# Optimal regularity for isoperimetric sets with density

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*CONFORMAL GEOMETRY AND NON-LOCAL OPERATORS*

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(joint work with L. Beck and C. Seis)

# The isoperimetric problem with density

Let us given two density functions $f$, $h\colon \mathbb{R}^n \to (0, +\infty)$.

For any measurable set $E \subset \mathbb{R}^n$, we set

$$V_f(E) := \int_E f(x)\, dx$$

and

$$P_h(E) := \int_{\partial^* E} h(x)\, d\mathcal{H}^{n-1}(x),$$

We are interested in the following minimization problem:

$$\inf \left\{ P_h(E) \colon E \subset \mathbb{R}^n \text{ with } V_f(E) = m \right\}.$$

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# Some References

- Existence: Morgan-Pratelli, De Philippis-Franzina-Pratelli, Pratelli-Saracco
- Regularity: Morgan, C.-Pratelli, Pratelli-Saracco, Beck-C.-Seis.

Many other contributions by: Cañete - Miranda - Vittone; Rosales - Cañete - Bayle - Morgan; Cañete - Rosales; Chambers; Cabré - Ros-Oton - Serra; Brock - Chiacchio - Mercaldo, .....

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# Regularity results

Theorem (Morgan, Trans. AMS 2003)

*Let $f = h$ be of class $C^{k,\alpha}(\mathbb{R}^n, \mathbb{R}^+)$ for some $k \geq 1$ and $\alpha \in (0,1]$. Then the boundary of any isoperimetric set is of class $C^{k+1,\alpha}$, except for a singular set of Hausdorff dimension at most $n - 8$.*

QUESTION: What about less regular densities?

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$$\sigma(\alpha, n) := \frac{\alpha}{2n(1-\alpha)+2\alpha}.$$

Remark

- We observe that the Hölder exponent

$$\sigma = \frac{\alpha}{2n(1-\alpha)+2\alpha} \to \frac{1}{2}, \quad \text{as } \alpha \uparrow 1;$$

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# Regularity result: $n = 2$

Theorem (C.-Pratelli, Math. Ann. 2017 )

*Let $n = 2$ and $f = h$ be of class $C^{0,\alpha}(\mathbb{R}^2, \mathbb{R}^+)$ for some $\alpha \in (0, 1]$. If $E$ is an isoperimetric set, then $\partial^* E \in C^{1,\alpha/(3-2\alpha)}$.*

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- We observe that the Hölder exponent

$$\frac{\alpha}{3-2\alpha} > \frac{\alpha}{4-2\alpha} =: \sigma(\alpha, 2)$$

- in particular

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# Idea of the proof

We show that the following implication holds true:

*If $E$ is an isoperimetric set w.r.t. the density $f \in C^{0,\alpha}$*

THEN

*$E$ is an $\omega$-minimal set for the classical perimeter*,

i.e. $\forall B_r$ and $\forall F$ such that $F \triangle E \subset\subset B_r$, we have

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Then, the regularity of $E$ follows by standard regularity theory for $\omega$-minimal sets:

**Theorem (Tamanini, 1984)**

*If $E$ is $\omega$-minimal with $\omega(r) = r^{2\sigma}$, then $\partial^* E \in C^{1,\sigma}$.*

**Main ingredient** in the proof of the previous implication: $\varepsilon - \varepsilon^\beta$ property.

We say that $F$ *fulfills the $\varepsilon - \varepsilon^\beta$ property with constant $C$* if for any ball $B$ such that $\mathcal{H}^{n-1}(B \cap \partial^* F) > 0$, there exists a constant $\bar{\varepsilon} > 0$ such that, for every $|\varepsilon| < \bar{\varepsilon}$, there is a set $G \subseteq \mathbb{R}^n$ such that

$$G \triangle F \subset\subset B\,, \qquad V_g(G) - V_g(F) = \varepsilon\,, \qquad P_f(G) \le P_f(F) + C|\varepsilon|^\beta\,.$$

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*Let $h$ be a density of class $C^{0,\alpha}(\mathbb{R}^n, \mathbb{R}^+)$ for some $\alpha \in (0,1]$ and $f$ be a locally bounded function. If $E$ is an isoperimetric set, then $\partial^* E \in C^{1,\alpha/(2n(1-\alpha)+2\alpha)}$.*

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# The main result

**Theorem (Beck, C., Seis, 2023)**

*Let $h$ be density of class $C^{0,\alpha}(\mathbb{R}^n, \mathbb{R}^+)$ and $f$ be a density of class $C^{0,\gamma}(\mathbb{R}^n, \mathbb{R}^+)$ for some $\alpha$ and $\gamma \in (0,1)$. Then the boundary of any isoperimetric set is of class $C^{1,\alpha/(2-\alpha)}$, except for a singular set of Hausdorff dimension at most $n-8$.*

**Remark**

- We achieve the Hölder exponent $\frac{\alpha}{2-\alpha}$ in any dimensions, improving all the previous results;
- observe that

$$\frac{\alpha}{2-\alpha} \to 1, \quad \text{as } \alpha \uparrow 1;$$

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# Idea of the proof

Thanks to the previous results, we know that $\partial^* E$ is locally the graph of a function $u$, which minimizes the functional

$$\int_{B_R} h(x,w)\left(1+|Dw|^2\right)^{\frac{1}{2}}\,dx$$

among all functions $w$ satisfying the constraint

$$\int_{B_R}\int_0^{w(x)} f(x,t)\,dt\,dx = m$$

for a given constant $m$ and with prescribed boundary values on $\partial B_R$.

# Idea of the proof

Since $h$ is merely Hölder continuous, we cannot write the associated Euler-Lagrange equation!!

**Idea**: we introduce a *comparison problem*, and consider $v$ to be the minimizer of

$$\int_{B_R} \left(1 + |Dv|^2\right)^{\frac{1}{2}} dx, \qquad v = u \quad \text{on } \partial B_R;$$

under the constraint

$$\int_{B_R} \int_0^{v(x)} f(x,t)\, dt\, dx = \int_{B_R} \int_0^{u(x)} f(x,t)\, dt\, dx = m.$$

Hence $v$ satisfies the E-L equation:

$$\int_{B_R} D_z a(Dv(x)) \cdot D\varphi(x)\, dx + \lambda \int_{B_R} f(x,v)\varphi(x)\, dx = 0.$$

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In order to prove that $u \in C^{1,\alpha/(2-\alpha)}$, it is enough to prove that for any ball $B_\rho(y) \subset B_R(0)$, we have

$$\int_{B_\rho(y)} |\partial_i u - (\partial_i u)_{B_\rho(y)}|^2 \, dx \leq C \rho^{n-1+\frac{2\alpha}{2-\alpha}} .$$

We will do this in two setps:

**Step 1.** We first prove suitable decay estimates for the solution $v$ of the comparison problem;

**Step 2.** We transfer such decay estimates from $v$ to the solution of our original problem $u$.

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# Step 2

We need to estimate the quantity

$$\int_{B_R} |Du - Dv|^2 \, dx.$$

In order to do that, we have to estimate the Lagrange multiplier $\lambda$.

**First easy estimates:**

$$\lambda \lesssim R^{-1};$$

$$\int_{B_R} |Du - Dv|^2 \, dx \lesssim R^{n-1+\frac{2\alpha}{2-\alpha}} + R^{n-1+\frac{2\gamma}{1-\gamma}}.$$

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If $\gamma \geq \alpha$ we are happy! Otherwise, the previous estimates needs an improvement.

**Lemma (Improved error estimate)**

*Let $u \in C^{1,\sigma}(B_R)$ with $\sigma \leq \frac{\alpha}{2-\alpha}$, then we have that*

$$|\lambda| \lesssim R^{\sigma-1}$$

*and*

$$\int_{B_R} |Du - Dv|^2\, dx \lesssim R^{n-1+\frac{2\alpha}{2-\alpha}} + R^{n-1+\frac{2}{1-\gamma}(\gamma+\sigma)}.$$

# Step 1: Decay estimates for $v$

**Proposition**

*Let $u \in C^{1,\sigma}(B_R)$ with $\sigma \leq \frac{\alpha}{2-\alpha}$ and let $v$ be a solution of the comparison problem.*

*Then, for any $0 < r < \rho$, we have*

$$\int_{B_r(x_0)} |\partial_i v - (\partial_i v)_r|^2 \, dx \lesssim \left(\frac{r}{\rho}\right)^{n-1+2(\gamma+\sigma)} \int_{B_\rho(x_0)} |\partial_i v - (\partial_i v)_\rho|^2 \, dx + r^{n-1+2(\gamma+\sigma)},$$

# Conclusion

Combining **Step 1** and **Step 2**, we improve the regularity of $u$

$$\text{from} \quad C^{1,\gamma} \quad \text{to} \quad C^{1,\gamma+\sigma}.$$

This allows to iterate the errore estimate Lemma and the deacy estimates for $v$, to conclude

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# Final Remarks

- Such regularity is optimal, we provide an explicit example;
- even if the final regularity does not depend on $\gamma$ we need to assume $f \in C^{0,\gamma}$;
- **Open question**: can we remove such assumption on $f$?

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**Muchas gracias!!**