P_6 f),$$

and so we may replace $D_A \circ D_B \circ D_C$ with $P_{ABC}$ in a construction of new $\delta$-operators and we retain polynomiality wrt $w$.
There is an algorithm for similarly modifying any power $D^k$ of the $D$-operator. From G.+Peterson 2003, 2006 we get:

## Proposition

*There is an explicit curvature modification of* $D_{A_1} \cdots D_{A_k}$

$$P_{A_1 \cdots A_k} : \mathcal{T}^\Phi[w] \to \mathcal{T}_{A_1 \cdots A_k} \otimes \mathcal{T}^\Phi[w-k] \ ,$$

*polynomial in* $w \in \mathbb{R}$*, which takes the following form:*

$$P_{A_1 \cdots A_k} = (-1)^k X_{A_1} \cdots X_{A_k} \tilde{P}_{2k} + (n + 2w - 2k) S_{A_1 \cdots A_k} \ ,$$

*where* $S$ *is also polynomial in* $w$*.* $P$ *is conformally invariant.*

How do we use above **optimally**?
The **example:** For $f \in \mathcal{E}[w]$ we have

$$P_{ABC} f := (0, \cdots, 0, -P_6 f) = -X_A X_B X_C P_6 f \quad \text{if} \quad \boxed{(d+2w-6)=0}.$$

Thus, the next **key observation:**

$$\delta_2 P_{ABC} f = X_{(A} \tilde{P}_{BC)} f \implies N^B N^C \delta_2 P_{ABC} f = X_A \tilde{\tilde{P}} f \text{ if } (d+2w-6)=0.$$

That is $N^B N^C \delta_2 P_{ABC} f = (0,0,*)$. But $N^A = (-H, n^a, 0)$, so

$$N^A N^B N^C \delta_2 P_{ABC} f = 0 \quad \text{if} \quad \boxed{(d+2w-6)=0}.$$

So for $f \in \mathcal{E}[w]$

$$N^A N^B N^C \delta_2 P_{ABC} f =: (d+2w-6)\delta_{2,3} f$$

defines

$$\delta_{2,3} : \mathcal{T}^\Phi[w] \to \mathcal{T}^\Phi[w-5], \quad w \in \mathbb{R}.$$

For example this has **transverse order** 5 (generically and) on **functions** along $\Sigma^5$.

In particular if (1) $d$ is odd, or (2) $d$ is even and $k \leq d/2$, or (3) $(M, \boldsymbol{c})$ is conformally flat, we obtain the following,:

**Lemma (Lemma 5.1,G-Peterson PJM 21)**

*For each $k \in \mathbb{Z} > 0$ and $0 \leq i \leq 2k-1$ there is*

$$\delta_i : \mathcal{E}[k - d/2] \to \overline{\mathcal{E}}[k - d/2 - i],$$

*natural, conformally invariant, and of order and transverse order $i$.*

where the $\delta_i$ notation has been simplified. Whence:

**Theorem (Theorem 5.7, G-Peterson PJM 21)**

*On the boundary of a conformal mfld with boundary $(\overline{M}, \boldsymbol{c}, \partial M)$, There are well-defined*[a] *D-to-N conformally maps*

$$P^k_{2m} : \overline{\mathcal{E}}[m - n/2] \to \overline{\mathcal{E}}[-m - n/2] \quad \text{by} \quad f \mapsto \delta_{2k-1-\ell} f$$

*where $m := k - 1/2 - \ell$ and $u$ solves*

$$P_{2k} u = 0, \quad \delta_\ell u = f, \delta_j u = 0 \ \text{ for } j \neq \ell \ \text{ and } \ 0 \leq j \leq k-1.$$

[a] fine print about uniqueness of solutions etc

# Scalar scattering as a DtoN

On a $\overline{M} = M \cup \partial M$ with $g = g_+ = r^{-2}\overline{g}$ we look at solns to $\left(\Delta^g + s(n-s)\frac{2J^g}{d}\right)f = 0$ of the form

Chang

$$f = F + r^{n+2w}G + (\text{potential } r^{n+2w}\log r \text{ terms if } n+2w \in \mathbb{Z}_{>0}) \quad (4)$$

To extract $G$ via a **differential operator** we need $n + 2w \in \mathbb{Z}_{>0}$. Then difficult as 1. Potential log terms, and 2. $G$ is at an order that can be absorbed into $F$ without changing smoothness.

Cases:

$n + 2w = 2k$ **even**. Then $w = k - n/2$ so we have a boundary (in general extrinsically coupled) GJMS operator $\overline{P}_{2k}$. Then $\overline{P}_{2k}F|_{\partial M}$ is the coefficient of the log term. Also the required $\delta_i$ operators DNE – at least the construction fails. In fact in the conf flat case where the $\delta_{2k}$ becomes $\overline{P}_{2k}$.

$n + 2w = 2k - 1$ **odd**. Then $w = k - d/2$ and we do have $\delta_{n+2w} = \delta_{2k-1}$ from the Lemma. Thus:

**Theorem**

*Consider a smooth conformally compact manifold* $(\overline{M}^d, g_+)$ *such that (1) d is odd, or (2) d is even and* $k \leq d/2$*, or (3)* $(M, \boldsymbol{c} = [g_+])$ *is conformally flat. If the equation* $\left(\Delta^g + 2s(n-s)\frac{J^g}{d}\right) f = 0$ *has* **smooth** *solutions of the form* (4)*, uniquely parametrised by* $F|_{\partial M} \in \overline{\mathcal{E}}[w]$*, with* $n + 2w = 2k - 1$ *odd, there is a well defined invariant Dirichlet-to-Neumann map*

$$DtoN : \overline{\mathcal{E}}[m - \frac{n}{2}] \to \overline{\mathcal{E}}[-m - \frac{n}{2}] \qquad m := \frac{n+2w}{2} = k - \frac{1}{2}$$

*given by*

$$f|_{\partial M} \mapsto (\delta_{n+2w} f)|_{\partial M}.$$

Cf. **Graham-Zworski 2003** for PE case using eveness in $r$ of the expansion.

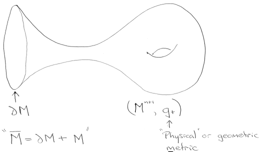
# Explanation

$n+2w=2k-1$ **odd**. Then $w$ an interior GJMS weight $k-d/2$ – and the GJMS is a good conformally invariant elliptic operator to the boundary (it doesn't see the scale singularity). So we "expect" corresponding conformal boundary operators (and they exist by the Lemma). These may be used for the $\left(\Delta^g + s(n-s)\frac{J^g}{d}\right)f = 0$ problem, and we obtain the non-local (pseudodifferential) operator

$$\overline{\mathcal{E}}[(k-1/2)-n/2] \to \overline{\mathcal{E}}[-(k-1/2)-n/2] \text{ by } f|_{\partial M} \mapsto (\delta_m f)|_{\partial M}.$$

In fact $\left(\Delta^g + s(n-s)\frac{2J^g}{d}\right) = I \cdot D$ and on PE manifolds $\sigma^k P_{2k} = (I \cdot D)^k$ see [G.- Math Ann 2006 ]. So in that setting the scattering problem is a special case of the Theorem above.

# The non-linear Poincaré-Einstein (PE) problem




Spse $g_+$ PE. Following [Graham- Oberw.-2003?] For $d = n+1$ even: Pick $\overline{g} \in \boldsymbol{c}_{\partial M}$. In suitable local coordinates

$$g_+ = r^{-2}(dr^2 + \overline{g}_r), \qquad \overline{g}_r \text{ a 1-para family metics on } \partial M,$$

the Taylor expansion of $\overline{g}_r$ is even to order $n$: there are $\varphi_r$ , and $\psi_r \in C^\infty(\partial M) \times [0, \epsilon)$ whose Taylor expansions at $r = 0$ contain only even terms s.t. $\overline{g}_r = \varphi_r + r^n \psi_r$. Moreover $\overline{g}_0 = \varphi_0 = \overline{g}$, $tr_{\overline{g}} \psi_0 = 0$ and $div_{\overline{g}} \psi_0 = 0$. Then $\psi_0$ is **locally formally undetermined** subject only to these two conditions and the term $\psi_0$ plays the role of Neumann data for this problem. So:

$$[\overline{g}] \mapsto \psi_0$$

a non-linear DtoN – when existence and uniqueness works (e.g. $S^n = \partial Ball$ with $\overline{g}$ "close" to round metric **[Graham-Lee 91]**).

# Capturing via curvature

An example: When $d = 4$ (so $n = 3$) then

$$\mathcal{D}_{ij} := (\overline{\nabla}^c W^T_{c(ij)\hat{n}}) + C^T_{\hat{n}(ij)} + HW_{\hat{n}ij\hat{n}} \in \overline{\mathcal{E}}_{(ij)_0}[-1]$$

is a local **conformal invariant** of embedded hypersurfaces [Blitz-G-Waldron 21] which, in the coordinates mentioned has leading symbol $tf_{\overline{g}}(\partial_r^3 \overline{g}_r)|_{\partial M}$. Thus **Thm:** If $n = 3$ then on PE manifolds

$$[\overline{g}] \mapsto \mathcal{D}_{ij}$$

is a conformally invariant map that captures the non-local data.

That is it recovers the DtoN map when it exists: $\mathcal{D} = \psi_0 + lots$. Higher dimension? : By analogy with linear cases we expect the idea to extend when $d$ is even – as we have a good conformal interior problem, namely $\mathcal{B}_{ab} = 0$ where $\mathcal{B}_{ab}$ is the Fefferman-Graham Obstruction tensor. (This provides an analogue of the GJMS operators – and is zero for Einstein metrics.)

# The PE Dirichlet-Neumann invariant

On conformal manfolds there is the conformally invariant $W$ tractor

$$W_{ABCD} = \begin{array}{l}(d-4)\big(Z_A{}^a Z_B{}^b Z_C{}^c Z_E{}^e W_{abce} - 2 Z_A{}^a Z_B{}^b X_{[C} Z_{E]}{}^e C_{abe} \\ -2 X_{[A} Z_{B]}{}^b Z_C{}^c Z_E{}^e C_{ceb}\big) + 4 X_{[A} Z_{B]}{}^b X_{[C} Z_{E]}{}^e B_{eb},\end{array}$$

The $\delta_i$ operators (with some adjustment) extend to tractors. In particular we can form $\delta_{d-5} W_{ABCD}$ and project to boundary tractors $\Pi(\delta_{d-5} W_{BCDE})$

## Proposition

*On the boundary of a smooth conformal manifold with boundary,*

$$\Pi(\delta_{d-5} W_{BCDE})$$

*is a conformally invariant.*

*On the boundary of a PE manifold* **with** $d$ **even** *this takes the form*

$$\overline{X}_{[B} \overline{Z}^c_{C]} \overline{X}_{[D} \overline{Z}^d_{E]} \mathcal{D}_{cd},$$

*where* $\mathcal{D}_{cd} \in \Gamma(\overline{\mathcal{E}}_{(cd)_0}[3-d])$.

# The PE Dirichlet-Neumann map

Sketch of proof/idea: By construction $\Pi(\delta_{d-5}W_{BCDE})$ is conformally invariant. Thus the first non-zero slot of $\Pi(\delta_{d-5}W_{BCDE})$ is a conformal invariant of odd weight. Using PE, all slots except for the $\overline{X}_{[B}\overline{Z}^{c}_{C]}\overline{X}_{[D}\overline{Z}^{d}_{E]}$ slot **can be expressed as intrinsic boundary invariants**. As at their orders $g_+$ is determined formally by $\overline{g}$. Thus the first non-zero slot of $\Pi(\delta_{d-5}W_{BCDE})$ above the bottom is an **intrinsic** to $(\partial M, \boldsymbol{c}_{\partial M})$ conformal invariant of **odd weight**. When $n$ odd, **no such non-trivial invariant**. □
Then $\mathcal{D}_{ab} = \nabla_n^{d-5}B_{ab} + lots$ – has the right leading symbol. Thus we have:

**Theorem**

*On PE manifolds the map*

$$[\overline{g}] \mapsto \mathcal{D}_{ab}$$

*provides a non-linear invariant PE Dirichlet-to-Neumann map.*

# THE END

# Examples

For any $w \in \mathbb{R}$, the operator $\delta_2 : \mathcal{T}^\Phi[w] \to \mathcal{T}^\Phi[w-2]$ is given by $\delta_2 := N^A \delta_1 D_A$. For all $f \in \mathcal{T}^\Phi[w]$,

$$\begin{aligned}\delta_2 f = &\\ &-(\Delta + w\mathrm{J})\mathrm{f} + (\mathrm{n} + 2\mathrm{w} - 2)\mathrm{n}^\mathrm{a}\mathrm{n}^\mathrm{b}\nabla_\mathrm{a}\nabla_\mathrm{b}\mathrm{f}\\ &-2(w-1)(n+2w-2)Hn^a\nabla_a f + (w-1)w(n+2w-2)H^2 f\\ &+w(n+2w-2)n^a n^b \mathrm{P}_{ab} f \ .\end{aligned}$$

Thus:

$$\boxed{T_g(\delta_2) = \mathrm{J} + (\mathrm{n}-2)\mathrm{H}^2 - (\mathrm{n}-2)\mathrm{n}^\mathrm{a}\mathrm{n}^\mathrm{b}\mathrm{P}_\mathrm{ab}}.$$

$E(\delta_2) = \{(3-n)/2\}$, so for all $n \geq 4$, $T_g(\delta_2)$ is a hypersurface $T$-curvature.

We see for $w = 1 - \frac{n}{2}$ (interior Yamabe weight)

$$\delta_2 = -(\Delta + w\mathrm{J})\mathrm{f} = -(\mathrm{Yamabe})\mathrm{f} \ .$$

# Third order

The operator $\delta_{1,2} : \mathcal{E}[w] \to \mathcal{E}[w-3]$ is given simply by

$$(n+2w-4)\delta_{1,2} = N^A N^B \delta_1 D_A D_B.$$

Expanding. For $f \in \mathcal{E}[w]$

$$\delta_{1,2} f = (n+2w-5)\delta_1 \Box f - (n+2w-2)\big(\Box_\Sigma \delta_1 f + \text{ lower order}\big).$$

Where $\Box := \Delta + w\mathrm{J}$ and $\Box_\Sigma$ is the intrinsic to $\Sigma$ equivalent. When $w = 1 - \frac{n}{2}$ this factors: $\delta_{1,2} f = -3\delta_1 \Box f$. When $w = 2 - \frac{n}{2}$ then $\delta_{1,2} f = -3\Box_\Sigma \delta_1 f + \star f$ where $\star$ is a manifestly invariant lower order operator.

The $T$-curvature is:

$$T_g(\delta_{1,2}) = 3n^a \nabla_a \mathrm{J} - (\mathrm{n}-2)\mathrm{n}^\mathrm{a}\mathrm{n}^\mathrm{b}\mathrm{n}^\mathrm{c}\nabla_\mathrm{a}\mathrm{P}_\mathrm{bc} + 6\mathrm{HJ}$$
$$-6(n-2)Hn^a n^b \mathrm{P}_{ab} + 2(n-2)H^3 \; .$$

# The end

1. State as in Robin's article
2. Give the $d = 4$ explicit operator
3. Perhaps the $d = 6$ operator in PE
4. W tractor

5 $\delta_{d-5}W$ when $d$ even is in the bottom slot. $\delta_{d-5}Bach + ..$ This has metric transverse derivatives of order $n$ and pairs with boundary metric variations . . . – can show that it has the right leading symbol

Help!

## Some linear and non-linear (conformal) Dirichlet-Neumann maps

Recall that the conformal-Robin operator and Yamabe operators leads to a conformally invariant first order Dirichlet-to-Neumann operator on conformal manifolds with boundary. We discuss the construction of canonical invariant differential operators along a hypersurface in a conformal manifold that are higher order analogues of the conformal-Robin operator and so facilitate the construction of higher odd order Laplacian powers on the boundary of conformal manifolds, and in particular on conformally compact manifolds including Poincare-Einstein geometries.

Using these tools, we then show how, in certain cases, the scalar field scattering map on conformally compact manifolds can also be recovered by an explicit conformal Dirichlet-to-Neumann map. Extending these ideas, we study the non-linear Dirichlet-to-Neumann map for the even-dimensional Poincaré–Einstein problem. In particular, we describe its range in terms of a natural rank two tensor along the boundary. In low dimensions we give this explicitly, while in higher dimensions we can provide a tractor formula.