Weierstrass Institute for
Applied Analysis and Stochastics

Leibniz
Association

Two examples of gradient flows in continuity equation format

Gradient Flows Face to Face
Granada, 18 September 2025

Georg Heinze
based on joint works with
Alexander Mielke, Jan-Frederik Pietschmann, André Schlichting, Artur Stephan

Overview

- Gradient systems in continuity equation format
  - Definition and heuristics
  - EDP convergence with embedding
- Example 1: Gradient flows on metric graphs with reservoirs
  - Model and gradient flow formulation
  - Discussion of analytic results
- Example 2: Reaction-diffusion systems
  - Model and gradient flow formulation
  - Discussion of chain rule inequality
- Summary and outlook

Gradient systems in CE format

Gradient systems in continuity equation format

Peletier-Schlichting '23: Gradient system $(X, Y, \nabla, \mathcal{E}, \mathcal{R}^*)$ in CE format:

- Base spaces X, Y and gradient operator $\nabla : C_c^\infty(X) \to C_c^\infty(Y)$
- Energy $\mathcal{E} : \mathcal{M}_+(X) \to \mathbb{R}$
- Dual dissipation potential $\mathcal{R}^* : \mathcal{M}_+(X) \times C_c^\infty(Y) \to [0, \infty]$; convex in 2nd argument

▶ Gradient flow (GF) equation induced by $(X, Y, \nabla, \mathcal{E}, \mathcal{R}^*)$:

$$\partial_t \mu = -\operatorname{div} \mathrm{D}_2 \mathcal{R}^*(\mu, -\nabla \mathrm{D}\mathcal{E}(\mu))$$

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- Example: The choices

$$X = Y = \mathbb{R}$$

$$\nabla = \partial_x$$

$$\mathcal{E}(\rho) = \mathcal{H}(\rho|\pi) := \int_{\mathbb{R}} \rho \log(\rho/\pi) - \rho + \pi \, \mathrm{d}x$$

$$\mathcal{R}^*(\rho, \xi) = \frac{1}{2} \int_{\mathbb{R}} |\xi|^2 \rho \, \mathrm{d}x$$

give us the Fokker-Planck equation

$$\partial_t \rho = \partial_x [\rho \partial_x \log(\rho/\pi)]$$

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$$\mathcal{D}(\mu, j) := \int_0^T \mathcal{R}(\mu, j) + \mathcal{R}^*(\mu, -\nabla \mathrm{D}\mathcal{E}(\mu)) \,\mathrm{d}t$$

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$$\mathcal{L}(\mu, j) := \mathcal{E}(\mu(T)) - \mathcal{E}(\mu(0)) + \mathcal{D}(\mu, j)$$

Energy dissipation principle (EDP)

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Heuristics:
$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{E}(\mu) &= \langle \mathrm{D}\mathcal{E}(\mu), \partial_t \mu \rangle = \langle \mathrm{D}\mathcal{E}(\mu), -\operatorname{div} j \rangle = -\langle -\nabla \mathrm{D}\mathcal{E}(\mu), j \rangle \\ &\geq -\mathcal{R}(\mu, j) - \mathcal{R}^*(\mu, -\nabla \mathrm{D}\mathcal{E}(\mu))\end{aligned}$$

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$$\mathcal{L}(\mu, j) \geq 0$$
* Energy dissipation principle (EDP):
$$\mathcal{L}(\mu, j) = 0 \iff \begin{cases} (\mu, j) \in \mathrm{CE} \\ j = \mathrm{D}_2\mathcal{R}^*(\mu, -\nabla \mathrm{D}\mathcal{E}(\mu)) \end{cases}$$
i.e. $\mathcal{L}(\mu, j) = 0$ if and only if μ is solution of the GF equation

Convergence of gradient systems

- Next goal: Study limits $(X_\varepsilon, Y_\varepsilon, \nabla_\varepsilon, \mathcal{E}_\varepsilon, \mathcal{R}^*_\varepsilon) \to (X_0, Y_0, \nabla_0, \mathcal{E}_0, \mathcal{R}^*_0)$

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- Idea: Embed curves satisfying continuity equation in shared space
  (cf. Disser-Liero '15, Hraivoronska-Tse '23, Hraivoronska-Schlichting-Tse '24, Esposito-H-Schlichting '24, H-Mielke-Stephan '25)

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Main steps:

* Compactness: $\exists$ embeddings $\Pi_\varepsilon : \mathrm{CE}_\varepsilon \to \mathrm{CE}_0$ s.t. for all $(\mu_\varepsilon, j_\varepsilon)_{\varepsilon>0}$ satisfying

$$(\mu_\varepsilon, j_\varepsilon) \in \mathrm{CE}_\varepsilon, \qquad \sup_{\varepsilon>0} \operatorname*{ess\,sup}_{t\in[0,T]} \mathcal{E}_\varepsilon(\mu_\varepsilon(t)) < \infty, \qquad \sup_{\varepsilon>0} \mathcal{D}_\varepsilon(\mu_\varepsilon, j_\varepsilon) < \infty$$

the family $(\Pi_\varepsilon(\mu_\varepsilon, j_\varepsilon))_{\varepsilon>0}$ is precompact in CE_0

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$$\liminf_{\varepsilon\to 0} \mathscr{L}_\varepsilon(\mu_\varepsilon, j_\varepsilon) \geq \mathscr{L}_0(\mu_0, j_0)$$

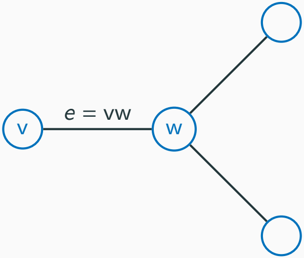
- ▶ Immediate consequence: Solutions converge to solutions, i.e., if $\mathscr{L}_\varepsilon(\mu_\varepsilon, j_\varepsilon) = 0$ and $\mathscr{L}_0 \geq 0$, then $\mathscr{L}_0(\mu_0, j_0) = 0$

Metric graphs with reservoirs
(GH, J-F Pietschmann, A Schlichting)

Setup

Undirected irreducible simple finite graph:

- Finite set of vertices V
- Set of edges $E \subset V \times V$



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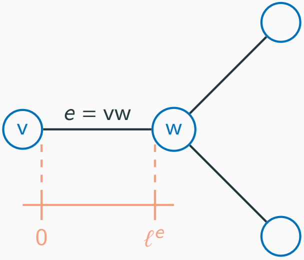
Metric Graph:

- Orientation: $e = vw \in E$ has starting vertex v and end vertex w
- Associate to each $e \in E$ an intervall $[0, \ell^e]$



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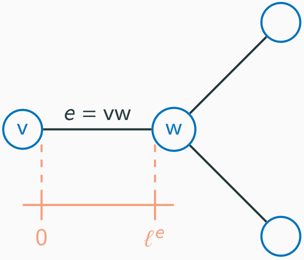

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- Associate to each $e \in \mathsf{E}$ an intervall $[0, \ell^e]$

Dynamics:

- Drift-diffusion along metric edges coupled with reservoirs on vertices

$$\partial_t \rho^e = d^e \partial_x \left(\partial_x \rho^e + \rho^e \partial_x V^e \right)$$

$$-d^e \left[\partial_x \rho^e + \rho^e \partial_x V^e \right]_{\mathsf{v}} \mathsf{n}^e_{\mathsf{v}} = r(e, \mathsf{v}) \rho^e|_{\mathsf{v}} - r(\mathsf{v}, e) \gamma_{\mathsf{v}}$$

$$\partial_t \gamma_{\mathsf{v}} = \sum_{e \in \mathsf{E}(\mathsf{v})} \left(r(e, \mathsf{v}) \rho^e|_{\mathsf{v}} - r(\mathsf{v}, e) \gamma_{\mathsf{v}} \right)$$

for potentials $V^e \in \mathrm{Lip}$, diffusivity constants $d^e > 0$, and jump rates $r > 0$

Some selective literature

- Freidlin-Wentzell '93: Diffusion on metric intervall obtained as vanishing-diameter limit of diffusion narrow tube using probabilistic approach
- Erbar-Forkert-Maas-Mugnolo '22: McKean-Vlasov-type equations on metric graphs with Kirchhoff-type conditions at vertices identified as GFs w.r.t. suitable dynamic Wasserstein distance
- Fazeny-Burger-Pietschmann '25: "Overdamped isothermal model 3" for gas transport in networks formally understood as dynamic 3-Wasserstein GF
- Burger-Humpert-Pietschmann '23: Dynamic Wasserstein-type distance defined on metric graphs with mass reservoirs at vertices exchanging mass with edges
- Mugnolo-Romanelli '07: Asymptotic behaviour and regularity of solutions studied for similar model to ours using semigroup techniques

Detailed balance

$$\partial_t \rho^e = d^e \partial_x \left(\partial_x \rho^e + \rho^e \partial_x V^e \right)$$

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- Introduce edge reference measures $\mathrm{d}\pi^e \coloneqq \exp(-V^e)\,\mathrm{d}x$ (abuse notation to also write π^e for its density)
- Assume detailed balance condition: $\exists\, \omega = (\omega_{\mathsf{v}})_{\mathsf{v} \in \mathsf{V}} \in \mathcal{M}_{\geq 0}(\mathsf{V})$ s.t.

$$r(e, \mathsf{v}) \pi^e|_{\mathsf{v}} = r(\mathsf{v}, e) \omega_{\mathsf{v}} \qquad \forall e \in \mathsf{E}, \mathsf{v} \in \mathsf{V}$$

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- Denote $\kappa_{\mathrm{v}}^e \coloneqq r(e,\mathrm{v}) \sqrt{\frac{\pi^e|_{\mathrm{v}}}{\omega_{\mathrm{v}}}} = r(\mathrm{v},e) \sqrt{\frac{\omega_{\mathrm{v}}}{\pi^e|_{\mathrm{v}}}}$

▶ Equations rewritten:

$$\partial_t \rho^e = d^e \partial_x \left(\rho^e \partial_x \log \frac{\rho^e}{\pi^e} \right)$$

$$-d^e \left[\rho^e \partial_x \log \frac{\rho^e}{\pi^e} \right]_{\mathrm{v}} \mathrm{n}_{\mathrm{v}}^e = \kappa_{\mathrm{v}}^e \sqrt{\pi^e|_{\mathrm{v}} \omega_{\mathrm{v}}} \left(\frac{\rho^e}{\pi^e}\bigg|_{\mathrm{v}} - \frac{\gamma_{\mathrm{v}}}{\omega_{\mathrm{v}}} \right)$$

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Contents of the paper

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$$-d^e \left[\rho^e \partial_x \log \frac{\rho^e}{\pi^e} \right]_v n_v^e = \mathscr{k}_v^e \sqrt{\pi^e|_v \omega_v} \left(\frac{\rho^e}{\pi^e}\bigg|_v - \frac{\gamma_v}{\omega_v} \right)$$

$$\partial_t \gamma_v = \sum_{e \in E(v)} \mathscr{k}_v^e \sqrt{\pi^e|_v \omega_v} \left(\frac{\rho^e}{\pi^e}\bigg|_v - \frac{\gamma_v}{\omega_v} \right)$$

Analytic results:

* Well-posedness
* Kirchhoff-type limit: Vanishing reservoirs without cutting dynamics (cf. Erbar-Forkert-Maas-Mugnolo '22)
* Fast-diffusion limit: Acceleration of edge dynamics (combinatorial graph)

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Analytic results:

* Well-posedness
* Kirchhoff-type limit: Vanishing reservoirs without cutting dynamics (cf. Erbar-Forkert-Maas-Mugnolo '22)
* Fast-diffusion limit: Acceleration of edge dynamics (combinatorial graph)

Strategy:

- Write as gradient flow in CE format
- EDP convergence with embedding

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Numerical simulations:

- Based on finite volume discretization
- Comparison to analytic results
- Highlighting further aspects beyond analysis

Gradient flow formulation

Gradient flow formulation – drift-diffusion terms

Drift-diffusion terms:

$$d^e \partial_x \left(\rho^e \partial_x \log \frac{\rho^e}{\pi^e} \right)$$

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$$\mathcal{E}_{\mathsf{E}}(\rho) \coloneqq \sum_{e \in \mathsf{E}} \mathcal{H}(\rho^e | \pi^e) \coloneqq \sum_{e \in \mathsf{E}} \int_0^{\ell^e} \rho^e \log(\rho^e / \pi^e) - \rho^e + \pi^e \, \mathrm{d}x$$

with variational derivatives $\mathrm{D}\mathcal{E}_{\mathsf{E}}(\rho)|_e = \log(\rho^e/\pi^e)$

Gradient and divergence: For $\varphi : \mathsf{L} \to \mathbb{R}$ (L disjoint union of metric edges)

$$\nabla \varphi|_e = \partial_x \varphi^e \qquad\qquad \operatorname{div} j|_e = \partial_x j^e$$

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$$\mathcal{R}^*_{\mathsf{E}}(\rho, \xi) \coloneqq \sum_{e \in \mathsf{E}} \frac{1}{2} d^e \int_0^{\ell^e} |\xi_e|^2 \, \mathrm{d}\rho^e$$

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- Drift-diffusion terms rewritten:

$$-\operatorname{div} \mathrm{D}_\xi \mathcal{R}^*_\mathsf{E}(\rho, -\nabla \mathrm{D}\mathcal{E}_\mathsf{E}(\rho))|_e$$

Gradient flow formulation – jump terms

Jump terms:

$$\sum_{e\in \mathsf{E}(\mathsf{v})} \mathscr{k}_{\mathsf{v}}^{e} \sqrt{\pi^{e}|_{\mathsf{v}}\omega_{\mathsf{v}}} \left(\frac{\rho^{e}}{\pi^{e}}\bigg|_{\mathsf{v}} - \frac{\gamma_{\mathsf{v}}}{\omega_{\mathsf{v}}} \right)$$

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$$\sum_{e\in\mathsf{E}}\int_0^{\ell^e} \varphi^e \mathrm{div}\, j^e \mathrm{d}x + \sum_{\mathsf{v}\in\mathsf{V}} \phi_{\mathsf{v}}\overline{\mathrm{div}}\,\bar{\jmath}_{\mathsf{v}} = \sum_{e\in\mathsf{E}}\int_0^{\ell^e} \varphi^e \partial_x j^e \mathrm{d}x + \sum_{\mathsf{v}\in\mathsf{V}} \phi_{\mathsf{v}}\left(-\sum_{e\in\mathsf{E}(\mathsf{v})} \bar{\jmath}_{\mathsf{v}}^{e}\right)$$

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&= -\sum_{e\in\mathsf{E}}\int_0^{\ell^e} \partial_x\varphi^e j^e \mathrm{d}x + \sum_{e\in\mathsf{E}}\sum_{\mathsf{v}\in\mathsf{V}(e)} \varphi^e|_{\mathsf{v}} j^e|_{\mathsf{v}} n_{\mathsf{v}}^e - \sum_{\mathsf{v}\in\mathsf{V}}\sum_{e\in\mathsf{E}(\mathsf{v})} \phi_{\mathsf{v}}\bar{\jmath}_{\mathsf{v}}^{e}\\
&= -\sum_{e\in\mathsf{E}}\int_0^{\ell^e} \partial_x\varphi^e j^e \mathrm{d}x - \sum_{\mathsf{v}\in\mathsf{V}}\sum_{e\in\mathsf{E}(\mathsf{v})} \left(\phi_{\mathsf{v}}\bar{\jmath}_{\mathsf{v}}^{e} - \varphi^e|_{\mathsf{v}} j^e|_{\mathsf{v}} n_{\mathsf{v}}^e\right)
\end{aligned}$$

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Problem: Missing logarithms to rewrite in terms of energy

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Gradient and divergence: For $\Phi = (\phi, \varphi) : \mathsf{V} \times \mathsf{L} \to \mathbb{R}$

$$\overline{\nabla}\Phi|_{\mathsf{v},e} = \phi_{\mathsf{v}} - \varphi^e|_{\mathsf{v}} \qquad\qquad \overline{\mathrm{div}}\,\bar{\jmath}|_{\mathsf{v}} = -\sum_{e\in\mathsf{E}(\mathsf{v})} \bar{\jmath}^e_{\mathsf{v}}$$

Nonlinear transformation: $\mathsf{C}^*(r) \coloneqq 4(\cosh(r/2) - 1)$ has derivative $(\mathsf{C}^*)'(r) = 2\sinh(r/2)$ which satisfies

$$r - s = \sqrt{rs}(\mathsf{C}^*)'(\log r - \log s)$$

Dual dissipation potential:

$$\mathcal{R}^*_{\mathsf{V},\mathsf{E}}(\mu, \zeta) \coloneqq \sum_{\mathsf{v}\in\mathsf{V}} \sum_{e\in\mathsf{E}(\mathsf{v})} \mathscr{k}^e_{\mathsf{v}} \sqrt{\rho^e|_{\mathsf{v}}\gamma_{\mathsf{v}}}\,\mathsf{C}^*(\zeta^e_{\mathsf{v}})$$

- Jump terms rewritten:

$$-\overline{\mathrm{div}}\mathrm{D}_\zeta \mathcal{R}^*_{\mathsf{V},\mathsf{E}}(\mu, -\overline{\nabla}\mathrm{D}\mathcal{E}(\mu))|_{\mathsf{v}}$$

Gradient flow formulation – full system

System in gradient flow form:

$$\partial_t \rho^e = -\operatorname{div} \mathrm{D}_\xi \mathcal{R}^*_{\mathsf{E}}(\rho, -\nabla \mathrm{D}\mathcal{E}_{\mathsf{E}}(\rho))|_e$$

$$\mathrm{D}_\xi \mathcal{R}^*_{\mathsf{E}}(\rho, -\nabla \mathrm{D}\mathcal{E}_{\mathsf{E}}(\rho))|_{e,\mathsf{v}} \mathsf{n}^e_{\mathsf{v}} = \mathrm{D}_\zeta \mathcal{R}^*_{\mathsf{V},\mathsf{E}}(\mu, -\overline{\nabla} \mathrm{D}\mathcal{E}(\mu))|_{e,\mathsf{v}}$$

$$\partial_t \gamma_{\mathsf{v}} = -\overline{\operatorname{div}}\, \mathrm{D}_\zeta \mathcal{R}^*_{\mathsf{V},\mathsf{E}}(\mu, -\overline{\nabla} \mathrm{D}\mathcal{E}(\mu))|_{\mathsf{v}}$$

Free energy (denoting $\mu \coloneqq (\gamma, \rho) \in \mathcal{P}(\mathsf{V} \times \mathsf{L})$):

$$\mathcal{E}(\mu) \coloneqq \sum_{\mathsf{v}\in\mathsf{V}} \gamma_{\mathsf{v}} \log\left(\frac{\gamma_{\mathsf{v}}}{\omega_{\mathsf{v}}}\right) - \gamma_{\mathsf{v}} + \omega_{\mathsf{v}} + \sum_{e\in\mathsf{E}} \int_0^{\ell^e} \rho^e \log\left(\frac{\rho^e}{\pi^e}\right) - \rho^e + \pi^e \,\mathrm{d}x$$

Dual dissipation potential:

$$\mathcal{R}^*_{\mathsf{E}}(\rho, \xi) \coloneqq \sum_{e\in\mathsf{E}} \frac{1}{2} d^e \int_0^{\ell^e} |\xi_e|^2 \,\mathrm{d}\rho^e$$

$$\mathcal{R}^*_{\mathsf{V},\mathsf{E}}(\mu, \zeta) \coloneqq \sum_{\mathsf{v}\in\mathsf{V}} \sum_{e\in\mathsf{E}(\mathsf{v})} 4\Bbbk^e_{\mathsf{v}} \sqrt{\rho^e|_{\mathsf{v}} \gamma_{\mathsf{v}}} (\cosh(\zeta^e_{\mathsf{v}}/2) - 1)$$

Gradient and divergence:

$$\nabla \varphi|_e = \partial_x \varphi^e \qquad \operatorname{div} j|_e = \partial_x j^e$$

$$\overline{\nabla} \Phi|_{\mathsf{v},e} = \phi_{\mathsf{v}} - \varphi^e|_{\mathsf{v}} \qquad \overline{\operatorname{div}}\, \bar{\jmath}|_{\mathsf{v}} = -\sum_{e\in\mathsf{E}(\mathsf{v})} \bar{\jmath}^e_{\mathsf{v}}$$

Gradient system

- Free energy functional (denoting $\mu \coloneqq (\gamma, \rho) \in \mathcal{P}(\mathsf{V} \times \mathsf{L})$):

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- Dissipation functional (denoting $\mathrm{j} \coloneqq (\bar{\jmath}, j) \in \mathcal{M}((\mathsf{V}\times\mathsf{E})\times\mathsf{L})$):

$$\begin{aligned}\mathcal{D}(\mu,\mathrm{j}) = &\sum_{e\in\mathsf{E}} \int_0^T \frac{1}{2d^e}\int_0^{\ell^e} \frac{|j^e|^2}{\rho^e}\,\mathrm{d}x\,\mathrm{d}t \\ &+ \sum_{e\in\mathsf{E}} \int_0^T \int_0^{\ell^e} 2d^e \left|\partial_x \sqrt{\frac{\rho^e}{\pi^e}}\right|^2 \pi^e\,\mathrm{d}x\,\mathrm{d}t \\ &+ \sum_{\mathsf{v}\in\mathsf{V}} \sum_{e\in\mathsf{E}(\mathsf{v})} \int_0^T \mathscr{k}_{\mathsf{v}}^e \sqrt{\rho^e|_{\mathsf{v}}, \gamma_{\mathsf{v}}}\,\mathsf{C}\big(\bar{\jmath}_{\mathsf{v}}^e / \mathscr{k}_{\mathsf{v}}^e \sqrt{\rho^e|_{\mathsf{v}}, \gamma_{\mathsf{v}}}\big)\,\mathrm{d}t \\ &+ \sum_{\mathsf{v}\in\mathsf{V}} \sum_{e\in\mathsf{E}(\mathsf{v})} \int_0^T 2\mathscr{k}_{\mathsf{v}}^e \sqrt{\pi^e|_{\mathsf{v}}, \omega_{\mathsf{v}}} \left| \sqrt{\frac{\rho^e}{\pi^e}}\bigg|_{\mathsf{v}} - \sqrt{\frac{\gamma_{\mathsf{v}}}{\omega_{\mathsf{v}}}} \right|^2 \mathrm{d}t\end{aligned}$$

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- Continuity equation: $(\mu, \mathrm{j}) \in \mathrm{CE}$ if for all $\Phi = (\phi, \varphi) \in C^1(\mathsf{V} \times \mathsf{L})$ and a.e. time

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[\sum_{\mathsf{v}\in\mathsf{V}} \phi_{\mathsf{v}}\gamma_{\mathsf{v}} + \sum_{e\in\mathsf{E}} \int_0^{\ell^e} \varphi^e \,\mathrm{d}\rho^e\right] = \sum_{e\in\mathsf{E}} \int_0^{\ell^e} \partial_x\varphi^e \,\mathrm{d}j^e + \sum_{\mathsf{v}\in\mathsf{V}} \sum_{e\in\mathsf{E}(\mathsf{v})} (\phi_{\mathsf{v}} - \varphi^e|_{\mathsf{v}})\bar{\jmath}_{\mathsf{v}}^e$$

Chain rule and well-posedness

Chain rule and well-posedness

* Chain rule identity:

$$\mathcal{E}(\mu(t)) - \mathcal{E}(\mu(s)) = \int_s^t \left[\langle \nabla \mathrm{D}\mathcal{E}_{\mathsf{E}}(\rho(\tau)), j(\tau) \rangle + \langle \overline{\nabla} \mathrm{D}\mathcal{E}(\mu(\tau)), \bar{j}(\tau) \rangle \right] \mathrm{d}\tau$$

Proof: Exploit decoupling due to CE format

- Metric edge terms: Special case of Erbar-Forkert-Maas-Mugnolo '22
- Transition terms: Special case of Peletier-Rossi-Savaré-Tse '23

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$$\text{for} \quad j^e = d^e \rho^e \partial_x \log \frac{\rho^e}{\pi^e} \quad \text{and} \quad \bar{\jmath}^e_{\mathsf{v}} = \hbar^e_{\mathsf{v}} \sqrt{\pi^e|_{\mathsf{v}} \omega_{\mathsf{v}}} \left[\frac{\rho^e}{\pi^e}\bigg|_{\mathsf{v}} - \frac{\gamma_{\mathsf{v}}}{\omega_{\mathsf{v}}} \right]$$

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- Existence of solutions:

Proof: Abstract EDP convergence applied with finite volume approximation (similar to Hraivoronska-Tse '23)

Chain rule and well-posedness

* Chain rule identity:

$$\mathcal{E}(\mu(t)) - \mathcal{E}(\mu(s)) = \int_s^t \Big[\langle \nabla \mathrm{D}\mathcal{E}_{\mathsf{E}}(\rho(\tau)), j(\tau)\rangle + \langle \overline{\nabla}\mathrm{D}\mathcal{E}(\mu(\tau)), \bar{\jmath}(\tau)\rangle \Big]\,\mathrm{d}\tau$$

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$$\text{for} \quad j^e = d^e \rho^e \partial_x \log \frac{\rho^e}{\pi^e} \quad \text{and} \quad \bar{\jmath}^e_{\mathsf{v}} = k^e_{\mathsf{v}} \sqrt{\pi^e|_{\mathsf{v}}\omega_{\mathsf{v}}}\left[\frac{\rho^e}{\pi^e}\bigg|_{\mathsf{v}} - \frac{\gamma_{\mathsf{v}}}{\omega_{\mathsf{v}}} \right]$$

* Existence of solutions:
Proof: Abstract EDP convergence applied with finite volume approximation (similar to Hraivoronska-Tse '23)

* Uniqueness of solutions:
Proof: Use that $(\mu, \mathrm{j}) \mapsto \mathcal{L}(\mu, \mathrm{j})$ is strictly convex w.r.t. affine interpolations

Kirchhoff limit

Kirchhoff limit

- Kirchhoff means no mass exchange with vertex reservoirs

▶ Remove mass from reservoirs

$$\pi^{\varepsilon} := \frac{\pi}{\pi(\mathsf{L}) + \varepsilon\omega(\mathsf{V})} \qquad \text{and} \qquad \omega^{\varepsilon} := \frac{\varepsilon\omega}{\pi(\mathsf{L}) + \varepsilon\omega(\mathsf{V})}$$

▶ Accelerate jump rates to retain non-trivial fluxes

$$\Bbbk_{\mathsf{v}}^{\mathsf{e},\varepsilon} := \frac{1}{\varepsilon} \Bbbk_{\mathsf{v}}^{\mathsf{e}}$$

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$$\mathscr{k}_{\mathsf{v}}^{\mathsf{e},\varepsilon} := \frac{1}{\varepsilon}\mathscr{k}_{\mathsf{v}}^{\mathsf{e}}$$

- Challenge: Lose control over $\bar{\jmath}_{\mathsf{v}}^{e} \implies$ Lose control over $\frac{\mathsf{d}}{\mathsf{d}t}\gamma_{\mathsf{v}}$

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∗ Steps to EDP limit:

  - Energy bound: $\gamma_{\mathsf{v}}^{\varepsilon} \to 0$ strongly in $L^1(0, T)$

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∗ Steps to EDP limit:
  - Energy bound: $\gamma_{\mathsf{v}}^{\varepsilon} \to 0$ strongly in $L^1(0, T)$
  - Slope term for jumps: $\left|\sqrt{\frac{\rho^{\mathsf{e},\varepsilon}}{\pi^{\mathsf{e},\varepsilon}}}\Big|_{\mathsf{v}} - \sqrt{\frac{\gamma_{\mathsf{v}}^{\varepsilon}}{\omega_{\mathsf{v}}^{\varepsilon}}}\right| \to 0$ weakly in $L^2(0, T)$

Kirchhoff limit

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▶ Accelerate jump rates to retain non-trivial fluxes

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- Limit model coincides with model in Erbar-Forkert-Maas-Mugnolo '23

▶ Limit model is gradient flow

Fast edge-diffusion limit

Fast edge-diffusion limit

- Rescale diffusivity constant:

$$d^{e,\varepsilon} := \frac{1}{\varepsilon} d^e$$

- Challenge: Lose control over $j^e \implies$ Lose control over $\frac{\mathrm{d}}{\mathrm{d}t}\rho^e$

Fast edge-diffusion limit

- Rescale diffusivity constant:

$$d^{e,\varepsilon} := \frac{1}{\varepsilon} d^e$$

- Challenge: Lose control over $j^e \implies$ Lose control over $\frac{\mathrm{d}}{\mathrm{d}t}\rho^e$
- Heuristics: ρ^e averaged out $\implies$ Metric edge becomes new vertex
  $\implies$ Jump fluxes suffice to control time derivatives

Fast edge-diffusion limit

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- Limit model is graph gradient flow on extended graph $(\hat{\mathsf{V}}, \hat{\mathsf{E}})$
- ∗ Can be further reduced to graph gradient flow on (V, E) by sending $\pi^e \to 0$ and $\kappa^e_\mathsf{v} \to \infty$ s.t. $\kappa^e_\mathsf{v}\sqrt{\pi^e|_\mathsf{v}\omega_\mathsf{v}} \in \mathcal{O}(1)$ (cf. also Peletier-Schlichting '23)

Reaction-diffusion systems
(GH, A Mielke, A Stephan)

Setup and gradient system

- Finitely many species $i \in I$ with concentrations $\rho = (\rho_i)_{i \in I} \in \mathcal{M}_{\geq 0}(\mathbb{T}^d; \mathbb{R}^I)$
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▶ RDS written as GF in CE format:

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  - Fischer-Hopf-Kniely-Mielke '22: $j \log j \approx \mathsf{C}(j) \le \frac{q}{q-1}\mathsf{C}(j|a) + \frac{4}{q-1}a^q$, $\forall q > 1$

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  - Fischer-Hopf-Kniely-Mielke '22: $j \log j \approx \mathsf{C}(j) \le \frac{q}{q-1}\mathsf{C}(j|a) + \frac{4}{q-1}a^q$, $\forall q > 1$
  - Can control exponent via diffusion if $|\alpha^r + \beta^r|_1 < 2p_{\mathrm{crit}}$

Chain rule inequality for RDS

- Idea: Test CE with $\mathrm{D}\mathcal{E}(\rho)$, use Fenchel's inequality for pair $(\mathcal{R}, \mathcal{R}^*)$
- Obstruction: Lack of regularity of $\mathrm{D}\mathcal{E}(\rho)_i = \log(\rho_i/\pi_i)$
- A priori bounds: Energy $\mathcal{E}$ and dissipation $\mathcal{D}$ finite
- Standard strategy: Truncate and mollify, then control errors using $\mathcal{D}$ and $\mathcal{E}$
- Convexity: Mollified dissipations bounded by unmollified of dissipations

▶ Works well for drift-diffusion terms

- Reactive rate terms $\mathcal{R}_{\mathrm{react}}(\rho, j)$: Contain non-concave terms $\rho^{(\alpha^r+\beta^r)/2}$

▶ Commutator estimate: $\left\| (\nu_\varepsilon * j)\left[(\nu_\varepsilon * \rho)^{(\alpha^r+\beta^r)/2} - \nu_\varepsilon * \rho^{(\alpha^r+\beta^r)/2}\right]\right\|_{\mathrm{L}^1} \to 0$

- Key tools:
    - Use the truncation to obtain weak-* convergence in L^∞ of differences
    - Hardy-Littlewood maximal function: Pointwise bound $|\nu_\varepsilon * j| \le \mathrm{M}j$ and integrability $\mathrm{M}j \in \mathrm{L}^1$ iff $j \in \mathrm{L}\log\mathrm{L}$
    - Fischer-Hopf-Kniely-Mielke '22: $j \log j \approx \mathsf{C}(j) \le \frac{q}{q-1}\mathsf{C}(j|a) + \frac{4}{q-1}a^q$, $\forall q > 1$
    - Can control exponent via diffusion if $|\alpha^r + \beta^r|_1 < 2p_{\mathrm{crit}}$
- Reactive slope terms $\mathcal{R}^*_{\mathrm{react}}(\rho, -\Gamma\mathrm{D}\mathcal{E}(\rho))$: Continuous if $|\alpha^r|_1, |\beta^r|_1 \le p_{\mathrm{crit}}$

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Some future perspectives:

- Coupling of metric graph model with dynamics on domains (with J Krautz, J-F Pietschmann)
- GF with other boundary conditions as limits (with A Schlichting)
- PME as GF of Boltzmann entropy: Approximation by spatially discrete reaction systems (with A Mielke, A Stephan)
- Rigorous GF formulation for nonlinear cross-diffusion with exclusion