The number of critical points of the Robin function in domains with a small hole.

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Joint result with

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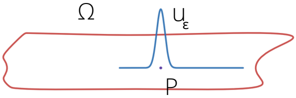
In the last years there has been a lot of interest around concentration problems such as

$$\begin{cases} -\Delta u = f_\varepsilon(u) & in\ \Omega \subset \mathbb{R}^N \\ u = 0 & on\ \partial\Omega \end{cases}$$

In this talk we only consider *positive* solutions.
Typical examples are

- $f_\varepsilon(s) = s^{\frac{N+2}{N-2}-\varepsilon}$, $N \geq 3$ (critical Sobolev exponent)
- $f_\varepsilon(s) = \varepsilon e^s$, $N = 2$ (Gelfand problem, Liouville problem)

A common feature of these problems is that it is possible to build solutions *concentrating at a finite number of points.*



Solution u_ε concentrating at P.

Concentration at k points

By concentration at k points we mean

$$-\Delta\big(C(\varepsilon)u_\varepsilon\big) \rightharpoonup \sum_{i=1}^{k} \alpha_i \delta_{P_i} \quad \text{with } \alpha_i \in \mathbb{R}^+$$

with $P_1, ..P_k \in \Omega$.
An immediate consequence of the previous statement and the standard regularity theory is the convergence of $C(\varepsilon)u_\varepsilon$ to the Green function of $-\Delta$, namely

$$C(\varepsilon)u_\varepsilon(x) \to \sum_{i=1}^{k} \alpha_i G(x, P_i) \quad \text{in } \Omega \setminus \{P_1, .., P_k\}$$

What about the location of P_i's?

Here the role of the Green function is more subtle, as showed in next definitions.

The Robin function

Let us recall the definitions of the *Robin* and *Kirchhoff-Routh* functions.
For $D \subset \mathbb{R}^N$, $N \geq 2$ we denote by $G_D(x,y)$ the Green function in D.

$$\begin{cases} -\Delta_x G_D(x,y) = \delta_y(x) & \textit{in } D \\ G_D(x,y) = 0 & \textit{on } \partial D. \end{cases}$$

We have the classical representation formula (for $N \geq 3$)

$$G_D(x,y) = \frac{1}{N(N-2)\omega_N} \frac{1}{|x-y|^{N-2}} - H_D(x,y),$$

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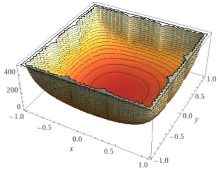
$H_D(x,y)$ is the *regular part of the Green function*. The *Robin* function is

$$\boxed{\mathcal{R}_D(x) := H_D(x,x) \quad \text{in } D}$$

Note that since $H_D(x,y) = \frac{C_N}{|x-y|^{N-2}}$ on $\partial\Omega$

then $\mathcal{R}_D(x) = +\infty$ on ∂D.



The Kirchhoff-Routh function

Once we have defined the Robin function it is immediate to define the Kirchhoff-Routh function,
For $D \subset \mathbb{R}^N$ and $k \geq 2$ set $\mathcal{KR}(x_1, .., x_k) : D \times \cdots \times D \to \mathbb{R}$ defined as

$$\boxed{\mathcal{KR}_D(x_1, .., x_k) = \sum_{i=1}^{k} \Lambda_i^2 \mathcal{R}_D(x_i) - \sum_{i \neq j \ \ i,j=1}^{k} \Lambda_i \Lambda_j G_D(x_i, x_j), \text{ for } \Lambda_i \in \mathbb{R}^+ \setminus \{0\}}$$

Note that in the definition some positive number $\Lambda_i \in \mathbb{R}$ appear.
They are related to the "speed of concentration" of the concentrated solutions.

The Robin function plays a crucial role in several concentration problems. We have the following *necessary condition*,

Theorem (Rey (1990), Han (1991))

Suppose that u_ε is a solution to

$$\begin{cases} -\Delta u = u^{\frac{N+2}{N-2}-\varepsilon} & \textit{in } \Omega \subset \mathbb{R}^N, \ N \geq 3 \\ u = 0 & \textit{on } \partial\Omega, \end{cases}$$

such that u_ε concentrates at one point $P \in \Omega$. Then P is a critical point of the Robin function.

An (almost) sufficient condition

Under a stability assumption the previous conditions on Robin function turn to be sufficient

Theorem (Rey (1990))

Assume that $P \in \Omega$ is a nondegenerate critical points of the Robin. Then there exist a solution u_ε to the problem

$$\begin{cases} -\Delta u = u^{\frac{N+2}{N-2}-\varepsilon} & in\ \Omega \subset \mathbb{R}^N,\ N \geq 3 \\ u = 0 & on\ \partial\Omega, \end{cases}$$

which concentrates at P.
Moreover this solution is locally unique, i.e. if v_ε is another solution which concentrate at P then $u_\varepsilon = v_\varepsilon$ for ε small.

CRITICAL POINTS OF THE ROBIN FUNCTION

$$\boxed{\mathcal{R}_D(x) := H_D(x,x) \quad \text{in } D}$$

KNOWN FACTS

Known facts on the Robin function

We recall some known facts about the Robin function.

- The only cases where the Robin function is explicitly known are the same as for the Green function (ball, half space, exterior of the ball). If $D = B_R$ we have that

$$\mathcal{R}_{B_R}(x) = \begin{cases} \frac{1}{2\pi} \ln \frac{R}{R^2 - |x|^2} & \text{if } N = 2, \\ \frac{1}{N(N-2)\omega_N} \frac{R^{N-2}}{\left(R^2 - |x|^2\right)^{N-2}} & \text{if } N \geq 3 \end{cases}$$

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- Unfortunately it is not known if the Robin function satisfies any differential equation. This is known only if $\Omega \subset \mathbb{R}^2$ and Ω is simply connected. In this case $u = 2\pi\mathcal{R}_\Omega$ solves the Liouville equation,

$$\begin{cases} \Delta u = 4e^{2u} & \textit{in } \Omega \\ u = +\infty & \textit{on } \partial\Omega \end{cases}$$

Known facts on the Robin function

- $\Omega \subset \mathbb{R}^2$ *convex* $\mathcal{R}_\Omega$ has one *nondegenerate* critical point (Caffarelli-Friedman, 1985)
- $\Omega \subset \mathbb{R}^N$, $N \geq 3$ *convex* $\mathcal{R}_\Omega$ has one critical point (Cardaliaguet-Tahraoui, 2002)
- Nondegenaracy of the critical points of $\mathcal{R}_\Omega$ holds *generically* with respect to $\Omega \subset \mathbb{R}^N$, $N \geq 2$, (Micheletti-Pistoia, 2014)
- Nondegeneracy for symmetric domains (Grossi, 2002)

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  Remark

  The Robin function plays a fundamental role in a great number of problems: conformal mappings, inner radius, capacity etc. A great source of references is the classical review by *Bandle* and *Flucher* (1996).

  Many questions are still unanswered and we are far from a complete understanding of its properties.

CRITICAL POINTS OF THE ROBIN FUNCTION

$$\boxed{\mathcal{R}_D(x) := H_D(x,x) \quad \text{in } D}$$

ADDITIONAL REMARKS FOR DOMAINS WITH HOLES

Domains with small holes

Here we consider the case where the domain Ω has *one* hole. Even in this particular case seems impossible to compute the exact number of critical points of the Robin function
For this reason we consider the case of one SMALL hole, namely
$\Omega_\varepsilon = \Omega \setminus B(P, \varepsilon)$
Since $R_{\Omega_\varepsilon} \to +\infty$ on $\partial\Omega_e$, by topological reasons we have the existence of at least 2 critical points.

Question Is this number sharp?

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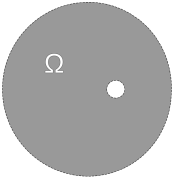
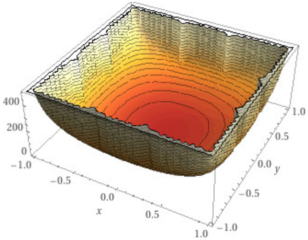
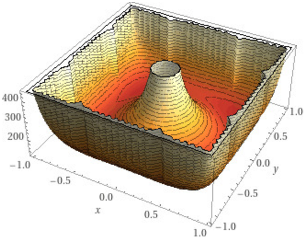


Convergence far away from the hole

Fix the setting of our problem. Denote by R_{Ω_ε} the Robin function in Ω_ε and by R_Ω the Robin function in Ω. Let us give a look to their graphs,



Robin function $\mathcal{R}_\Omega$ *in* Ω



Robin function $\mathcal{R}_{\Omega_\varepsilon}$ *in* $\Omega_\varepsilon = \Omega \setminus B(P,\varepsilon)$

Note that $\mathcal{R}_{\Omega_\varepsilon} \to \mathcal{R}_\Omega$ far away from the hole $B(P,\varepsilon)$.
On the other hand, since $\mathcal{R}_{\Omega_\varepsilon} \to +\infty$ on $\partial B(P,\varepsilon)$, there is a big *gap* between the two functions near $\partial B(P,\varepsilon)$.

Convergence far away from the hole

Remark

The standard regularity theory gives

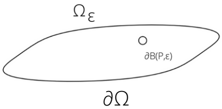
$$\mathcal{R}_{\Omega_\varepsilon} \to \mathcal{R}_\Omega \text{ far away from } B(P,\varepsilon).$$

Hence if $Q \neq P$ is a nondegenerate critical point of $\mathcal{R}_\Omega$ then there exists a critical point $Q_\varepsilon \to Q$ of $\mathcal{R}_{\Omega_\varepsilon}$. In the next slide we try to give some idea about this fact
The difficult part is to study what happens close to the boundary of the hole. Here an additional (and delicate!) analysis is needed.

Convergence far away from the hole

The regular part of the Green function $H_{\Omega_\varepsilon}(x,y)$ solves (for $N \geq 3$),

$$\begin{cases} \Delta_x H_{\Omega_\varepsilon}(x,y) = 0 & in\ \Omega_\varepsilon \\ H_{\Omega_\varepsilon}(x,y) = \frac{1}{N(N-2)\omega_N}\frac{1}{|x-y|^{N-2}} & on\ \partial\Omega_\varepsilon \end{cases}$$



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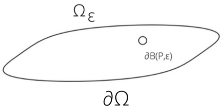

Using the representation formula for *harmonic* function we have

$$H_{\Omega_\varepsilon}(x,y) = -\frac{1}{N(N-2)\omega_N}\int_{\partial\Omega_\varepsilon}\frac{\partial G_{\Omega_\varepsilon}(x,t)}{\partial\nu_t}\frac{1}{|t-y|^{N-2}}d\sigma_t \Rightarrow$$

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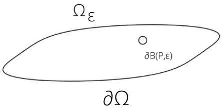

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and

$$\boxed{\mathcal{R}_{\Omega}(x) = H_{\Omega}(x,x) = -\frac{1}{N(N-2)\omega_N}\int_{\partial\Omega}\frac{\partial G_{\Omega}(x,t)}{\partial\nu_t}\frac{1}{|x-t|^{N-2}}d\sigma_t}$$

An important consequence

Remark

Let us consider the unit ball $B(0) \subset \mathbb{R}^N$ and $B_\varepsilon = B(0) \setminus B(P,\varepsilon)$ with $P \neq 0$. By the previous computations we have that, far away from P, $\mathcal{R}_{B(P)_\varepsilon} \to \mathcal{R}_B(0)$.
Since 0 is the only critical point of $\mathcal{R}_B(0)$ we have that the additional critical points $\mathcal{R}_{B(P)_\varepsilon}$ must collapse at P!
This is a general fact, **all additional critical points of the Robin function in $\Omega \setminus B(P,\varepsilon)$ collapse at P.**

What happens near the hole?

Remark (Schiffer-Spencer for $N=2$ (1952), Ozawa for $N\geq 3$ (1981))

The following expansion for the Robin function is known,

$$\mathcal{R}_{\Omega_\varepsilon}(x)=\mathcal{R}_\Omega(x)+\frac{\varepsilon^{N-2}G_\Omega^2(x,P)}{N(N-2)\omega_N\left(1-\varepsilon^{N-2}\mathcal{R}_\Omega(P)\right)}+O\left(\varepsilon^{N-1}\right),$$

(an analogous formula holds for $N=2$). However the remainder term $O\left(\varepsilon^{N-1}\right)$ is not uniform with respect to x .
We will improve the previous estimate in this way,

$$\mathcal{R}_{\Omega_\varepsilon}(x)=\mathcal{R}_\Omega(x)+\mathcal{R}_{B_\varepsilon^c}(x)+\begin{cases} O\left(\frac{\varepsilon^{N-2}}{|x-P|^{N-2}}\right)+O(\varepsilon) & \text{for } N\geq 3,\\ \frac{1}{2\pi}\ln\frac{\varepsilon}{|x-P|^2}+O\left(\frac{1}{|\ln\varepsilon|}\right) & \text{for } N=2,\end{cases}$$

where the remainder terms are uniform with respect to $x\in\Omega_\varepsilon$.

CRITICAL POINTS OF THE ROBIN FUNCTION

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THE MAIN RESULTS

The main result

Our main result is to compare the number of critical points of $\mathcal{R}_{\Omega_\varepsilon}$ in $\Omega \setminus B(P, \varepsilon)$ with $\mathcal{R}_\Omega$ in Ω.
The most interesting phenomenon is that the location of the point P is important! So it is crucial where you place the center of the hole!
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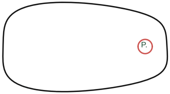
- If P is not a critical point of $\mathcal{R}_\Omega$ we have exactly "one more" critical point (a saddle point). 

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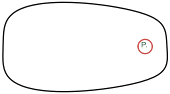
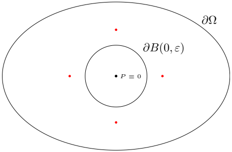
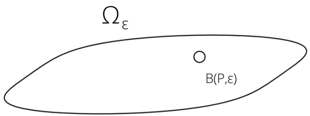
- If P is not a critical point of $\mathcal{R}_\Omega$ we have exactly "one more" critical point (a saddle point). 

- If P is a critical point of $\mathcal{R}_\Omega$ then we have more critical points. In particular if the eigenvalues of the Hessian matrix of $\mathcal{R}_\Omega(P)$ are simple we have $2N$ critical points close to P. 

Theorem (Gladiali, Grossi, Luo, Yan, to appear in JEMS)

Assume that $\Omega_\varepsilon = \Omega \setminus B(P,\varepsilon)$ *and*

$$\boxed{\nabla \mathcal{R}_\Omega(P) \neq 0}$$



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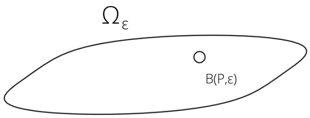

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then for ε *small enough,*

$$\sharp\Big\{\text{critical points of } \mathcal{R}_{\Omega_\varepsilon} \text{ in } B(P,r)\setminus B(P,\varepsilon)\Big\} = 1,$$

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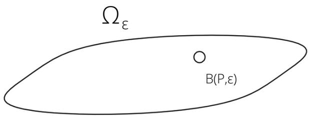

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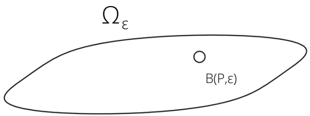

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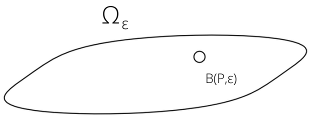

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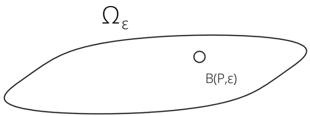

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- $\mathcal{R}_{\Omega_\varepsilon}(x_\varepsilon) \to \mathcal{R}_\Omega(P).$

The case $\Omega = B(0,1)$

Let us recall that in this case we have that (for $N \geq 3$)

$$\mathcal{R}_{B(0,1)}(x) = \frac{C_N}{\left(1 - |x|^2\right)^{N-2}}$$

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Corollary

Assume that $\Omega = B(0,1) \subset \mathbb{R}^N$, $N \geq 2$ *and* $\Omega_\varepsilon = B(0,1) \backslash B(P,\varepsilon)$. *Then, for* ε *small enough,*

$$\sharp\left\{\textit{critical points of } \mathcal{R}_{\Omega_\varepsilon} \textit{ in } B(0,1) \backslash B(P,\varepsilon)\right\} = \begin{cases} 2 & \textit{if } P \neq 0 \\ \infty & \textit{if } P = 0. \end{cases}$$

and if $P \neq 0$ *the two critical points are nondegenerate.*

The case $\Omega = B(0,1)$

Let us recall that in this case we have that (for $N \geq 3$)

$$\mathcal{R}_{B(0,1)}(x) = \frac{C_N}{\left(1 - |x|^2\right)^{N-2}}$$

Corollary

Assume that $\Omega = B(0,1) \subset \mathbb{R}^N$, $N \geq 2$ *and* $\Omega_\varepsilon = B(0,1) \backslash B(P,\varepsilon)$. *Then, for* ε *small enough,*

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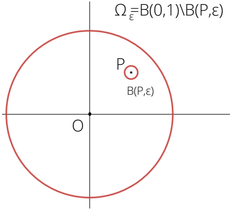
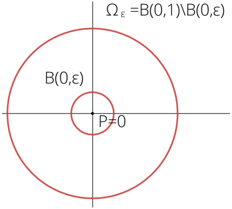


Two critical points for $\mathcal{R}_{\Omega_\varepsilon}$ *in* Ω_ε



Infinitely many critical points for $\mathcal{R}_{\Omega_\varepsilon}$ *in* Ω_ε

The previous result allow to compute di exact number of single-peak solutions for some suitable problem,

Corollary

Assume that $\Omega = B(0,1) \subset \mathbb{R}^N$, $N \geq 2$ *and* $\Omega_\varepsilon = B(0,1) \backslash B(P, \varepsilon)$. *Then, for* ε *small enough and* $P \neq 0$ *the problem*

$$\begin{cases} -\Delta u = u_\varepsilon^{\frac{N+2}{N-2}-\delta} & \text{in } \Omega_\varepsilon \subset \mathbb{R}^N \\ u > 0 & \text{in } \Omega_\varepsilon \\ u = 0 & \text{on } \partial\Omega_\varepsilon \end{cases}$$

admits exactly two single-peak solutions for δ *small enough.*

Idea of the proof

We have the expansion

$$\mathcal{R}_{\Omega_\varepsilon}(x) = \mathcal{R}_\Omega(x) + \underbrace{\mathcal{R}_{\mathbb{R}^N \setminus B(P,\varepsilon)}(x)}_{= \frac{C_N \varepsilon^{N-2}}{\left(|x-P|^2-\varepsilon^2\right)^{N-2}}} + \underbrace{O\left(\frac{\varepsilon^{N-2}}{|x-P|^{N-1}}\right) + O(\varepsilon)}_{delicate\ \ computation} \Rightarrow$$

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Computing the gradient of $\mathcal{R}_{\Omega_\varepsilon}$ we get

$$\nabla\mathcal{R}_{\Omega_\varepsilon}(x) \sim \nabla\mathcal{R}_\Omega(P) + \nabla\mathcal{R}_{\mathbb{R}^N\setminus B(P,\varepsilon)}(x)$$

and then if $\nabla\mathcal{R}_{\Omega_\varepsilon}(x_\varepsilon) = 0$ we deduce

$$\boxed{\underbrace{\nabla\mathcal{R}_{\mathbb{R}^N\setminus B(P,\varepsilon)}(x_\varepsilon)}_{this\ \ is\ \ explicit=C_N\varepsilon^{N-2}\frac{x_\varepsilon-P}{|x_\varepsilon-P|^{2N-4}}} = -\nabla\mathcal{R}_\Omega(P)}$$

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Solving the equation we get $|x_\varepsilon - P| \sim \varepsilon^{\frac{N-2}{2N-3}}$.
Observe that the equation

$$\boxed{\nabla\mathcal{R}_{\mathbb{R}^N\setminus B(P,\varepsilon)}(x_\varepsilon) = -\nabla\mathcal{R}_\Omega(P)}$$

has *exactly* one nonsingular zero and this implies the *nondegeneracy* of the critical point x_ε of $\mathcal{R}_{\Omega_\varepsilon}$.

The case $\nabla \mathcal{R}_\Omega(P) = 0$ for $N \geq 3$

Theorem, $\big($***Gladiali, Grossi, Luo, Yan***$\big)$**.** Assume that

- $\nabla \mathcal{R}_\Omega(P) = 0$ and P is nondegenerate, i.e. $det\big(Hess\mathcal{R}_\Omega(P)\big) \neq 0$.
- The Hessian matrix $Hess\big(\mathcal{R}_\Omega(P)\big)$ has N *positive simple* eigenvalues $0 < \lambda_1 < .. < \lambda_m$ for $i = 1, .., m$.

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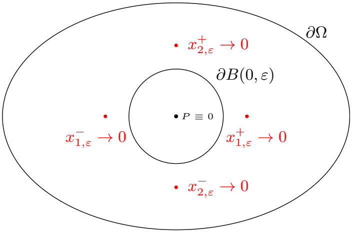
Then we have that for $\varepsilon > 0$ small enough,
$\sharp\Big\{\text{critical points of } \mathcal{R}_{\Omega_\varepsilon}(x) \text{ in } B(P,r) \setminus B(P,\varepsilon)\Big\} = 2N,$



Similar estimates for the critical points
$x^+_{1,\varepsilon}, x^-_{1,\varepsilon}, \ldots, x^+_{m,\varepsilon}, x^-_{m,\varepsilon}$
hold as when $\nabla\mathcal{R}_\Omega(O) \neq 0$.

In particular $x^\pm_{i,\varepsilon} \to P$ and
$\mathcal{R}_{\Omega_\varepsilon}(x^\pm_{i,\varepsilon}) \to \mathcal{R}_\Omega(0)$ for $i = 1, \cdots, m$.

Example

Assume that Ω is an ellipsoid in $\mathbb{R}^N$, $N \geq 2$ centered at 0. Then it is possible to choose the N axis such that the conditions

- $\nabla \mathcal{R}_\Omega(P) = 0$ and P is nondegenerate, i.e. $det\big(Hess\mathcal{R}_\Omega(P)\big) \neq 0$.
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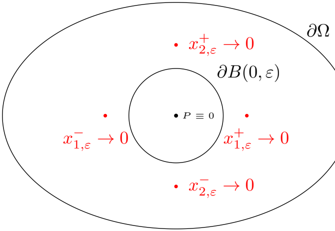

result hold.

The previous result allow to compute di exact number of single-peak solutions for some suitable problem,

Corollary

Assume that Ω *verifies*

- $\nabla \mathcal{R}_\Omega(P) = 0$ *and* P *is nondegenerate, i.e.* $det\big(Hess\mathcal{R}_\Omega(P)\big) \neq 0$.
- *The Hessian matrix* $Hess\big(\mathcal{R}_\Omega(P)\big)$ *has* N positive simple *eigenvalues* $0 < \lambda_1 < .. < \lambda_m$ *for* $i = 1, .., m$.

hence for $\Omega_\varepsilon = \Omega \backslash B(0, \varepsilon)$ *the problem*

$$\begin{cases} -\Delta u = u_\varepsilon^{\frac{N+2}{N-2}-\delta} & in\ \Omega_\varepsilon \subset \mathbb{R}^N \\ u > 0 & in\ \Omega_\varepsilon \\ u = 0 & on\ \partial\Omega_\varepsilon \end{cases}$$

admits exactly $2N$ *single-peak solutions for* ε *small enough.*

Analogous results

Similar results about the number of critical points of solutions of nonlinear problems were obtained jointly with P. Luo,

Theorem (Grossi,Luo, to appear in IUMJ)

Assume that $\Omega_\varepsilon = \Omega \setminus B(P,\varepsilon)$ *and and let* u_ε, u_0 *be solutions of the problems*

$$\begin{cases} -\Delta u_\varepsilon = 1 & \text{in } \Omega_\varepsilon \\ u_\varepsilon = 0 & \text{on } \partial\Omega_\varepsilon \end{cases} \qquad \begin{cases} -\Delta u_0 = 1 & \text{in } \Omega \\ u_0 = 0 & \text{on } \partial\Omega \end{cases}$$

Moreover,

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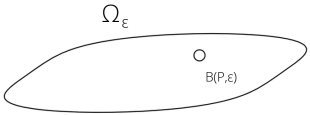

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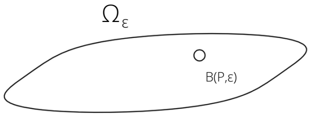

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$$\sharp\left\{\text{critical points of } u_\varepsilon \text{ in } B(P,r) \setminus B(P,\varepsilon)\right\} = 1,$$

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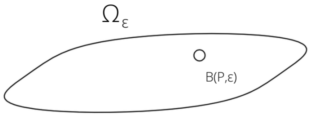

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We would like to extend our results to the following problems,

- The Kirchhoff-Routh function,

$$\mathcal{KR}_{\Omega_\varepsilon}(x_1, .., x_k) = \sum_{i=1}^{k} \Lambda_i^2 \mathcal{R}_{\Omega_\varepsilon}(x_i) - \sum_{i \neq j \;\; i,j=1}^{k} \Lambda_i \Lambda_j G_{\Omega_\varepsilon}(x_i, x_j)$$

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Extensions to the Kirchhoff-Routh function

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  which gives the location of the peaks of multi-bump solutions.

- If $k = 2$ the Kirchhoff-Routh function, becomes

$$\mathcal{KR}_{\Omega_\varepsilon}(x_1, x_2) = \Lambda_1^2 \mathcal{R}_{\Omega_\varepsilon}(x_1) + \Lambda_2^2 \mathcal{R}_{\Omega_\varepsilon}(x_2) - 2\Lambda_i \Lambda_j G_{\Omega_\varepsilon}(x_1, x_2).$$

  Even this simplified problem requires A LOT OF computations (at least 100 pages, in preparation with F. Gladiali, P.Luo and S.Yan).

Gracias a todos!