Critical points of the Moser-Trudinger functional

Francesca De Marchis
Sapienza, Università di Roma

work in collaboration with A. Malchiodi, L. Martinazzi, P-D. Thizy

International doctoral summer school in conformal geometry and non-local operators

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and there exists $\bar{\Lambda} > 0$ such that for any $\Lambda \in (0, \bar{\Lambda}]$

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$\bar{\Lambda} = 4\pi$ is sharp in (MT), in particular

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Critical points of the Moser Trudinger functional

Moser Trudinger Functional

$$F(u) := \int_\Omega e^{u^2}\,dx$$

$$M_\Lambda := \{u \in H_0^1(\Omega) \,:\, \|u\|^2_{H_0^1(\Omega)} = \Lambda\}$$

Question: Given $\Lambda > 0$ does $F_{|M_\Lambda}$ admit positive critical points, i.e. positive solutions to

$$-\Delta u = 2\lambda u e^{u^2} \quad \text{in } \Omega, \qquad \Lambda = \int_\Omega |\nabla u|^2 dx = \int_\Omega 2\lambda u^2 e^{u^2}\,dx\ ?$$

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For $\boldsymbol{\Lambda \le 4\pi}$ there are maximizers ([Carleson-Chang, '86], [Struwe, '88], [Flucher, '92]).

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If $\Lambda \in (4\pi, 4\pi + \varepsilon)$, then $F_{|M_\Lambda}$ has at least two critical points: a local maximum and a saddle point.

$\Omega = B_1(0)$

[Malchiodi-Martinazzi, JEMS '14]

There exists $\Lambda_\# > 4\pi$ such that $F_{|M_\Lambda}$ has:

$\triangleright$ at least one positive critical point for $\Lambda \in (0, 4\pi] \cup \{\Lambda_\#\}$;

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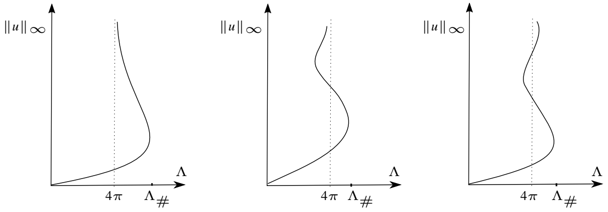

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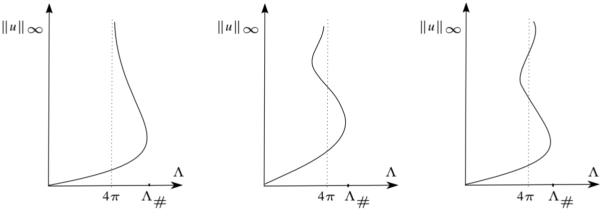

[Mancini-Martinazzi, Calc. Var. '17]

Blow-up only occurs from the right, namely if $\mu := \|u_\mu\|_\infty = u_\mu(0) \to +\infty$

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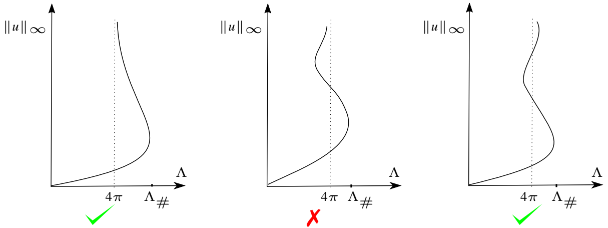

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Ω not simply connected

[del Pino-Musso-Ruf, J. Funct. Anal. '10]

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Remarks

- ▷ at $4\pi k$, $k \in \mathbb{N}$, blow-up can occur
- ▷ topology seems to play a role

Blow-up and compactness of positive critical points

[**Druet**, Duke '06]

Let $(u_j)_j \in H_0^1(\Omega)$ be a sequence of positive solutions to

$$-\Delta u_j = 2\lambda_j u_j e^{u_j^2}; \qquad \Lambda_j = \int_\Omega |\nabla u_j|^2 dx \to \Lambda$$

Then (up to a subsequence) either

(i) $u_j \to u_\infty$ in $C^\infty(\bar{\Omega})$ or

(ii) $u_j \to u_\infty$ in $C^1_{loc}(\Omega \setminus \{x_1, \dots, x_N\})$

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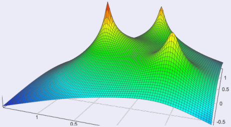

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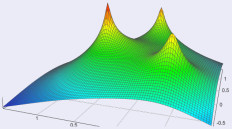

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[Druet-Thizy, JEMS '20] ([Malchiodi-Martinazzi, JEMS '14] for the radial case)

In case (ii): $u_\infty = 0$ and $k = N$.

Consequence: compactness holds provided $\Lambda \notin 4\pi\mathbb{N}$.

A degree formula for positive solutions

[Malchiodi-Martinazzi-Thizy, preprint '22]

Let $\Lambda \in (4\pi k, 4\pi(k+1))$. Then

$$d_\Lambda = \frac{(1-\chi(\Omega))(2-\chi(\Omega))\dots(k-\chi(\Omega))}{k!}$$

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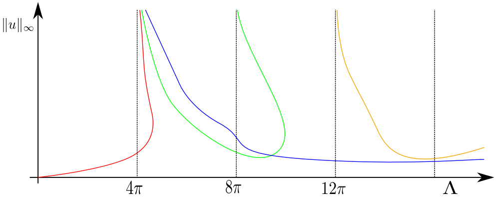

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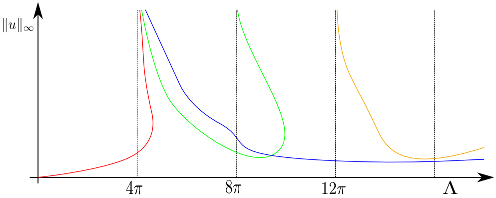

Same degree formula as for the equation $-\Delta u = 2\Lambda \frac{e^u}{\int_\Omega e^u \, dx}$ [Chen-Lin, '03].

Moser-Trudinger functional on surfaces

[**Moser**, Indiana ’71], [**Fontana**, Comment. Math. Helv. ’93], [**Y. Li**, J. PDE ’01]

(Σ, g_0) compact Riemannian surface. For any $\Lambda \leq 4\pi$

$$\sup_{\|u\|^2_{H^1(\Sigma)}=\Lambda} \int_\Sigma e^{u^2} dv_{g_0} = C_{\Sigma,\Lambda} \qquad \text{and} \qquad \sup_{\substack{\|\nabla u\|^2_{L^2(\Sigma)}=\Lambda \\ \int u\, dv_{g_0}=0}} \int_\Sigma e^{u^2} dv_{g_0} = C_{\Sigma,\Lambda},$$

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[DM-Malchiodi-Martinazzi-Thizy, Inventiones '22]

For every $\Lambda > 0$ and for any K smooth positive function the functional

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Introducing the norm

$$\|u\|_h^2 = \int_\Sigma (|\nabla u|^2 + hu^2)\, dv_g$$

where $h := \frac{1}{K}$ and $g := Kg_0$

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A known compactness result

[Y. Yang, Calc. Var. '15]

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How to get existence?

Ingredients

▷ New compactness results (the idea is to prove that $u_\infty = 0$ and $k = N$, but an extra parameter is introduced)

▷ A min-max scheme (in the spirit of [Djadli-Malchiodi, '08])

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▷ A min-max scheme (in the spirit of [Djadli-Malchiodi, '08])

Problem

Min-max schemes produce Palais-Smale sequences, but the Palais-Smale condition it is known to hold for $\Lambda < 4\pi$ [Adimurthi, '90], [Adimurthi-Struwe, '00] and it fails for $\Lambda > 4\pi$ [Costa-Tintarev, '14].

How to get existence?

For $p \in (1,2)$, $\Lambda > 0$, we introduce the functional

$$J_{p,\Lambda}(u) := \frac{2-p}{2}\left(\frac{p\|u\|_h^2}{2\Lambda}\right)^{\frac{p}{2-p}} - \log \int_\Sigma (e^{u_+^p} - 1)\, dv_g$$

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Any non trivial critical point of $J_{p,\Lambda}$ is a positive solution to

$$-\Delta_g u + hu = p\lambda u^{p-1}e^{u^p} \qquad (*)_p$$

where $\lambda > 0$ is given by the relation

$$\Lambda = \frac{\lambda p^2}{2}\left(\int_\Sigma (e^{u^p} - 1)\,dv_g\right)^{\frac{2-p}{p}}\left(\int_\Sigma u^p e^{u^p}\,dv_g\right)^{\frac{2(p-1)}{p}} \qquad (**)_p$$

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Strategy

Solve for $p < 2$ (via a min-max method) and prove compactness for $p \nearrow 2$.
Indeed if $(u_p)_p$ are positive solutions to $(*)_p$ and $(**)_p$ converging strongly, as $p \nearrow 2$, to u, then u is a positive solution to $(*)_2$ and it satisfies

$$\Lambda = 2\lambda \int_\Sigma u^2 e^{u^2}\, dv_g.$$

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Let $p \in (1,2)$ and $\Lambda > 0$

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Conversely:

- if $\Lambda > 4\pi$: $J_{p,\Lambda}$ is unbounded from below.

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Some improved inequalities allow to perform an analysis of the topology of low sublevels of $J_{p,\Lambda}$ and to prove that:

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Using $\boldsymbol{p < 2}$, via the Struwe's monotonicity trick, we are able to prove for a.e. Λ the existence of a bounded Palais-Smale sequence and thus to find **a positive critical point of $J_{p,\Lambda}$ for a.e. Λ**, namely a positive solution to

$$-\Delta u + hu = p\lambda u^{p-1}e^{u^p}, \qquad \lambda = \lambda(p, \Lambda, u).$$

Existence for $\Lambda \notin 4\pi\mathbb{N}$

Compactness for $p < 2$ fixed

Let $\Lambda_j \to \Lambda$ and let $(u_j)_j$ be positive solutions to

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and so of **a positive critical point of $J_{p,\Lambda}$ for every $\Lambda \notin 4\pi\mathbb{N}$ and for every $p \in (1, 2)$**.

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Compactness as $p \nearrow 2$

Let u_j be critical points of $J_{p_j,\Lambda}$ with $p_j \nearrow 2$.
If $\Lambda \notin 4\pi\mathbb{N}$, then (no blow-up so) $u_j \to u_\infty$, $u_\infty > 0$, solution to

$$-\Delta_g u_\infty + hu_\infty = 2\lambda u_\infty e^{u_\infty^2}, \qquad \|u_\infty\|_h^2 = \Lambda,$$

namely u_∞ is **a positive critical point of the M-T functional $\hat{F}$ on $\hat{M}_\Lambda = \{\|u\|_h^2 = \Lambda\}$**.

Existence for any $\Lambda > 0$

Let $\Lambda = 4\pi N$ and let $(u_j)_j$ be positive solutions to

$$-\Delta u_j + hu_j = 2\lambda_j u_j e^{u_j^2}, \qquad \|u_j\|_h^2 = \Lambda_j \to \Lambda,$$

blowing up. Then u_j blows up at N points $x_1, \dots, x_N$ and

$$\|u_j\|_h^2 \geq 4\pi N + \sum_{i=1}^{N} \frac{4\pi + o(1)}{\mu_{i,j}^4} > 4\pi N \qquad \mu_{i,j} = \max_{B_\delta(x_i)} u_j.$$

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For $\Lambda_j \nearrow 4\pi N$ no blow up , hence $u_j \to u$ positive solution for $\boldsymbol{\Lambda = 4\pi N}$.
Thus u **is a positive critical point of the M-T functional** $\hat{F}$ **on** $\hat{M}_\Lambda = \{\|u\|_h^2 = \Lambda\}$**.**

A remark on the case $p = 1$

Analogously, let $\Lambda \notin 4\pi\mathbb{N}$ and let u_j be positive critical points to J_{p,Λ_j}, then as $p_j \searrow 1$ $u_j \longrightarrow u$ positive critical point of $J_{1,\Lambda}$.

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Moreover we can exclude blow up from the left hand side for any $p \in (1, 2]$ but not for $p = 1$. Indeed if u_j is a sequence of positive blowing-up critical points of J_{p,Λ_j} (when p is fixed and $\Lambda_j \longrightarrow \Lambda = 4\pi N$) then

$$\Lambda_j \geq 4\pi N + \sum_{i=1}^{N} \frac{16\pi(p-1) + o(1)}{p^2 \mu_{i,j}^{2p}} \overset{\text{if } p>1}{>} 4\pi N = \Lambda, \qquad \mu_{i,j} = \max_{B_\delta(x_i)} u_j.$$

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For the widely studied functional

$$I_\Lambda(u) := \frac{1}{2}\int_\Sigma |\nabla u|^2 \, dv_{g_0} + 2\Lambda ⨍_\Sigma u \, dv_{g_0} - 2\Lambda \log \int_\Sigma K e^u \, dv_{g_0}$$

the side condition (which depends on the Gauss curvature of the surface) was determined in [Chen-Lin, '02].

Open problems

▷ What about multiplicity?
Is there generic multiplicity (in h and g) of solutions for any fixed $p \in [1, 2]$ and $\Lambda \notin 4\pi\mathbb{N}$, as proved for I_Λ in [DM, '10]?

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• Existence of 2 peaks solutions $\Lambda_j \longrightarrow 8\pi^+$ [Figueroa-Musso, '09]

• Quantization seems to be very hard, indeed there exist blowing-up sequences u_ε of critical points of

$$F_{h_\varepsilon}(u) = \int_\Omega h_\varepsilon e^{u^2}\, dx, \text{ where } h_\varepsilon \to 1 \text{ in } C^2(\Omega) \text{ as } \varepsilon \to 0,$$

such that $\|\nabla u_\varepsilon\|^2_{L^2} \to \Lambda$, $u_\varepsilon \rightharpoonup u_0 \in H^1_0(\Omega)$, where u_0 is a sign changing critical point of $F(u) = \int_\Omega e^{u^2} dx$ [Martinazzi-Thizy-Vétois, '22]

Muchas gracias!


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Blow-up analysis for positive critical points

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then v_j satisfies

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It is possible to prove that $v_j \to V$ in $C^2_{loc}(\mathbb{R}^2)$, where V solves

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Solutions were classified in [Chen-Li, '91]: $V(x) = -\log(1 + |x|^2)$ and $\int_{\mathbb{R}^2} 4e^{2V} dx = 4\pi$.

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$$\int_{B_{Rr_j}(x_j)} 2\lambda_j u_j^2 e^{u_j^2}\, dx = \int_{B_R} 4\left(1 + \frac{v_j}{\mu_j^2}\right)^2 e^{2v_j + \frac{v_j}{\mu_j^2}} dx \overset{j\to\infty}{\longrightarrow} \int_{B_R} 4e^{2V} dx \overset{R\to\infty}{\longrightarrow} 4\pi.$$

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Hard part of the analysis: what happens at larger scales?

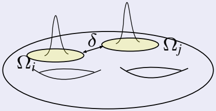
How to get existence?

Proposition (after [Moser,'73], [Aubin,'76], [Chen-Li,'91])

Let $\Omega_1, \dots, \Omega_{k+1} \subset \Sigma$ satisfy $\mathrm{dist}(\Omega_i, \Omega_j) \geq \delta > 0$ and let $\gamma > 0$.

Suppose that

$$\frac{\int_{\Omega_i} (e^{|u|^p} - 1) dv_g}{\int_\Sigma (e^{|u|^p} - 1) dv_g} \geq \gamma \qquad \forall i \in \{1, \dots, k+1\}.$$



Then for any $\varepsilon > 0$ there exists $C = C(\varepsilon, \delta, \gamma) > 0$ (independent of u) such that

$$\log \int_\Sigma (e^{|u|^p} - 1) dv_g \leq \frac{2-p}{p} \left(\frac{p \|u\|_h^2}{8\pi(k+1) - \varepsilon} \right)^{\frac{p}{2-p}} + C$$

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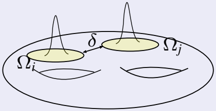

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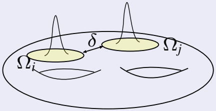

Then for any $\varepsilon > 0$ there exists $C = C(\varepsilon, \delta, \gamma) > 0$ (independent of u) such that

$$\log \int_\Sigma (e^{|u|^p} - 1) dv_g \leq \frac{2-p}{p} \left(\frac{p \|u\|_h^2}{8\pi(k+1) - \varepsilon} \right)^{\frac{p}{2-p}} + C$$

How to get existence?

Proposition (after [Moser,'73], [Aubin,'76], [Chen-Li,'91])

Let $\Omega_1, \ldots, \Omega_{k+1} \subset \Sigma$ satisfy $\mathrm{dist}(\Omega_i, \Omega_j) \geq \delta > 0$ and let $\gamma > 0$.

Suppose that

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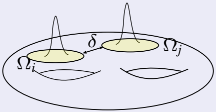

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Corollary

If $\Lambda \in (4\pi k, 4\pi(k+1))$ and $\mu_p(u)$ spreads in $(k+1)$ regions, then

$$J_{p,\Lambda}(u) = \frac{2-p}{2} \left(\frac{p\|u\|_h}{2\Lambda} \right)^{\frac{p}{2-p}} - \log \int_\Sigma (e^{u_+^p} - 1)\, dv_g > -L.$$

Step 2. Topology of low sublevels of $J_{p,\Lambda}$

$$\text{If } \Lambda \in (4\pi k, 4\pi(k+1)) \text{ and } u \in \{J_{p,\Lambda} \leq -L\}$$

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$$\frac{(e^{|u|^p} - 1)dv_g}{\int_\Sigma (e^{|u|^p} - 1)dv_g} \quad \text{concentrates around at most } k \text{ points}$$

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$$\Psi : \{J_{p,\Lambda} \leq -L\} \longrightarrow \Sigma_k := \{\sigma = \sum_{i=1}^k t_i \delta_{x_i} : \sum_{i=1}^k t_i = 1,\ t_i \geq 0,\ x_i \in \Sigma\}$$

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Viceversa we can map Σ_k into arbitrarily low sublevels of $J_{p,\Lambda}$.

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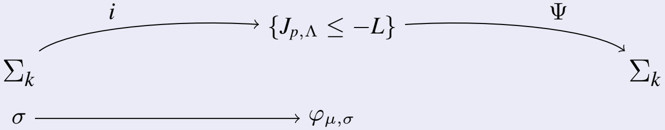

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it is continuous and it is homotopically equivalent to the identity on Σ_k (for μ large enough). Hence, being Σ_k non contractible, $i(\Sigma_k)$ is non contractible in $\{J_{p,\Lambda} \leq -L\}$.

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$$\mathcal{C}_k := \Sigma_k \times [0,1]/\sim \qquad \text{cone over } \Sigma_k$$
$$\mathcal{A}_k := \{\gamma \in C^0(\mathcal{C}_k; H^1(\Sigma)) \,:\, \gamma(\sigma,1) = \varphi_{\mu,\sigma}\}$$

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$$\max_{(\sigma,t)\in\mathcal{C}_k} J_{p,\Lambda}(\gamma(\sigma,t)) > -L \qquad \text{for any } \gamma \in \mathcal{A}_k$$

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Back to the same problem!

Even for the approximate problem it is not known whether the Palais-Smale condition holds or not. But in the subcritical case, $p < 2$, it is standard to prove convergence of bounded (in H^1-norm) Palais-Smale sequences.

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