Critical double phase problems

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The double phase problem

$$(P)\ \begin{cases} -\mathrm{div}(|\nabla u|^{p-2}\nabla u + a(x)|\nabla u|^{q-2}\nabla u) = f(x,u) & \Omega \\ u = 0 & \partial\Omega \end{cases}$$

$\Omega \subset \mathbb{R}^N$ bdd $\qquad 1 < p < q < N \qquad 0 \le a \in \mathrm{Lip}(\bar{\Omega})$

$$\boxed{f(x,u) = \mu|u|^{p^*-2}u + b(x)|u|^{q^*-2}u + g(x,u)}$$

g subcritical

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* **Brezis-Nirenberg** '83, Cerami-Fortunato-Struwe, Capozzo-Fortunato-Palmieri, Gazzola-Ruf...
* $-\Delta_p$: Azorero-Peral, Guedda-Véron, Egnell, de Figueireido-Gossez-Ubilla, Cingolani-Vannella, Ariola-Gazzola...
* $(-\Delta)^s_p$: Servadei-Valdinoci, Barrios-Del Pezzo-Melián-Quaas, Mawhin-Molica Bisci, Mosconi-Perera-Squassina-Yang...
* $-\Delta_p - \Delta_q$: Ho-Perera-Sim

Nonstandard growth

In models for strongly anisotropic materials the following functionals arise

$$\int_\Omega (|\nabla u|^p + a(x)|\nabla u|^q)dx$$

p, q: hardening exponents of the materials

$a(x) = 0$ somewhere $\rightsquigarrow$ two different elliptic behaviors

$*$ '80-'90-s: V. V. Zhikov

- regularity of minimizers
    - $*$ P. Marcellini, P. Baroni, M. Colombo, G. Mingione

Musielak-Orlicz spaces

We consider the Φ-function (≥ 0; in t: continuous, convex, $\nearrow$; $L^1_{\text{loc},x}$)

$$\boxed{\mathcal{A}(x,t) := t^p + a(x)t^q \quad \forall (x,t) \in \overline{\Omega} \times [0,\infty)}$$

▷ $1 < p < q < N, \quad \frac{q}{p} < 1 + \frac{1}{N} \;\; (\Rightarrow q < p^*)$

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$$L^{\mathcal{A}}(\Omega) := \left\{ u : \Omega \to \mathbb{R} \text{ meas. } : \rho_{\mathcal{A}}(u) := \int_\Omega \mathcal{A}(x, |u(x)|)dx < \infty \right\}$$ *modular*

$$\|u\|_{\mathcal{A}} := \inf \left\{ \gamma > 0 : \int_\Omega \mathcal{A}(x, |u(x)/\gamma|)\, dx \leq 1 \right\}$$ *Luxemburg norm*

▷ $\min\{\|u\|^p, \|u\|^q\} \leq \rho_{\mathcal{A}}(u) \leq \max\{\|u\|^p, \|u\|^q\}$

$$W^{1,\mathcal{A}}(\Omega) := \{ u \in L^{\mathcal{A}}(\Omega) : |\nabla u| \in L^{\mathcal{A}}(\Omega) \}$$
$$\|u\|_{1,\mathcal{A}} := \|\nabla u\|_{\mathcal{A}} + \|u\|_{\mathcal{A}}$$

$$W_0^{1,\mathcal{A}}(\Omega) := \overline{C_0^\infty(\Omega)}^{\|\cdot\|_{1,\mathcal{A}}}, \qquad \|u\| \overset{\textit{Poincaré}}{:=} \|\nabla u\|_{\mathcal{A}}$$

* J. Musielak, *Springer*, 1983

Critical growth

▷ Fan, Nonlinear Anal. '12:

- $W^{1,\mathcal{A}}(\Omega) \hookrightarrow L^{\mathcal{A}_*}(\Omega)$, $\mathcal{A}_*(x,\cdot)$, *Sobolev conjugate function of* $\mathcal{A}$, is the inverse of

$$\mathcal{A}_*^{-1}(x,\tau) := \int_0^\tau \frac{\mathcal{A}^{-1}(x,t)}{t^{\frac{N+1}{N}}} dt$$

- $\Phi \ll \mathcal{A}_*$, i.e. $\lim_{t\to\infty} \frac{\Phi(x,kt)}{\mathcal{A}_*(x,t)} = 0 \ \forall\ k > 0$, uniformly in x $\rightsquigarrow$ $W_0^{1,\mathcal{A}}(\Omega) \hookrightarrow\hookrightarrow L^\Phi(\Omega)$

∗ '70s: Donaldson-Trudinger, Adams
$\rightsquigarrow$ for Orlicz spaces , i.e. $\Phi = \Phi(t)$ indep. of x

Critical growth

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  - No results for best Sobolev constants $W^{1,\mathcal{A}}(\mathbb{R}^N) \hookrightarrow L^{\mathcal{A}_*}(\mathbb{R}^N)$ (not even for Orlicz spaces)

  - $\mathcal{A}_*(x,t) \sim \begin{cases} t^{p^*} & \text{if } a(x) = 0 \\ t^{q^*} & \text{if } a(x) \neq 0 \end{cases}$ as $t \to \infty$

Our nonlinearity

$$\boxed{f(x,u) = \mu|u|^{p^*-2}u + b(x)|u|^{q^*-2}u + \lambda|u|^{r-2}u + c(x)|u|^{s-2}u}$$

- $0 \leq \lambda,\ \mu,\ b(x),\ c(x) \in L^\infty(\Omega),\ p \leq r < p^* \leq s < q^*$
- $a_0 := \inf_{\mathrm{supp}(b)} a(x) > 0$

Embedding 1

- $$\int_\Omega |u|^{p^*} \leq \frac{1}{S_p^{p^*/p}} \left(\int_\Omega |\nabla u|^p \right)^{p^*/p}$$
- $$\int_\Omega b(x)|u|^{q^*} \leq \frac{b_\infty}{(a_0 S_q)^{q^*/q}} \left(\int_\Omega a(x)|\nabla u|^q \right)^{q^*/q}, \quad b_\infty := \|b\|_\infty$$
- $W_0^{1,\mathcal{A}}(\Omega) \hookrightarrow L^{\mathcal{B}}(\Omega),\ \mathcal{B}(x,t) := t^{p^*} + b(x)t^{q^*}$

Our nonlinearity

$$\boxed{f(x,u) = \mu|u|^{p^*-2}u + b(x)|u|^{q^*-2}u + \lambda|u|^{r-2}u + c(x)|u|^{s-2}u}$$

- $0 \leq \mu$, $b(x)$, $c(x) \in L^\infty(\Omega)$, $p \leq r < p^* \leq s < q^*$
- $c(x) \leq Ca(x)^{s/q}$

Embedding 2

$W_0^{1,\mathcal{A}}(\Omega) \hookrightarrow\hookrightarrow L^{\mathcal{C}}(\Omega)$

$\mathcal{C}(x,t) := t^r + c(x)t^s$, $1 < r < p^*$, $r < s < q^*$

- $c(x) = 0 \Leftrightarrow a(x) = 0 \rightsquigarrow \mathcal{C} \ll \mathcal{A}_*$, i.e. $\lim_{t\to\infty} \frac{\mathcal{C}(x,kt)}{\mathcal{A}_*(x,t)} = 0 \ \forall$ $k > 0$, uniformly in $x \rightsquigarrow W_0^{1,\mathcal{A}}(\Omega) \hookrightarrow\hookrightarrow L^{\mathcal{C}}(\Omega)$

Our nonlinearity

$$f(x,u) = \mu|u|^{p^*-2}u + b(x)|u|^{q^*-2}u + \lambda|u|^{r-2}u + c(x)|u|^{s-2}u$$

- $0 \le \mu,\ b(x),\ c(x) \in L^\infty(\Omega),\ p \le r < p^* \le s < q^*$
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EMBEDDING 2

$W_0^{1,\mathcal{A}}(\Omega) \hookrightarrow\hookrightarrow L^{\mathcal{C}}(\Omega)$

$\mathcal{C}(x,t) := t^r + c(x)t^s,\ 1 < r < p^*,\ r < s < q^*$

- $c(x) = 0 \Leftrightarrow a(x) = 0 \rightsquigarrow \mathcal{C} \ll \mathcal{A}_*$, i.e. $\lim_{t\to\infty} \frac{\mathcal{C}(x,kt)}{\mathcal{A}_*(x,t)} = 0\ \forall$ $k > 0$, uniformly in x $\rightsquigarrow W_0^{1,\mathcal{A}}(\Omega) \hookrightarrow\hookrightarrow L^{\mathcal{C}}(\Omega)$
- $\tilde{c}(x) := c(x) + a(x)^{s/q} \rightsquigarrow \tilde{c}(x) = 0 \Leftrightarrow a(x) = 0 \rightsquigarrow$ $W_0^{1,\mathcal{A}}(\Omega) \hookrightarrow\hookrightarrow L^{\tilde{\mathcal{C}}}(\Omega)$
- $\tilde{c}(x) \ge c(x) \rightsquigarrow \tilde{\mathcal{C}} \ge \mathcal{C} \rightsquigarrow L^{\tilde{\mathcal{C}}}(\Omega) \hookrightarrow L^{\mathcal{C}}(\Omega)$

* Ho, Winkert, ArXiv '22, $c(x) = Ca(x)^\alpha$

Existence results

$$(P_1)\begin{cases}-\mathrm{div}(|\nabla u|^{p-2}\nabla u + a(x)|\nabla u|^{q-2}\nabla u) & \\ \qquad = \mu|u|^{p^*-2}u + b(x)|u|^{q^*-2}u + \lambda|u|^{r-2}u & \Omega \\ u = 0 & \partial\Omega\end{cases}$$

▷ $p \leq r < p^*, \quad \mu > 0$

Theorem 1

If $\exists\, B_\rho(x_0) \subset \Omega$ where $a(x) \equiv 0$, then $\exists\, b_* > 0$ s.t. problem (P_1) has a nontrivial weak solution when $b_\infty < b_*$, in each of the following cases:

- $N \geq p^2$, $r = p$, $0 < \lambda < \lambda_1(p)$
- $N \geq p^2$, $p < r < p^*$, $\lambda > 0$
- $N < p^2$, $\frac{(Np-2N+p)p}{(N-p)(p-1)} < r < p^*$, $\lambda > 0$

∗ *Critical dimensions* for $-\Delta_p$: $N \in (p, p^2)$ [Azorero,Peral, CPDE '87]

∗ Farkas, Fiscella, Winkert '22:

$1 < r < p$, $b \equiv 0 \rightsquigarrow$ negative energy solutions

Existence results

$$(P_2)\begin{cases} -\mathrm{div}(|\nabla u|^{p-2}\nabla u + a(x)|\nabla u|^{q-2}\nabla u) \\ \qquad\qquad = \mu|u|^{p^*-2}u + b(x)|u|^{q^*-2}u + c(x)|u|^{s-2}u & \Omega \\ u = 0 & \partial\Omega \end{cases}$$

▷ $p^* \le s < q^*, \quad b \not\equiv 0$

Theorem 2

If $\exists\ B_\rho(x_0) \subset \Omega$ where

$$a(x) \equiv a_0, \quad b(x) \equiv b_\infty, \quad c(x) \ge c_0 > 0$$

then $\exists\ \mu_* > 0$ s.t. problem (P_2) has a nontrivial weak solution when $0 \le \mu < \mu_*$, in each of the following cases:

- $1 < p < \frac{N(q-1)}{N-1}$, $\frac{N^2(q-1)}{(N-1)(N-q)} < s < q^*$
- $\frac{N(q-1)}{N-1} \le p < q$, $\frac{Np}{N-q} < s < q^*$

* Ha, Ho, ArXiv, '23: both μ and b small via Concentration Cpt Principle

Strategy of proofs

For all $u \in W_0^{1,\mathcal{A}}(\Omega)$

$$E(u) := \int_\Omega \left(\frac{|\nabla u|^p}{p} + a(x)\frac{|\nabla u|^q}{q} - \frac{\mu}{p^*}|u|^{p^*} - \frac{b(x)}{q^*}|u|^{q^*} - G(x,u) \right)$$

$$G(x,t) - \frac{t}{\sigma} g(x,t) \leq \mu\left(\frac{1}{\sigma} - \frac{1}{p^*}\right)|t|^{p^*} + c_1,\ q < \sigma < p^*,\ \forall\ (x,t)$$

✓ Mountain pass geometry

For Brezis-Nirenberg's problem

For $-\Delta u = \lambda u + u^{2^*-1}$

1. E satisfies $(PS)_\beta$ for all $\beta \in (0, \frac{1}{N}S^{N/2})$
2. Mountain pass energy level $c < \frac{1}{N}S^{N/2} \rightsquigarrow$ use U_ε realizing S

We do not know if $\inf_{W^{1,\mathcal{A}}(\mathbb{R}^N)\setminus\{0\}} \|u\|_{W^{1,\mathcal{A}}(\mathbb{R}^N)}/\|u\|_{L^{\mathcal{A}_*}(\mathbb{R}^N)}$ is realized (the problem is open even for Orlicz spaces)

Compactness result

$\exists$ a threshold $\beta^* > 0$ for compactness:

Compactness

If

$$0 < \beta < \beta^*(\mu, b_\infty)$$

then every $(PS)_\beta$ sequence for E has a subsequence that converges weakly to a nontrivial weak solution of (P).

Due to inhomogeneity, we do not have
a closed form formula for the threshold β^*

Threshold for compactness

$$\beta^*(\mu, b_\infty) := \inf_{(X,Y,Z,W)\in\mathcal{S}(\mu,b_\infty)} I(X,Y,Z,W)$$

- $I(X,Y,Z,W) := \frac{1}{p}\,X + \frac{1}{q}\,Y - \frac{1}{p^*}\,Z - \frac{1}{q^*}\,W$
- $S(\mu, b_\infty)$ is the set of points $(X,Y,Z,W) \in (\mathbb{R}_+)^4$ s.t.
  1. $I(X,Y,Z,W) > 0$
  2. $X + Y = Z + W$
  3. $Z \leq \dfrac{\mu}{S_p^{p^*/p}}\,X^{p^*/p}, \quad W \leq \dfrac{b_\infty}{(a_0\,S_q)^{q^*/q}}\,Y^{q^*/q}$

Asymptotics:

$$\beta^*(\mu, b_\infty) \geq \begin{cases} \frac{1}{N}\,\frac{S_p^{N/p}}{\mu^{(N-p)/p}} + o(1) & \text{as } b_\infty \to 0, \quad \text{if } \mu > 0, \\ \frac{1}{N}\,\frac{(a_0\,S_q)^{N/q}}{b_\infty^{(N-q)/q}} + o(1) & \text{as } \mu \to 0, \quad \text{if } b_\infty > 0, \end{cases}$$

Idea of proof for compactness

1. (u_j) a $(PS)_\beta$ sequence $\overset{\text{shape of } g}{\Rightarrow}$ bounded $\Rightarrow u_j \rightharpoonup u$
2. $u_j \rightharpoonup u$ and $E'(u_j) \to 0 \Rightarrow \nabla u_j \to \nabla u$ a.e.
3. u weak sol

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It remains to prove that $u \neq 0$, we argue by contradiction:

4. (u_j) a $(PS)_\beta$ $\overset{g \text{ subcritical}}{\Rightarrow}$
$$I\left(\int |\nabla u_j|^p, \int |\nabla u_j|^q, \int |u_j|^{p^*}, \int |u_j|^{q^*}\right) = \beta + o(1) > 0$$
$$\int |\nabla u_j|^p + \int |\nabla u_j|^q = \int |u_j|^{p^*} + \int |u_j|^{q^*} + o(1)$$
5. $\left(\int |\nabla u_j|^p, \int |\nabla u_j|^q, \int |u_j|^{p^*}, \int |u_j|^{q^*}\right) \to (X, Y, Z, W)$
6. $\int |u_j|^{p^*} \leq \frac{\mu}{S_p^{p^*/p}} \left(\int |\nabla u_j|^p\right)^{\frac{p^*}{p}}$, $\int |u_j|^{q^*} \leq \frac{b_\infty}{(a_0 S_q)^{q^*/q}} \left(\int |\nabla u_j|^q\right)^{\frac{q^*}{q}}$
7. 4. - 6. $\Rightarrow (X, Y, Z, W) \in \mathcal{S}(\mu, b_\infty)$ and $I(X, Y, Z, W) = \beta$
β^* is infimum $\Rightarrow$ $\beta \geq \beta^*$

Idea of proof for existence

$$\beta := \inf_{\gamma\in\Gamma} \max_{t\in[0,1]} E(\gamma(t))$$

asymptotics on β^* $\rightsquigarrow$ we prove $\beta < \begin{cases} \frac{1}{N}\,\frac{S_p^{N/p}}{\mu^{(N-p)/p}} & \text{for } b_\infty \text{ small, if } \mu > 0 \\ \frac{1}{N}\,\frac{(a_0\,S_q)^{N/q}}{b_\infty^{(N-q)/q}} & \text{for } \mu \text{ small, if } b_\infty > 0 \end{cases}$

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Focus on Thm 1

1. $v_\varepsilon := \frac{\psi(x)U_{\varepsilon,p}}{\|\psi(x)U_{\varepsilon,p}\|_{p^*}}$, ψ cutoff supported in $B_\rho(x_0)$, $tv_\varepsilon \in \Gamma$, $\beta \le \max_{t\ge0} E(tv_\varepsilon)$ ($t_{\max}$ not explicit due to inhomogeneity)
2. by contradiction let $\varepsilon_j \to 0$ be s.t. $\max_{t\ge0} E(tv_{\varepsilon_j}) \ge \frac{1}{N}\frac{S_p^{N/p}}{\mu^{(N-p)/p}}$
3. $\Rightarrow$ (t_j) (maximum points) bdd $\Rightarrow t_j \to t_0 > 0$
4. ingredients: t_j maximum points, estimates for $U_{\varepsilon,p} \rightsquigarrow t_0 = 0$ Absurd □

Pohožaev-type identity

$$(P)\ \begin{cases} -\mathrm{div}(|\nabla u|^{p-2}\nabla u + a(x)|\nabla u|^{q-2}\nabla u) = f(x,u) & \Omega \\ u = 0 & \partial\Omega \end{cases}$$

If $u \in W_0^{1,\mathcal{A}}(\Omega) \cap W^{2,\mathcal{A}}(\Omega)$ is a weak solution of (P), then we have the identity

$$\left(\frac{1}{p} - \frac{1}{q}\right) \int_\Omega |\nabla u|^p \, dx + \frac{1}{Nq} \int_\Omega |\nabla u|^q (\nabla a \cdot x) \, dx$$
$$+ \frac{1}{N} \int_{\partial\Omega} \left[\left(1 - \frac{1}{p}\right) |\partial_\nu u|^p + \left(1 - \frac{1}{q}\right) a(x)|\partial_\nu u|^q\right] (x \cdot \nu) \, d\sigma$$
$$= \int_\Omega \left[F(x,u) - \frac{1}{q^*} f(x,u)u\right] dx$$

Nonexistence results

Theorem

Let $\Omega \subset \mathbb{R}^N$ be a bounded and starshaped C^1-domain, $1 < p < q < N$, $q/p < 1 + 1/N$, and $0 \le a \in C^1(\Omega)$ be radial and radially nondecreasing. Let $f(x,u) = \mu|u|^{p^*-2}u + b(x)|u|^{q^*-2}u + c(x)|u|^{r-2}u$, with $p \le r < q^*$, $\mu \le 0$, and b, $c \in L^\infty(\Omega)$. Problem (P) does not admit a non-zero weak solution $u \in W_0^{1,\mathcal{A}}(\Omega) \cap W^{2,\mathcal{A}}(\Omega)$ in each of the following cases:

(I) $c(x) \le 0$ *a.e. in* Ω;

(II) $r = p$ *and* $0 < \|c\|_{L^\infty(\Omega)} < \lambda_1(p)\frac{N(q-p)}{N(q-p)+pq}$;

(II) Ω *is strictly starshaped,* $r = p$, *and* $\|c\|_{L^\infty(\Omega)} = \lambda_1(p)\frac{N(q-p)}{N(q-p)+pq}$.

Nonexistence results for small solutions

Theorem

Let $\Omega \subset \mathbb{R}^N$ *be a bounded and starshaped* C^1*-domain,* $1 < p < q < N$, $q/p < 1 + 1/N$, $0 \leq a \in C^1(\Omega)$ *be radial and radially nondecreasing, and let* $f(x,u) = c(x)|u|^{r-2}u + \mu|u|^{p^*-2}u + b(x)|u|^{q^*-2}u$ *with* $\mu \geq 0$, $0 \leq b \in L^\infty(\Omega)$ *satisfying* $\inf_{\mathrm{supp}(b)} a(x) > 0$, *and* $0 \leq c \in L^\infty(\Omega) \setminus \{0\}$. *Then there exists a positive constant* $\kappa = \kappa(\Omega, p, q, r, a, c, \mu, b_\infty)$ *such that* (P) *does not admit a weak solution* $u \in W_0^{1,\mathcal{A}}(\Omega) \cap W^{2,\mathcal{A}}(\Omega)$ *belonging to the ball* $\{u \in W^{1,\mathcal{A}}(\Omega) \, : \, \|u\| \leq \kappa\}$ *in each of the following cases:*

(I) $p < r \leq p^*$;

(II) $p^* < r < q^*$ *and* $c(x)$ *satisfies* $c(x) \leq Ca(x)^{s/q}$;

(III) $p^* < r < q^*$ *and* $c(x)$ *satisfies* $a_0' := \inf_{\mathrm{supp}(c)} a(x) > 0$.

Open problems

- Critical dimensions
- Is $L^{\mathcal{A}_*}(\Omega)$ the smallest Musielak-Orlicz space in which $W^{1,\mathcal{A}}(\Omega)$ is embedded?
  - * `Cianchi`, Indiana, '96: for Orlicz spaces
    - *optimal* embedding $W^{1,\Phi} \hookrightarrow L^{\Phi_N}$
    - L^{Φ_*} is not the smallest in the limiting Sobolev case
  - * `Ho, Winkert`, ArXiv '22: $t^{p^*} + a(x)^{q^*/q} t^{q^*}$ is optimal among $t^r + a(x)^\alpha t^s$
- Existence of minimizers for $W^{1,\Phi}(\mathbb{R}^N) \hookrightarrow L^{\Phi_N}(\mathbb{R}^N)$

Thank you for your attention!