p-Laplace equations, p-superharmonic functions, and their applications in conformal geometry

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The intermediate Schouten curvature tensor

In conformal geometry one often encounters the Schouten curvature tensor on manifolds (M^n, g)

$$A = \frac{1}{n-2}(Ric - \frac{1}{2(n-1)}R\, g)$$

where Ric stands for the Ricci curvature tensor and $R = \mathrm{Tr}_g Ric$ is the scalar curvature. For a good reason, one uses notation $J = \frac{R}{2(n-1)}$. In this talk, we want to call the attention to **the intermediate Schouten curvature tensor**

$$A^{(p)} = (p-2)A + J\, g$$

for $p \in (1, \infty)$.

The transformation under conformal changes

For $p \in (1, n)$ and $\bar{g} = u^{\frac{4(p-1)}{n-p}} g$,

$$A^{(p)}[\bar{g}] = A^{(p)} - \frac{2(p-1)}{n-p}\left[\frac{\Delta u}{u} g + (p-2)\frac{D^2 u}{u}\right]$$

$$+\frac{2(p-1)}{n-p}[(1-(n+p-4)\frac{p-1}{n-p})\frac{|\nabla u|^2}{u^2} g$$

$$+(p-2)(1+\frac{2(p-1)}{n-p})\frac{\nabla u \otimes \nabla u}{u^2}].$$

p-Laplace equations in conformal geometry

Multiplying $u|\nabla u|^{p-2}\frac{u_i}{|\nabla u|}\frac{u_j}{|\nabla u|}$ to and summing up on both sides, we arrive at **the p-Laplace equations** in conformal geometry

$$-\Delta_p u + \frac{n-p}{2(p-1)}S^{(p)}(\nabla u)u = \frac{n-p}{2(p-1)}(S^{(p)}(\nabla u))[\bar{g}]u^q$$

where $\bar{g} = u^{\frac{4(p-1)}{n-p}}g$,

$$S^{(p)}(\nabla u) = |\nabla u|^{p-2}A^{(p)}(\nabla u), \quad q = \frac{2p(p-1)}{n-p} + 1,$$

and $A^{(p)}(\nabla u)$ is the $A^{(p)}$ curvature in the direction ∇u. Recall

$$\Delta_p u = \mathrm{div}(|\nabla u|^{p-2}\nabla u).$$

$p \in [2, n]$

For $p = 2$, the intermediate Schouten curvature goes back to the scalar curvature and the p-Laplace equation goes back to the scalar curvature equation

$$-\Delta u + \frac{n-2}{4(n-1)} R u = \frac{n-2}{4(n-1)} R[\bar{g}] u^{\frac{n+2}{n-2}}$$

where $\bar{g} = u^{\frac{4}{n-2}} g$. For $p = n$, the intermediate Schouten curvature becomes the Ricci and we recover the n-Laplace equation

$$-\Delta_n \phi + |\nabla\phi|^{n-2} Ric(\nabla\phi) = (|\nabla\phi|^{n-2} Ric(\nabla\phi))[\bar{g}] e^{n\phi}$$

where $\bar{g} = e^{2\phi} g$, which was recently introduced.

$p \in (n, \infty]$

For $p \in (n, \infty)$, the p-Laplace equation for the intermediate Schouten curvature still holds

$$-\Delta_p u + \frac{n-p}{2(p-1)} S^{(p)}(\nabla u) u = \frac{n-p}{2(p-1)} (S^{(p)}(\nabla u))[\bar{g}] u^q$$

for $\bar{g} = u^{-\frac{4(p-1)}{p-n}} g$ and $q = -\frac{2p(p-1)}{p-n} + 1 < 0$. And, when taking $p \to \infty$, we arrive at the infinite Laplace equation on Schouten curvature A

$$-\Delta_\infty u - \frac{1}{2} |\nabla u|^2 A(\nabla u) u = -\frac{1}{2} (|\nabla u|^2 A(\nabla u))[\bar{g}] u^{-7}$$

for $\bar{g} = u^{-4} g$. Recall $\Delta_\infty u = u_{ij} u_i u_j$.

Positivity cones

For the curvature tensor $A^{(p)}$ we consider the cones

$$\mathcal{A}^{(p)} = \{\lambda \in \mathbb{R}^n : \min_k \{(p-2)\lambda_k + \sum_{i=1}^n \lambda_i\} \geq 0\}$$

Recall, for fully nonlinear equations, we often consider

$$\Gamma^k = \{\lambda \in \mathbb{R}^n : \sigma_1(\lambda) \geq 0, \sigma_2(\lambda) \geq 0, \cdots, \sigma_k(\lambda) \geq 0\}.$$

To apply the Böchner formula on r-forms on locally conformally flat n-manifolds, we consider, for $r \leq \frac{n}{2}$,

$$\mathcal{R}^{(r)} = \{\lambda \in \mathbb{R}^n : \min\{(n-r)\sum_{k=1}^r \lambda_{i_k} + r \sum_{k=r+1}^n \lambda_{i_k}\} \geq 0\}$$

Properties of cone $\mathcal{A}^{(p)}$

Lemma

- $\mathcal{A}^{(p_2)} \subset \mathcal{A}^{(p_1)}$ *for* $p_1 < p_2$*;*
- $\mathcal{A}^{(2)} = \{(\lambda_1, \lambda_2, \cdots, \lambda_n) \in \mathbb{R}^n : \sum_{i=1}^n \lambda_i \geq 0\} = \Gamma^1$ *is the baseline;*
- $\mathcal{A}^{(n)}$ *stands for* $Ric \geq 0$ *when* $(\lambda_1, \lambda_2, \cdots, \lambda_n)$ *represents the Schouten curvature tensor* A*;*
- $\mathcal{A}^{(p)}$ *approaches the nonnegative cone* Γ^n *as* $p \to \infty$*.*

Therefore, we define, for any cone Γ in between Γ^1 and Γ^n,

$$p_\Gamma = \max\{p : \Gamma \subset \mathcal{A}^{(p)}\}.$$

cone comparisons

Lemma

$$p_{\Gamma^k} = \frac{n(k-1)}{n-k} + 2 \in [2, n]$$

for $1 \leq k \leq \frac{n}{2}$ *and* $p_{\Gamma^{\frac{n}{2}}} = n$.

We also observe

Lemma

$$\mathcal{R}^{(s)} \subset \mathcal{R}^{(r)} \quad \text{for } 0 < s \leq r \leq \frac{n}{2}$$

and

$$\mathcal{A}^{(p)} \subset \mathcal{R}^{(r)} \quad \text{for } \frac{n-p}{2} + 1 \leq r \leq \frac{n}{2}.$$

Obviously $\mathcal{A}^{(2)} = \mathcal{R}^{(\frac{n}{2})} = \Gamma^1$ *is the baseline.*

Vanishing of Betti numbers

Theorem (Liu-Ma-Q-Zhong 2023)

Let (M^n, g) be a compact locally conformally flat manifold with $A^{(p)} \geq 0$ for $p \in [2, n)$. Suppose that the scalar curvature is positive somewhere on M^n. Then, for $\frac{n-p}{2} + 1 \leq k \leq \frac{n+p}{2} - 1$, the Betti numbers $\beta_k = 0$, unless $(\tilde{M}^n, g) \overset{iso}{\sim} \mathbb{H}^r \times \mathbb{S}^{n-r}$.

The Böchner formula on r-forms (cf. Guan-Lin-Wang 2005)

$$\Delta\omega = \nabla^*\nabla\omega + \mathcal{R}(\omega)$$

$$\mathcal{R}(\omega) = ((n-r)\sum_{i=1}^{r}\lambda_i + r\sum_{i=r+1}^{n}\lambda_i)\omega$$

for $\omega = \omega_1 \wedge \omega_2 \cdots \wedge \omega_r$ and $\{\omega_k\}$ is the orthonormal basis under which A is diagonalized on locally conformally flat n-manifolds.

Huber's theorem

Suppose that (M^2, g) is a complete surface and that

$$\int_M K^- dvol < \infty.$$

Then M is a closed surface with finitely many points removed.

On the analysis side, Huber's theorem includes the statement:

For a domain Ω in a surface (M, g) and a compact subset $S \subset \Omega$, if there is a conformal metric $\bar{g} = e^{2u} g$ on $\Omega \setminus S$, which is complete near S and satisfies

$$\int_\Omega (K^- dvol)[\bar{g}] < \infty.$$

Then S consists of finitely many points.

The development map

Suppose that (M^n, g) is a locally conformally flat manifold and that the conformal immersion from a covering $(\tilde{M}^n, \tilde{g})$ to $(\mathbb{S}^n, g_{\mathbb{S}})$ is injective.

$$\begin{array}{ccc} \tilde{M}^n & \xrightarrow{\ \phi\ } & \mathbb{S}^n \\ {\scriptstyle\pi}\downarrow & & \\ M^n & & \end{array}$$

Then, on $\phi(\tilde{M}^n) \subset \mathbb{S}^n$, there is a complete conformal metric $\tilde{g} = e^{2u} g_{\mathbb{S}}$. One is interested in the size of $\mathbb{S}^n \setminus \phi(\tilde{M}^n)$, or specifically, the Hausdorff dimension of $\mathbb{S}^n \setminus \phi(\tilde{M}^n)$. Smaller the Hausdorff dimension is, "less" the topology of M^n has. And, more "positive" the curvature is, smaller the Hausdorff dimension is.

p-capacity

Definition

For a compact subset K of a domain Ω in $\mathbb{R}^n$, we define

$$cap_p(K,\Omega) = \inf\{\int_\Omega |\nabla u|^p dx : \ u \in C_0^\infty(\Omega) \text{ and } u \geq 1 \text{ on } K\}.$$

Then p-capacity for arbitrary subset E of Ω is

$$cap_p(E,\Omega) = \inf_{E\subset G \,\&\, G\subset\Omega \text{ open}} \ \sup_{K\subset G \text{ compact}} cap_p(K,\Omega).$$

p-thineness

Definition

A set $E \subset \mathbb{R}^n$ is said to be p-thin for $p \in (1, n)$ at $x_0 \in \mathbb{R}^n$ if

$$\sum_{i=1}^{\infty} \frac{cap_p(E \cap \omega_i(x_0), \Omega_i(x_0))}{cap_p(\partial B(x_0, 2^{-i}), B(x_0, 2^{-i+1}))} < +\infty.$$

E is said to be n-thin at $x_0 \in \mathbb{R}^n$ if

$$\sum_{i=1}^{\infty} i^{n-1} \, cap_n(E \cap \omega_i(x_0), \Omega_i(x_0)) < +\infty.$$

Lemma

Suppose E is p-thin at x_0 for $p \in (1, n)$. Then there is a ray from x_0 that avoids E in some neighborhhod of x_0.

The Wolff potential and p-Laplace equation

For a nonnegative Radon measure μ on a bounded domain $\Omega \subset \mathbb{R}^n$ and $p \in (1, n]$, let

$$W_{1,p}^{\mu}(x, r) = \int_0^r \left(\frac{\mu(B(x_0, t))}{t^{n-p}}\right)^{\frac{1}{p-1}} \frac{dt}{t}$$

Theorem (Kilpeläinen and Malý 1994)

Suppose that u is a nonnegative p-superharmonic function satisfying $-\Delta_p u = \mu$. Then

$$c_1 W_{1,p}^{\mu}(x, r) \leq u(x) \leq c_2 \left(\inf_{B(x,r)} u + W_{1,p}^{\mu}(x, 2r)\right)$$

for some constants $c_1(n, p)$ and $c_2(n, p)$ for $p \in (1, n]$.

The asymptotic behavior

Theorem (Liu-Ma-Q-Zhong 2023)

Let μ be a nonnegative finite Radon measure in Ω and $p \in (1, n]$, and let $B(x_0, 3r_0) \subset \Omega$. Then there is a subset E that is p-thin at x_0 such that

$$\lim_{x \to x_0 \text{ and } x \notin E} |x - x_0|^{\frac{n-p}{p-1}} W^{\mu}_{1,p}(x,\ r_0) = \frac{p-1}{n-p} \mu(\{x_0\})^{\frac{1}{p-1}}$$

for $p \in (1, n)$. Similarly, there is a subset E that is n-thin at x_0 such that

$$\lim_{x \to x_0 \text{ and } x \notin E} \frac{W^{\mu}_{1,n}(x,\ r_0)}{\log \frac{1}{|x - x_0|}} = \mu(\{x_0\})^{\frac{1}{n-1}}.$$

The improved asymptotic behavior

Theorem (Liu-Ma-Q-Zhong 2023)

Suppose μ is a nonnegative finite Radon measure in Ω. Assume that, for a point $x_0 \in \Omega$ and some number $m \in (0, n-p)$,

$$\mu(B(x_0, t)) \leq Ct^m$$

for all $t \in (0, 3r_0)$ with $B(x_0, 3r_0) \subset \Omega$. Then, for $\varepsilon > 0$, there are a subset $E \subset \Omega$, which is p-thin at x_0, and a constant $C > 0$ such that

$$W^{\mu}_{1,p}(x,\ r_0) \leq C|x - x_0|^{-\frac{n-p-m+\varepsilon}{p-1}} \quad \text{for all } x \in \Omega \setminus E$$

for $p \in [2, n)$

Asymptotic behavior at singularities

Theorem (Liu-Ma-Q-Zhong 2023)

Suppose that u is a nonnagetive p-superharmonic function in $\Omega \subset \mathbb{R}^n$ for a nonnegative finite Radon measure on Ω and $p \in (1, n]$. Then, for $x_0 \in \Omega$, there is a subset E that p-thin at x_0 such that

$$\lim_{x \to x_0 \text{ and } x \notin E} \frac{u(x)}{G_p(x, x_0)} = m \geq 0.$$

Moreover $u(x) \geq mG_p(x, x_0) - c_0$ for some c_0 and all x in a neighborhood of x_0, where

$$G_p(x, x_0) = \begin{cases} |x - x_0|^{-\frac{n-p}{p-1}} & \text{when } p \in (1, n) \\ -\log |x - x_0| & \text{when } p = n. \end{cases}$$

Improved estimates on the asymptotic

Corollary

Suppose u is a nonnegative p-superharmonic function satisfying $-\Delta_p u = \mu$ *for a nonnegative finite Radon measure in* Ω. *Assume that, for a point* $x_0 \in \Omega$ *and some number* $m \in (0, n-p)$,

$$\mu(B(x_0, t)) \leq Ct^m$$

for all $t \in (0, 3r_0)$ *with* $B(x_0, 3r_0) \subset \Omega$. *Then, for* $\varepsilon > 0$, *there are a subset* $E \subset \Omega$, *which is p-thin at* x_0, *and a constant* $C > 0$ *such that*

$$u(x) \leq C|x - x_0|^{-\frac{n-p-m+\varepsilon}{p-1}} \text{ for all } x \in \Omega \setminus E$$

for $p \in [2, n)$

A generalized Lebesgue Theorem

Lemma (Kpata 2019) Let μ be a nonnegative Radon measure on a complete Riemannian manifold (M^n, g) and let

$$G_d^\infty = \{x \in M^n : \limsup_{r \to 0} r^{-d} \mu(B_r(x)) = +\infty\}$$

for any $d \in [0, n]$. Then

$$\mathcal{H}_d(G_d^\infty) = 0$$

where $\mathcal{H}_d$ is the Hausdorff measure of dimension d.

On Hausdorff dimensions and consequences

Theorem (Liu-Ma-Q-Zhong 2023)

Suppose that S is a closed subset of the sphere $\mathbb{S}^n$. And suppose that there is a metric $\bar{g}$ on $\mathbb{S}^n \setminus S$ that is conformal to the standard round metric $g_{\mathbb{S}}$. Assume that it is geodesically complete near S and that $A^{(p)}[\bar{g}] \geq 0$ for some $p \in [2, n)$. Then

$$dim_{\mathcal{H}}(S) \leq \frac{n-p}{2}.$$

Corollary

Suppose that (M^n, g) is locally conformally flat with $A^{(p)} \geq 0$ for $p \in [2, n)$. Then, for $1 < k < \frac{n+p}{2} - 1$, the homotopy groups $\pi_k(M^n)$ are trivial.

On a class of fully nonlinear equations

Corollary (Extension of Labutin 2002)

Suppose that u is nonnegative and $u \in C^2(\Omega \setminus S)$ for a compact $S \subset \Omega \subset \mathbb{R}^n$. Assume $\lim_{x \to S} u(x) = \infty$, and $-\lambda(D^2 u(x)) \in \Gamma^k$ for $1 \leq k \leq \frac{n}{2}$. Then, for $x_0 \in S$, there is E that is p_{Γ^k}-thin at x_0

$$\lim_{x \to x_0 \text{ and } x \notin E} \frac{u(x)}{\Gamma^k(x, x_0)} = m \geq 0$$

Moreover $u(x) \geq m\Gamma^k(x, x_0) - c_0$ in some neighborhood of x_0. Here $p_{\Gamma^k} = \frac{n(k-1)}{n-k} + 2$ and

$$\Gamma^k(x, x_0) = \begin{cases} |x - x_0|^{2 - \frac{n}{k}} \text{ when } 1 \leq k < \frac{n}{2} \\ -\log |x - x_0| \text{ when } k = \frac{n}{2}. \end{cases}$$

Thank you!