Finite volumes for the local sensing chemotaxis system

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Gradient Flows Face to Face, Granada, September 16, 2025

Outline

Presentation of the local sensing Keller-Segel model

Chemotaxis

Chemotaxis: directed movement of an organism in response to a chemical stimulus (chemoattractant)

Example:

- production of chemoattractant (signal)
- aggregation and formation of clusters

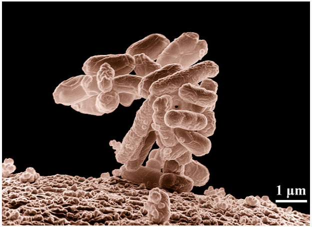


Figure 1: Bacteria E. Coli

Gradient sensing and local sensing

Gradient sensing: the cells can "compute" a gradient of the concentration of chemoattractant $v(x,t)$. For instance the cells possess two receptors

$$\frac{1}{|\alpha|}\left(v(x+\alpha/2,t)-v(x-\alpha/2,t)\right)\approx\frac{\alpha}{|\alpha|}\cdot\nabla v(x,t),$$

where α is the vector between receptors ($|\alpha|\ll 1$)

Local sensing: the cells can only sense the local concentration

$$v(x,t).$$

The Keller-Segel system (1971)

In two papers in *J. Theoret. Biol.*, Keller and Segel introduced

$$\partial_t u = \nabla \cdot (\gamma(v)\,\nabla u - u\,\chi(v)\,\nabla v)$$
$$\varepsilon \partial_t v = \delta \Delta v - \beta v + u$$

with u the cell density, v the chemical concentration, γ the diffusivity, χ the chemosensitivity, $\varepsilon \geq 0$, $\delta > 0$, $\beta > 0$.

- Minimal Keller-Segel model:

$$\gamma = \chi = 1$$

- Two receptors model [Keller, Segel (71) *"Models for Chemotaxis"*]:

$$\chi(s) = (\alpha - 1)\,\gamma'(s)$$

  with $\alpha \geq 0$ the effective body length.

- Local sensing model: case $\alpha = 0$

$$\gamma(v)\,\nabla u + u \nabla \gamma(v) = \nabla(\gamma(v)\,u)$$

The local sensing Keller-Segel system

In $\Omega \subset \mathbb{R}^2$ be a bounded domain and for $t \in (0, T)$, we consider

$$\partial_t u = \Delta(\gamma(v)u),$$
$$\varepsilon \partial_t v = \delta \Delta v - \beta v + u$$

with initial conditions $u(\cdot, 0) = u^0$ and $v(\cdot, 0) = v^0$ and the no-flux boundary conditions

$$\nabla[\gamma(v)u] \cdot \nu = \nabla v \cdot \nu = 0 \quad \text{on } \partial\Omega$$

In this talk we will assume that

$$\gamma(v) = \exp(-v).$$

Existence theory

Consider the LSKS system with Neumann boundary conditions

$$\partial_t u = \Delta(\gamma(v)u)$$
$$\varepsilon\partial_t v = \delta\Delta v - \beta v + u$$

- Local in time: [Amann]
- Global in time for γ upper and lower bounded or small: [Tao, Winkler ('17) *M3AS*], [Kim, Yoon ('17) *Acta Appl Math*]
- Global in time for γ power laws: [Desvillettes, Kim, Trescases, Yoon ('20) Nonlin. Anal.], [Fujie, Senba ('22) *Nonlin. Anal.*]...
- Global in time for $\gamma(v) = \exp(-v)$: [Jin, Wang ('20), PAMS], [Burger, Laurençot, Trescases ('21) *J. London Math. Soc.*], [Fujie, Jiang ('21) *Calc. Var.*]

Entropy and gradient flow structure

Entropy:

$$H(u,v) = \int_\Omega \left[u \log u - u + 1 + \frac{\beta}{2}|v|^2 - uv + \frac{\delta}{2}\left|\nabla v\right|^2 \right] dx.$$

Gradient flow structure:

- Minimal Keller-Segel (KS)

$$\partial_t u = \nabla \cdot (u \nabla \partial_u H)$$
$$\varepsilon \partial_t v = -\partial_v H$$

- Local sensing Keller-Segel (LSKS)

$$\partial_t u = \nabla \cdot (u e^{-v} \nabla \partial_u H)$$
$$\varepsilon \partial_t v = -\partial_v H$$

Entropy dissipation equality for LSKS: for $t > 0$

$$\frac{d}{dt} H(u(t), v(t)) + 4 \int_\Omega \left| \nabla \sqrt{u e^{-v}} \right|^2 dx + \varepsilon \int_\Omega \left| \partial_t v \right|^2 dx = 0$$

Duality estimate for Kolmogorov equation

Let $\mu \equiv \mu(t,x)$ be given and consider

$$\partial_t z - \Delta(\mu z) = 0$$

Which type of *a priori* estimate?

$$\begin{aligned} \partial_t z - \mu\Delta z &= 0 & &\Rightarrow \Delta z \in L^2(\mu \mathrm{d}t\mathrm{d}x) \\ \partial_t z - \nabla\cdot(\mu\nabla z) &= 0 & &\Rightarrow \nabla z \in L^2(\mu \mathrm{d}t\mathrm{d}x) \\ \partial_t z - \Delta(\mu z) &= 0 & &\Rightarrow z \in L^2(\mu \mathrm{d}t\mathrm{d}x) \end{aligned}$$

The $L^2(\mu \mathrm{d}t\mathrm{d}x)$ bound may be shown using the dual equation or applying $(-\Delta)^{-1}$ to the equation and integrating against z.

[Pierre, Schmitt (97) *SIMA*] [Desvillettes, Lepoutre, Moussa (14) *SIMA*] [Canizo, Desvillettes, Fellner (14) *CDPE*] [Moussa (20) *SIMA*]...

Duality estimate for LSKS

Following [Burger, Laurençot, Trescases ('21) *J. London Math. Soc.*], there exists $C_T > 0$ such that

$$\sup_{[0,T]} \|u - \langle u^0 \rangle\|^2_{(H^1)'(\Omega)} + 2\int_0^T \int_\Omega \gamma(v)\, u^2 \leq C_T$$

where

$$\|u - \langle u^0 \rangle\|_{(H^1)'(\Omega)} = \|\nabla(-\Delta)^{-1}(u - \langle u^0 \rangle)\|_{L^2(\Omega)}$$

Then, we can bound from below the entropy as

$$\begin{aligned} H(u,v) &= \int_\Omega \left[u \log u - u + 1 + \frac{\beta}{2}|v|^2 - uv + \frac{\delta}{2}\left|\nabla v\right|^2 \right] dx \\ &\geq -\frac{C_T^2}{\delta^2} - \frac{\langle u^0 \rangle^2}{\beta} + \int_\Omega \left[u \log u - u + 1 + \frac{\beta}{4}\left|v\right|^2 + \frac{\delta}{4}|\nabla v|^2 \right] \end{aligned}$$

$\Rightarrow$ Existence of nonnegative global weak solutions

A numerical scheme for LSKS

Main objectives

- Design of a structure-preserving scheme (Mass conservation, non-negativity, entropy decay)
- Numerical analysis guaranties (unconditional stability, convergence, AP properties)
- Good tradeoff between stability and computational cost

Methods/tools

- Finite volume scheme in space
- Implicit/explicit scheme in time (linearly implicit)
- Compactness techniques based on *a priori* estimates

State of the art for numerical KS: review by [Arumugam, Tyagi (21) *Acta Appl. Math.*] and essentially nothing for LSKS

Numerical scheme and main results

Preliminaries

Inspirations for the scheme:

- Finite volume scheme in space [Filbet ('06) *Numer. Math.*]
- Implicit/explicit (IMEX) scheme in time (linearly implicit) / convex splitting [Liu, Wang, Zhou ('18) *Math. Comp.*]

Consider the sequence $t_n = n\,\Delta t$ and the (semi) discrete entropy

$$H(u^n, v^n) = \int_\Omega \left[h(u^n) + \frac{|v^n|^2}{2} - u^n\, v^n + \frac{1}{2}\left|\partial_x v^n\right|^2 \right] dx.$$

with u^n, $v^n \approx u(t_n)$, $v(t_n)$ and $h(x) = x\log(x) - x + 1$

Let us find a discretization in time s.t.

$$H(u^n, v^n) + \text{dissip. terms} \leq H(u^{n-1}, v^{n-1})$$

Discretization in time

$$u^n - u^{n-1} = \Delta t\, \Delta(u^n \gamma(v^n)) \qquad \times \log(u^n \gamma(v^n))$$

$$\frac{v^n - v^{n-1}}{\Delta t} = \Delta v^n + u^{n-1} - v^n \qquad \times (v^n - v^{n-1})$$

Integrate in space, use integration by parts and convexity of h:

$$\int (h(u^n) - h(u^{n-1})) - \int (u^n - u^{n-1}) v^n + 4\Delta t \int |\nabla_x \sqrt{u^n \gamma(v^n)}|^2 \leq 0$$

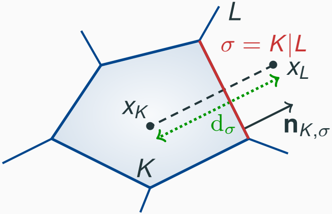
$$\int \frac{(v^n - v^{n-1})^2}{\Delta t} + \int \nabla_x v^n \cdot (\nabla_x v^n - \nabla_x v^{n-1}) - \int u^{n-1}(v^n - v^{n-1}) + \int v^n(v^n - v^{n-1}) = 0$$

Sum both expressions and use $a \cdot (a - b) \geq (|a|^2 - |b|^2)/2$

Mesh and discretization

Time discretization of $[0, T]$: $(t_n = n\,\Delta t)_{1\le n\le N_T}$

Mesh in space



- $\mathcal{T}$ set of cells K, L, ...
- $\mathcal{E}$ set of edges ($\sigma = K|L$)
- $\mathcal{E}_K$ set of edges (cell K)
- m Lebesgue measure (1 or 2d)

Discrete unknowns

$$u^n = (u_K^n)_{K\in\mathcal{T},1\le n\le N_T}, \quad v^n = (v_K^n)_{K\in\mathcal{T},1\le n\le N_T}$$

with $\mathrm{m}(K)u_K^n \approx \int_K u(\cdot, t_n)$

Discretization in space for u

Starting from the semi-discrete equation on u

$$\frac{u^n - u^{n-1}}{\Delta t} = \nabla \cdot \underbrace{\nabla(\gamma(v^n)\, u^n)}_{\text{flux}}$$

On a cell K one has

$$\mathrm{m}(K)\frac{u_K^n - u_K^{n-1}}{\Delta t} = \sum_{\sigma \in \mathcal{E}_K} \mathcal{F}_{K,\sigma}^n$$

with a numerical flux

$$\mathcal{F}_{K,\sigma}^n \approx \int_\sigma \nabla\left[\gamma(v(s,t_n))\, u(s,t_n)\right] \cdot \mathbf{n}_{K,\sigma}\, ds$$

which can be taken as

$$\mathrm{m}(K)\, \frac{u_K^n - u_K^{n-1}}{\Delta t} = \sum_{\sigma \in \mathcal{E}_K} \frac{\mathrm{m}(\sigma)}{\mathrm{d}_\sigma}\, \left(\gamma(v_L^n)\, u_L^n - \gamma(v_K^n)\, u_K^n\right)$$

Full numerical scheme for (u, v)

For $w = (w_K)_{K\in\mathcal{T}}$, we set

$$\mathrm{D}_{K,\sigma} w = w_L - w_K \quad \text{for } \sigma = K|L$$

Then, we get for $1 \leq n \leq N_T$

$$\mathrm{m}(K)\,\frac{u_K^n - u_K^{n-1}}{\Delta t} = \sum_{\sigma\in\mathcal{E}_K} \tau_\sigma\,\mathrm{D}_{K,\sigma}\gamma(v^n)\,u^n$$

$$\varepsilon\,\mathrm{m}(K)\frac{v_K^n - v_K^{n-1}}{\Delta t} = \delta \sum_{\sigma\in\mathcal{E}_K} \tau_\sigma\,\mathrm{D}_{K,\sigma} v^n + \mathrm{m}(K)\,(u_K^{n-1} - \beta v_K^n)$$

where $\tau_\sigma = \mathrm{m}(\sigma)/d_\sigma$.

Consequence: the scheme is linearly implicit, can be rewritten as

$$\mathbb{M}_v\, v^n = b_v^{n-1}, \qquad \mathbb{M}_u^n\, u^n = b_u^{n-1}$$

Existence of solutions

Theorem ([H., Trescases, Zurek (2024)])

Assume $u^0, v^0 \geq 0$ and $\varepsilon \geq 0$

- *Existence unique positive solutions at each time step*
- *Mass estimates for u and v are preserved at the discrete level*
- *Discrete entropy dissipation inequality (for any $1 \leq n \leq N_T$)*

$$H(u^n, v^n) + 4\Delta t \sum_{\sigma\in\mathcal{E}} \tau_\sigma \left(\mathrm{D}_{K,\sigma}\sqrt{u^n\gamma(v^n)}\right)^2 + \Delta t\varepsilon \sum_{K\in\mathcal{T}} \mathrm{m}(K) \left|\frac{v_K^n - v_K^{n-1}}{\Delta t}\right|^2 \leq H(u^{n-1}, v^{n-1})$$

with

$$H(u^n, v^n) = \sum_{K\in\mathcal{T}} \mathrm{m}(K) \left(h(u_K^n) + \frac{\beta}{2}|v_K^n|^2 - u_K^n v_K^n\right) + \frac{\delta}{2}\sum_{\sigma\in\mathcal{E}} \tau_\sigma \left(\mathrm{D}_{K,\sigma} v^n\right)^2.$$

Convergence of the scheme

Denote by u_m, v_m the constant by parts reconstructions on the mesh and ∇^m, ∂_t^m the discrete differential operators

Theorem ([H., Trescases, Zurek (2024)])

Let $u^0 \in L^2(\Omega)$ *and* $v^0 \in H^1(\Omega)$. *Let* $\gamma(v) = e^{-v}$. *If* $\varepsilon > 0$, $\eta_m = \max(\Delta x_m, \Delta t_m) \to 0$ *then (up to a subsequence)*

$$\begin{aligned} u_m &\rightharpoonup u \quad \text{weakly in } L^1(Q_T) \\ v_m &\to v \quad \text{strongly in } L^2(Q_T) \\ \nabla^m v_m &\rightharpoonup \nabla v \quad \text{weakly in } L^2(Q_T) \\ \partial_t^m v_m &\rightharpoonup \partial_t v \quad \text{weakly in } L^2(Q_T) \\ u_m \gamma(v_m) &\rightharpoonup u \gamma(v) \quad \text{weakly in } L^2(Q_T) \end{aligned}$$

with (u, v) *very weak solution to the system.*

Sketch of the convergence proof

1. Mass estimates
2. Discrete entropy inequality and discrete duality estimate

$$\max_{1\leq n\leq N_T} N(u^n - \langle u^0\rangle)^2 + 2\sum_{n=1}^{N_T} \Delta t \, \|u^n \sqrt{\gamma(v^n)}\|^2_{L^2(\Omega)} \leq C_T$$

where

$$N(u^n - \langle u^0\rangle)^2 = \sum_{\sigma\in\mathcal{E}} \tau_\sigma (\mathrm{D}_{K,\sigma} z)^2$$

with $z = (z_K)_{K\in\mathcal{T}}$ the unique solution, s.t. $\langle z\rangle = 0$, to

$$\underbrace{-\sum_{\sigma\in\mathcal{E}_K} \tau_\sigma \mathrm{D}_{K,\sigma} z}_{\approx -\Delta z} = \mathrm{m}(K)(u^n - \langle u^0\rangle), \quad \forall K \in \mathcal{T}$$

3. Obtain a bound from below on the discrete entropy
4. Kolmogorov $+$ Dunford-Pettis theorem $\Rightarrow$ convergence properties
5. Identification of the limit functions as weak solutions

Rigorous Asymptotic Preserving (AP) property

We investigate the quasi-stationary (parabolic to elliptic) limit

Theorem ([Herda, Trescases, Z. ('24)])

If $(\varepsilon_m) \subset [0,\infty)$, $\eta_m \to 0$ and well-prepared initial conditions then (up to a subsequence)

$$\begin{aligned} u_m &\rightharpoonup u \quad \textit{weakly in } L^1(Q_T) \\ v_m &\to v \quad \textit{strongly in } L^2(Q_T) \\ \nabla^m v_m &\rightharpoonup \nabla v \quad \textit{weakly in } L^2(Q_T) \\ \varepsilon_m \partial_t v_m &\rightharpoonup 0 \quad \textit{weakly in } L^2(Q_T) \end{aligned}$$

with (u, v) very weak solution to the parabolic-elliptic system ($\varepsilon = 0$).

Remarks:

- if $\varepsilon_m = 0\ \forall m$, result $\Rightarrow$ convergence of the scheme with $\varepsilon = 0$
- If $u_0 \in L^\infty$, convergences hold for the whole sequences (1 and 2d)

Main ingredients of the proof (at the continuous level)

Main objective: get uniform w.r.t. ε estimate on $\partial_t v$

Reminder: entropy inequality

$$\frac{d}{dt}H(u(t),v(t)) + 4\int_\Omega \left|\nabla\sqrt{ue^{-v}}\right|^2 dx + \varepsilon\int_\Omega |\partial_t v|^2 dx = 0$$

$\Rightarrow$ estimate on L^2 norm of $\sqrt{\varepsilon}\,\partial_t v$, not enough!

Main idea: derive in time the equation on v

$$\varepsilon\partial_{tt}v + \beta\partial_t v = \partial_t u + \delta\Delta\partial_t v$$

and establish the estimate

$$\int_0^T\int_\Omega (\partial_t v - \langle\partial_t v\rangle)^2 \leq C'_T$$

Last step: estimate the L^2 norm of $\langle\partial_t v\rangle$

Well-prepared initial data

One can show that

$$\int_0^T \int_\Omega \langle \partial_t v \rangle^2 = \frac{\mathrm{m}(\Omega)}{2\beta} \left(\frac{\langle u^0 \rangle - \beta \langle v^0 \rangle}{\sqrt{\varepsilon}} \right)^2 \left(1 - e^{-\frac{2\beta}{\varepsilon} T} \right)$$

Therefore, if

$$\langle u^0 \rangle - \beta \langle v^0 \rangle = O(\sqrt{\varepsilon})$$

Then, $\exists C'_T$ independent of ε such that

$$\|\partial_t v\|^2_{L^2(\Omega \times (0,T))} \leq C'_T$$

At the discrete level:

- Adapt all the previous computations
- Follow the lines of the convergence proof ($\varepsilon > 0$)

Numerical experiments

Implementation and convergence

General remarks

- Scheme implemented in 1 and 2d with Matlab

  `https://gitlab.inria.fr/herda/fvlocsens`

- In 2d, unstructured triangular meshes from Gmsh

  `https://gitlab.inria.fr/herda/fvmeshes`

Experimental convergence

- IC $u^0(x) = 15x^2(1-x)^2$, $v^0(x) = 0$
- $\Delta t = 10^{-4}$, reference solution 3200 cells
- Second order experimental convergence in space in L^2 and L^∞ norms at $T = 20$ (TPFA superconvergence [Droniou, Nataraj ('18) IMAJNA])

Entropy decay $\Omega = [0, 1]$

- IC $u^0(x) = 15x^2(1-x)^2$, $v^0(x) = 0$
- $\Delta t = 10^{-2}$, mesh in space 3200 cells

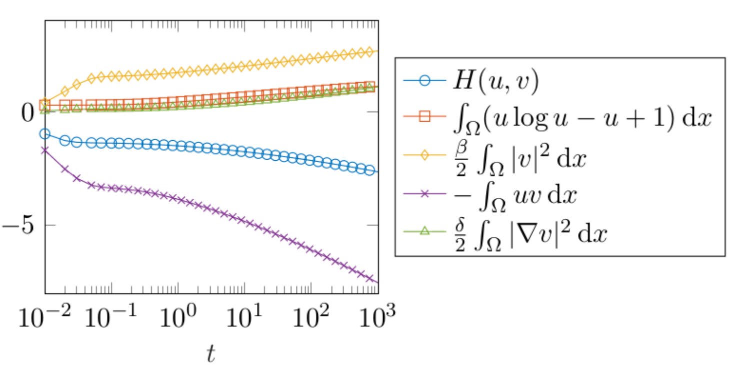


Figure 2: Time evolution of the entropy and related quantities

Stability/Instability in 2d, $\Omega = \mathbb{D}$

Linear stability analysis: let $\mu > 0$ be given, then

$$(u^*, v^*) = (\mu, \mu/\beta) \text{ linearly stable } \Leftrightarrow \mu < \mu_c^s = \beta + \lambda_1(\Omega)\delta$$

Remark: same stability condition at the discrete level (independent of Δt) with $\lambda_1(\Omega)$ replaced by the first $\neq 0$ eig. of the "FV operator" $(-\Delta)$

Critical mass: $M_c := 4\pi\delta/\mathrm{m}(\Omega)$

- if $\langle u^0 \rangle < M_c \Leftrightarrow$ bounded solutions
- if $\langle u^0 \rangle > M_c \Leftrightarrow$ solutions may blow-up

Testcase:

- $\varepsilon = \delta = 1$ ($M_c = 4$), $\beta = \lambda_1(\Omega) \approx 3.39$, $\mu_c^s \approx 6.78$
- mesh in space 11790 triangles, $\Delta t = 0.1$
- $u^0 = \mu$ ($\langle u^0 \rangle = \mu$) and v^0 perturbation of v^*

Stability in 2d, $\Omega = \mathbb{D}$

$$\mu = 0.9\mu_c^s < \mu_c^s \text{ and } \langle u^0 \rangle \approx 6.1 > 4 \quad \Rightarrow \quad (u^*, v^*) \text{ linearly stable}$$

$H((u(t), v(t)); (u_*, v_*))$

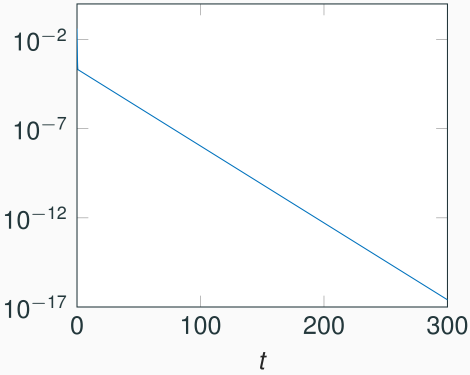
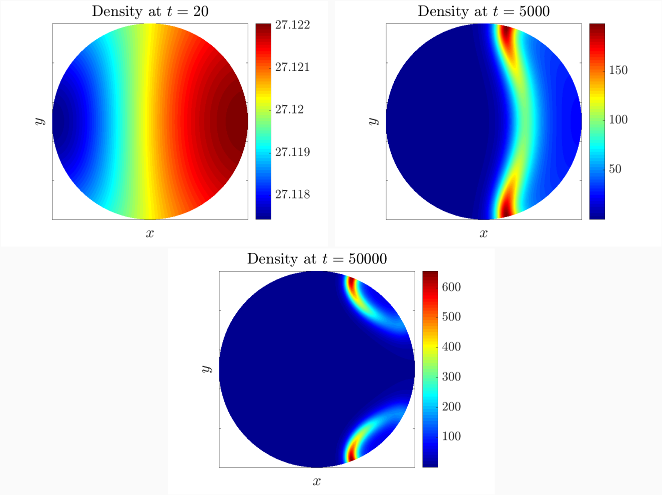


Figure 3: Time evolution of the relative entropy

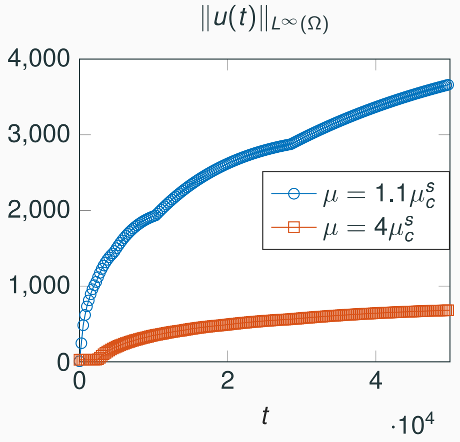
$\mu = 4\mu_c^s > \mu_c^s$ and $\langle u^0 \rangle \approx 27.12 > 4$

$\Rightarrow \quad (u^*, v^*)$ instability expected and may blow-up

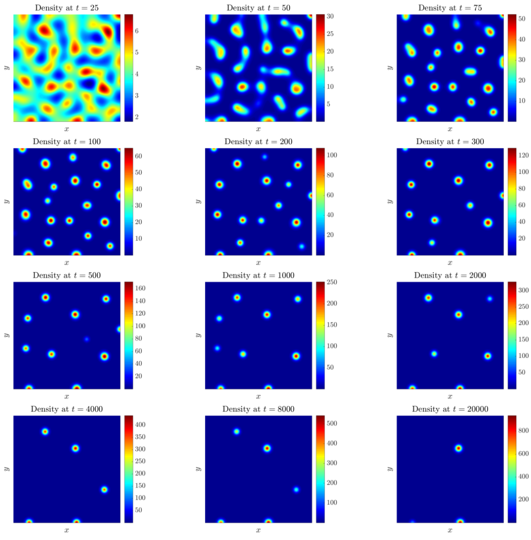


To be or not to be a blow-up...

Growth of L^∞ norm for two unstable solutions.



Aggregation patterns with $\gamma(v) = (1 + v^2)^{-1}$, $\Omega = [0, 10]^2$



[Desvillettes, Kim, Trescases, Yoon ('19) *Nonlin Anal. Real World Appl.*]

Conclusions and perspectives

Conclusions and perspectives

Conclusions:

- Construction of a linearly implicit FV scheme
- Scheme preserves the main features of the system (mass, nonnegativity, entropy...)
- Full convergence analysis and AP properties
- Implementation and some numerical validation

Perspectives:

- Numerical investigation of blowup mechanisms
- Relations with discrete functional inequalities and optimal constants

Thank you for your attention!

[H., Trescases, Zurek, *A finite volume scheme for the local sensing chemotaxis model*. arXiv:2412.13143]