Compactness of Dirac-Einstein Structures in Dimensions 3 and 4

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Outline

- Intro to the problem
- Main Results
- Key Ideas
- Ingredients

The functional

Let (M, g, Σ) be a smooth Riemannian, spin manifold and consider the following functional

$$E(g, \psi) = \int_M R_g + \langle D_g\psi, \psi\rangle - \lambda|\psi|_g^2 \; dv_g.$$

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The critical points of this functional are solutions of the system

$$\begin{cases} Ric_g - \frac{R_g}{2} g = T^{g,\psi} \\ \\ D_g\psi = \lambda\psi \end{cases} \tag{1}$$

where $T^{g,\psi}$ denotes the energy-momentum tensor

$$T^{g,\psi}(X,Y) = -\frac{1}{4}\langle X\cdot\nabla_Y\psi + Y\cdot\nabla_X\psi, \psi\rangle \ , \tag{2}$$

here $\cdot$ and ∇ denote the Clifford multiplication and the metric connection extended to the spin bundle ΣM.
Such a spinor will be called Einstein spinor.

Coupling

The structure is similar to some familiar problems such as:

- Schrödinger-Newton gravity. Actually, one could derive one from the other, under certain circumstances.
- Yang-Mills-Higgs-Dirac model (supersymmetric gauge theory).
- Seiberg-Witten equations.
- Supersymmetric nonlinear Sigma model.

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- Einstein spinors on product manifolds $M^6 \times N^r$, where M^6 is a six-dimensional simply connected nearly Kähler manifold and N^r is a manifold admitting Killing spinors.
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- WK-spinors on quasi-Einstein Sasakian manifolds.
- WK-spinors on the three-dimensional sphere S^3 with non-standard merics.
- WK-spinors on the three-dimensional Euclidean space R^3 with non-constant scalar curvature.

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Objective

We propose to study the properties of this set:

$$\mathcal{E}(D,c,K)=\Big\{(g,\psi)\in Crit_c(E);\quad diam(M,g)\leq D, -\Delta_g R_g \geq -KR_g\Big\}$$

in terms of compactness and local structure. In the spirit of:

- M. Anderson, Ricci curvature bounds and Einstein metrics on compact manifolds. J. Amer. Math. Soc. 2 (1989).
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Main results

We will focus on dimensions 3 and 4 but similar results can be proved under a stronger bound on the Riemann tensor in higher dimensions.

Theorem

Let $n = 3$, then the space $\mathcal{E}(D, c, K)$ is compact in the topology induced by the Hausdorff distance. That is, if $(g_k, \psi_k) \in \mathcal{E}(D, c, K)$ then there exists a subsequence again denoted by (g_k, ψ_k) that converges in $C^{\ell,\alpha}(M)$ to (g_∞, ψ_∞) for all $\ell > 0$ and $0 < \alpha < 1$ and ψ_∞ is an Einstein spinor on (M, g_∞).

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Let $n = 4$, then there exist a compact orbifold (M_∞, g_∞) with a finite set S of orbifold singularities, a spinor $\psi_\infty \in \Sigma(M_\infty \setminus S)$ and a sequence of C^∞ embeddings $F_k : (M_\infty \setminus S, g_\infty) \to (M, g_k)$, for k large enough, such that

- *$((F_k)^* g_k, (F_k)^* \psi_k)$ converges uniformly on compact subsets in the $C^{\ell,\alpha}$ topology on $M_\infty \setminus S$, to (g_∞, ψ_∞) for every $\ell > 0$ and $0 < \alpha < 1$ and ψ_∞ is an Einstein spinor on $(M_\infty \setminus S, g_\infty)$.*
- *For each $p_i \in S$, there exist a sequence of real numbers r_k and a sequence of points $x_k \in M$ such that $(M, r_k g_k, x_k)$ converges in the pointed Gromov-Hausdorff sense to $(Y_i, \overline{g}_i, x_\infty)$, where $(Y_i, \overline{g}_i)$ is a Ricci flat non-flat manifold.*

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continued...

Theorem

Moreover, there exists a parallel spinor $\psi_{i,\infty} \in \Gamma(\Sigma Y_i)$ *such that* $H_k^*\psi_k$ *converges in the* $C^{\ell,\alpha}$ *topology in* $B(p_i, r)$, *to the spinor* $\psi_{i,\infty}$ *for every* $\ell > 0$ *and* $0 < \alpha < 1$ *and*

$$\liminf_{k\to\infty} \int_M |Rm_{g_k}|^2 \, dv_{g_k} \geq \int_M |Rm_{g_\infty}|^2 \, dv_{g_\infty} + \sum_{i=1}^{|S|} \int_{Y_i} |Rm_{\overline{g}_i}|^2 \, dv_{\overline{g}_i}.$$

Second Variation

Theorem

Let h be a symmetric two tensor transverse to the diffeomorphism action, that is $\delta h = 0$. If (g, ψ) is a critical point of E, then we have

$$\nabla^2 E(g,\psi)[(h,\varphi),(h,\varphi)] = \int_M \frac{1}{2}\langle \Delta_L h + \nabla\nabla tr(h), h\rangle$$
$$+ \frac{1}{2}\left(-\Delta tr(h) - \langle Ric_g, h\rangle\right) tr(h) + \frac{R_g}{2}|h|^2$$
$$+ \frac{1}{2}\langle T^{g,\psi}, h\rangle tr(h) + \frac{1}{2}\langle h \times T^{g,\psi}, h\rangle + \frac{1}{2}\langle \nabla tr(h)\cdot\psi, \varphi\rangle$$
$$+ \langle \mathcal{D}^h\varphi, \psi\rangle + 2\langle D_g\varphi - \lambda\varphi, \varphi\rangle dv$$

Example 1: Killing Spinors

We consider a real Killing spinor ψ, that is $\nabla_X \psi = -\mu X \cdot \psi$ for a real constant $\mu > 0$, for all tangent vector X.
The second variation reads

$$\nabla^2 E(g,\psi)[(h,\varphi),(h,\varphi)] \\ = \int_M \frac{1}{2}\langle \Delta_L h, h\rangle + \frac{3n-2}{4n} R|h|^2 + 2\langle D\varphi - \lambda\varphi, \varphi\rangle dv.$$

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Exemple 2: Quasi-Killing Spinors

consider a Sasakian spin-manifold $(M^{2m+1}, g, \phi, \eta)$. A quasi-Killing spinor satisfies

$$\nabla_X \psi = aX \cdot \psi + b\eta(X)\xi \cdot \psi,$$

where ξ is the Reeb vector field of η. One can construct an Einstein Spinor from a quasi-Killing spinors with

$$a = \pm\frac{1}{2}, \quad b = \mp\frac{2m^2 - m - 2}{4(m-1)}.$$

In this case, we have

$$\begin{aligned}&\nabla^2 E(g,\psi)[(h,\varphi),(h,\varphi)]\\&= \int_M \frac{1}{2}\langle \Delta_L h, h\rangle + \frac{m+1}{m-1}|h|^2 + \frac{2m^2-m-2}{2(m-1)}|h(\xi)|^2\\&- \frac{2m^2-m-1}{4(m-1)}\langle h(\xi)\cdot\xi\cdot\psi,\varphi\rangle + 2\langle D\varphi - \lambda\varphi, \varphi\rangle \, dv.\end{aligned}$$

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\end{aligned}$$

Corollary

Let the pair (g, ψ) be a critical point for the functional E. Then the symmetric 2-tensor h belongs to the space of horizontal Dirac-Einstein deformations, if it solves the following system of equations

$$\begin{cases} \delta h = 0 \\ \left\langle Ric_g - \frac{R_g}{2} g, h \right\rangle = \langle T^{g,\psi}, h \rangle = 0 \\ \Delta h + R_g h + T^{g,\psi} \times h = 0 \end{cases}$$

In particular, the space of horizontal Dirac-Einstein perturbations is finite dimensional.

Tools

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Key Estimates

Proposition

Let $(g,\psi) \in \mathcal{E}(D,K,c)$, $p \in M$ *and* $r > 0$ *sufficiently small. There exists* $\varepsilon_0(\lambda, K, D) > 0$ *such that, if*

$$\int_{B_{2r}} |Rm|^2 dv < \varepsilon_0,$$

then there exists $C(\lambda, K, D) > 0$ *such that*

$$\|Rm\|_{L^4(B_r)} \leq C\left(\|Rm\|_{L^2(B_{2r})} + vol(B_{8r})^{\frac{1}{2}}\right)$$

and

$$\|\nabla Rm\|_{L^2(B_r)} \leq C\left[\|Rm\|^2_{L^2(B_{2r})} + \|Rm\|_{L^2(B_{2r})} + vol(B_{8r}) + vol(B_{8r})^{\frac{1}{2}}\right],$$

ε-regularity

Proposition

There exist $\varepsilon_1(\lambda, K, D) > 0$ *and* $0 < r_0 < 1$ *such that, if*

$$\int_{B_{16r}} |Rm|^2 dv < \varepsilon_1, \quad r < r_0,$$

then there exists $C(\lambda, K, D)$ *such that*

$$\sup_{B_{\frac{r}{2}}} |Rm| \leq C(\lambda, K, D).$$

The Good, the Bad

Let $r < \frac{r_0}{4}$ as defined in above and consider a covering of (M, g_i) by balls $B(x_k, r)$ such that $B(x_k, \frac{r}{2})$ are disjoint. We let

$$I = \left\{ k \in \mathbb{N}; \int_{B(x_k, 16r)} |Rm_{g_i}|^2 dv_{g_i} < \varepsilon_1 \right\},$$

and we define

- $G_i(r) = \bigcup_{k \in I} B(x_k, r)$.

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The Singular Set

The singular set S can then be defined by $S = M_\infty \setminus M^0$. It can be characterized by

$$S = \left\{ \begin{array}{l} p \in M, \text{ such that there exists } x_k \in M, r > 0, \varepsilon_1 > 0 \text{ with} \\ \\ x_k \to p \text{ and} \liminf_{r \to 0} \liminf_{k \to \infty} \int_{B(x_k,r)} |Rm_{g_k}|^2 dv_{g_k} \geq \varepsilon_1 \end{array} \right\}.$$

Around the Singularity

Around the singularity, we rescale the metric in order to have a uniform bound on the curvature tensor.

- Tracking the various quantities (as in the Einstein setting), we get a non-flat Ricci flat metric at the limit. In fact, it is ALE.
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Thank you for your attention!