Uniqueness results for local and non-local Dirichlet problems

Isabella Ianni

SAPIENZA
UNIVERSITÀ DI ROMA

IMAG Granada
June 28, 2023

Positive solutions for the problem

$$\begin{cases} -\Delta u + \lambda u = u^p & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \\ u > 0 & \text{in } \Omega \end{cases} \qquad (*)_p$$

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where

$\Omega \subset \mathbb{R}^N$, $N \geq 2$, smooth bounded domain

$1 < p < p_c$

$$p_c := \begin{cases} \frac{N+2}{N-2} & \text{if } N \geq 3 \\ +\infty & \text{if } N = 2 \end{cases}$$

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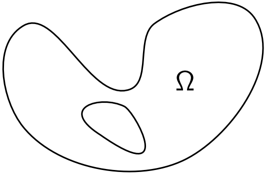

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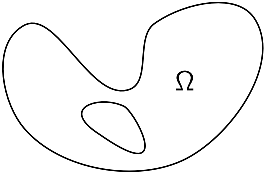

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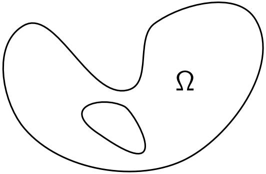

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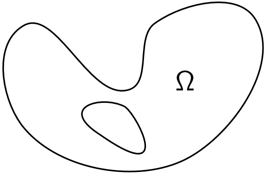

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- the Lane-Emden equation ($\lambda = 0$) in convex domains $\Omega \subset \mathbb{R}^2$

  F. De Marchis, M. Grossi, F. Pacella - *Sapienza University*, Roma (Italy)

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- outline of the known uniqueness results for this problem
- the Lane-Emden equation ($\lambda = 0$) in convex domains $\Omega \subset \mathbb{R}^2$

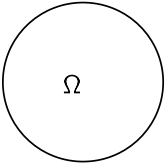
  F. De Marchis, M. Grossi, F. Pacella - *Sapienza University*, Roma (Italy)

- Fractional Laplacian case

  A. Dieb - *Université Abou Bakr Belkaïd*, Tlemcen (Algeria)
  A. Saldaña - *UNAM*, Mexico City (Mexico)

Uniqueness in the ball if $\lambda = 0$

if $\Omega = B$ then there is a unique solution to $(*)_p$



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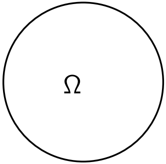

if $\Omega = B$ then there is a unique solution to $(*)_p$

PROOF: any solution of $(*)_p$ is radial by Gidas, Ni & Nirenberg, CMP 1979

so $(*)_p$ reduces to an ODE problem

$$\begin{cases} u'' - \frac{N-1}{r} u' + u^p = 0 & \text{in } (0, R) \\ u'(0) = 0 \end{cases}$$

and $u(R) = 0$, $u > 0$ in $(0, R)$. If by contradiction v is another solution then

$$w(r) := a^{\frac{2}{p-2}} v(ar), \qquad a := \left[\frac{u(0)}{v(0)}\right]^{\frac{p-1}{2}},$$

solves the Initial Value Problem (by the homogeneity of u^p)

$$\begin{cases} w'' - \frac{N-1}{r} w' + w^p = 0 & \text{in } (0, \frac{R}{a}) \\ w'(0) = 0 \\ w(0) = u(0) \end{cases}$$

and $w(\frac{R}{a}) = 0$. So $w \equiv u$ by uniqueness and as a consequence, using the boundary condition, $v \equiv u$.

Uniqueness in the ball if $\lambda \neq 0$

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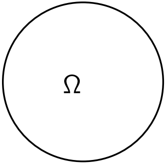
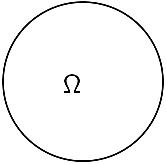


if $\Omega = B$ then there is a unique solution to $(*)_p$

- Ni, JDE 1983
- Ni & Nussbaum, Comm. Pure Appl. Math. 1985
- Kwong & Li, Trans. AMS 1992
- Zhang, Comm. PDE 1992
- Srikanth, Diff. Int. Eq. 1993
- Adimurthi & Yadava, ARMA 1994
- Aftalion & Pacella, JDE 2003

Non-uniqueness: the role of the nonlinearity

$$\begin{cases} -\Delta u = f(u) & \text{in } B \\ u = 0 & \text{on } \partial B \\ u > 0 & \text{in } B \end{cases}$$



An example for which uniqueness fails:

$$f(u) = \mu u^q + u^p$$

$$p \in \left(1, \tfrac{N+2}{N-2}\right), \quad q \in (0,1) \qquad (\mu > 0 \text{ small})$$

Ambrosetti, Brezis & Cerami, JFA 1994

$\Omega \neq B$?

$$\begin{cases} -\Delta u + \lambda u = u^p & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \\ u > 0 & \text{in } \Omega \end{cases} \qquad (*)_p$$

The question of the uniqueness of the solutions of $(*)_p$ in domains Ω other then the ball was raised already in Gidas, Ni & Nirenberg, CMP 1979 :

2.8. Theorem 1 yields a positive response to a question put us by C. Holland. For $p>1$, is the positive solution u of

$$\Delta u + u^p = 0 \quad \text{in} \quad |x| < R, \qquad u = 0 \quad \text{on} \quad |x| = R \tag{2.8}$$

unique? (The question is still open for other domains.) According to Theorem 1

$\Omega \neq B$. Known uniqueness results

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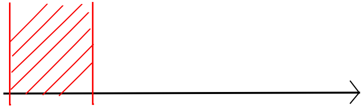
- for any Ω when $p \in (1, 1+\delta)$, where $\delta = \delta(\Omega) > 0$



(uniqueness and nondegeneracy)

C.-S. Lin, Manuscr. Math 1994

Damascelli, Grossi & Pacella AIHP 1999

Dancer, Math. Ann. 2003

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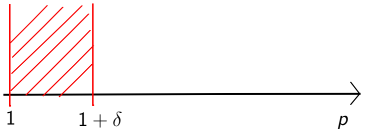

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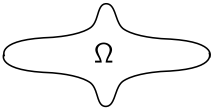
- for Ω symmetric and convex with respect to N orthogonal directions, $\lambda = 0$



(uniqueness and nondegeneracy)

$N = 2$, $\forall p > 1$

$N \geq 3$, $p = \frac{N+2}{N-2} - \varepsilon$

Dancer, JDE 1988 ($\Omega \sim B$)

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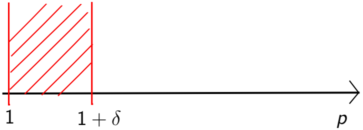

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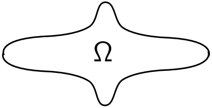

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$N \geq 3$, $p = \frac{N+2}{N-2} - \varepsilon$

Grossi, ADE 2000

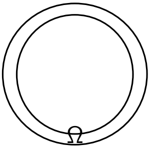
- for Ω unit square, $N = 2$ $p = 2, 3$

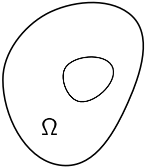
McKenna, Pacella, Plum & Roth, JDE 2009/ Inter. Ser. Numer. Math 2012

Multiplicity results in non-convex domains

annular shaped domains (effect of topology):

- Brezis & Nirenberg, CPAM 1983 annulus & $p = \frac{N+2}{N-2} - \varepsilon$, $N \geq 3$
- Lin, TAMS 1992 thin annulus, non-radial bifurcation
- Y.Y. Li, JDE, 1990 expanding annulus, non-radial bifurcation
- Byeon, JDE 1997
- Catrina & Wang, JDE 1999
- Gladiali, Grossi, Pacella & Srikanth. Calc. Var. 2011 expanding annulus, non-radial bifurcation
- Esposito, Musso & Pistoia, JDE 2006 $N = 2$, not simply connected, p large, $\exists$ multi-spike solutions
- Bartsch, Clapp, Grossi & Pacella, Math. Ann. 2012 expanding annular domains, $\exists$ asympt. radial solution
- Dancer & Yan, CPDE 2002 expanding annular domains, $\exists$ multi-bump solutions





dumb-bell shaped domains (contactible, star-shaped, etc):

- Dancer, JDE 1988 & JDE 1990
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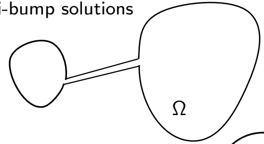

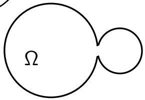

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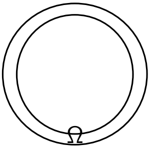

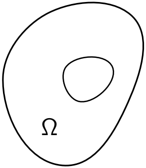

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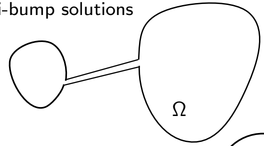

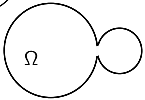

in all these cases the domain is **NOT CONVEX!**

The uniqueness conjecture in convex domains

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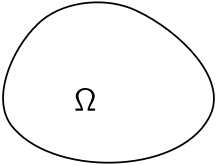
CONJECTURE (Kawohl '85 / Dancer '88)

If Ω is **convex** then there is a **unique solution** to $(*)_p$



Kawohl, Lect. Notes in Math. 1985

Open problem

Prove uniqueness of solutions to (3.62) (3.63) for $\lambda > 0$, $1 < q < \frac{n+2}{n-2}$.

If Ω is a ball, there is uniqueness [85]; if Ω is an annulus, there is no uniqueness [38]. A geometric assumption on Ω which induces uniqueness might be convexity. This problem plays a role in "fast dif-

Dancer, JDE 1988

Last, we obtain some very simple results on the uniqueness of positive solutions on certain highly symmetric domains. This problem seems to require much more work. Indeed, we conjecture that uniqueness holds for $f(y) = y^p$ if Ω is convex and $1 < p < (m+2)(m-2)^{-1}$.

The uniqueness conjecture in convex domains

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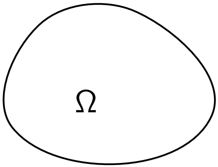


Very challenging problem, solved only when $\Omega = B$ is a ball

Remark. Convexity of Ω is not necessary for uniqueness

$\Omega \neq B$ convex. A partial uniqueness result for the Lane-Emden problem (i.e. $\lambda = 0$)

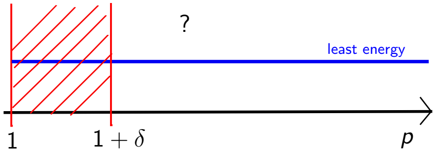
$$\begin{cases} -\Delta u = u^p & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \\ u > 0 & \text{in } \Omega \end{cases} \qquad (LE)_p$$

- for any Ω convex, $N = 2$, $\forall p > 1$ uniqueness of least energy solutions of $(LE)_p$

C.S. Lin, Manuscr. Math. 1994

Our result for the Lane-Emden problem

We consider problem $(LE)_p$ in dimension $N = 2$. When Ω is convex:



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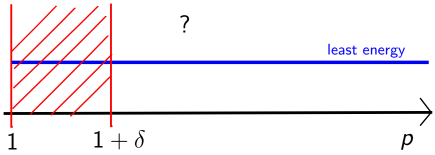

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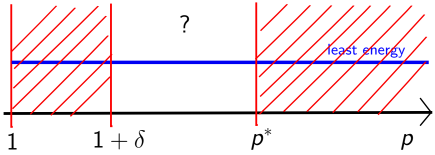

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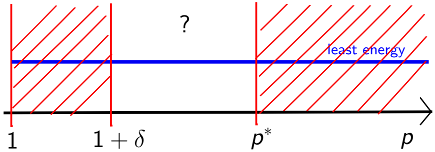

We prove the conjecture for any p large enough:

Theorem [De Marchis, Grossi, I., Pacella]

Let $\Omega \subset \mathbb{R}^2$ be a smooth bounded and convex domain, then there exists $p^\star = p^\star(\Omega) > 1$ such that

$(LE)_p$ has a unique solution for any $p \geq p^\star$
and it is nondegenerate

One of the main tools of our proof: the Morse index

The Morse index of a solution u_p of $(LE)_p$ is

$$m(u_p) = \#\{k \in \mathbb{N} \ : \ \lambda_{k,p} < 1\}$$

where $(0 <)\ \lambda_{1,p} < \lambda_{2,p} \leq \lambda_{3,p} \leq \dots$ is the sequence of eigenvalues for the linear problem

$$\begin{cases} -\Delta v = \lambda \ p u_p^{p-1} \ v & \text{in } \Omega \\ v = 0 & \text{on } \partial\Omega \end{cases} \qquad (Lin)_p$$

- $m(u_p) \geq 1$ $\quad (\forall N \geq 2,\ \forall \Omega \subset \mathbb{R}^N,\ \forall 1 < p < p_c)$

$$\lambda_{1,p} = \frac{1}{p} < 1$$

- problem: computation/a priori bounds for the Morse index

Uniqueness – non-degeneracy – Morse index

C.S. Lin, Manuscr. Math. 1994

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non-degeneracy $\forall p \Rightarrow$ uniqueness $\quad (\forall N \geq 2,\ \forall \Omega \subset \mathbb{R}^N,\ \forall 1 < p < p_c)$

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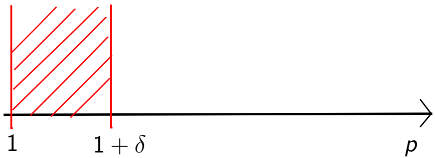

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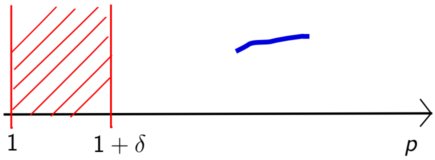

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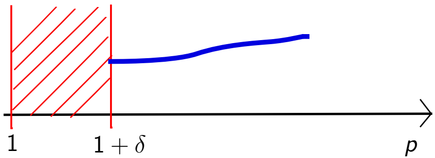

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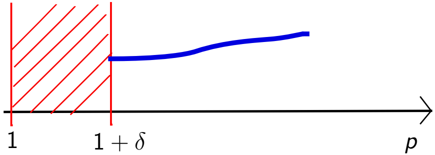

If **N = 2** and **Ω convex** then
any Morse index 1-solution is non-degenerate

(any least energy solution has Morse index 1, $\forall p$)

Our proof

Key Theorem [De Marchis, Grossi, I., Pacella]

If $\mathbf{N = 2}$ and **Ω convex** then

$\exists\ p^\star = p^\star(\Omega) > 1$ such that **any solution** of $(LE)_p$ has

Morse index=1 if $p \geq p^\star$

Morse index computation for p large

asymptotic analysis as $p \to +\infty$
for families u_p of solutions to $(LE)_p$

$\Downarrow$

computation of the Morse index
$m(u_p)$ for p *large*

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$$\begin{cases} -\Delta v = \lambda\, p u_p^{p-1}\, v & \text{in } \Omega \\ v = 0 & \text{on } \partial\Omega \end{cases}$$

Asymptotic characterization of the solutions. $N = 2$, any Ω

Theorem [De Marchis, I., Pacella]

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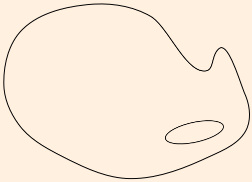

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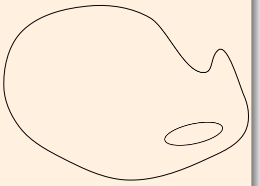

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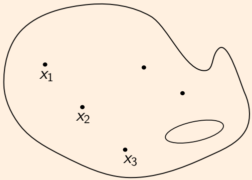

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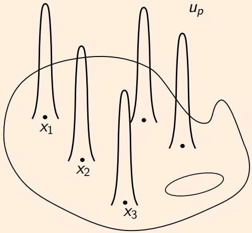

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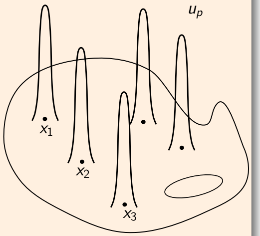

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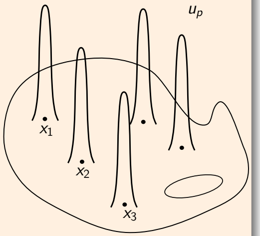

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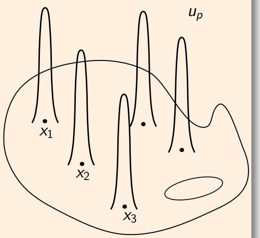

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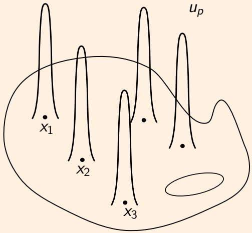

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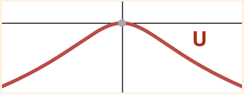

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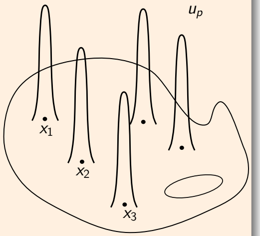

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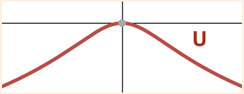

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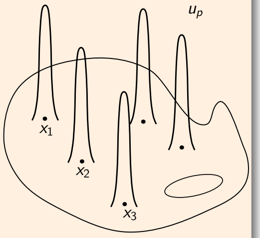
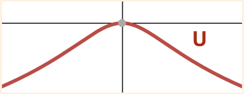


k-peaks solution



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Theorem ($k \geq 2$: [I., Luo, Yan], $k = 1$: [De Marchis, Grossi, I., Pacella])

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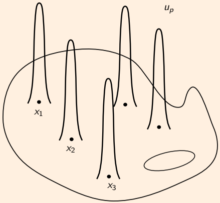

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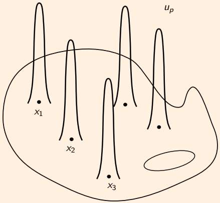

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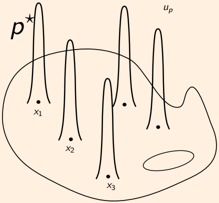
Then $\exists\ p^\star > 1$ such that

$$(k \leq)\ k + m(\bar{x}) \leq m(u_p) \leq m_0(u_p) \leq k + m_0(\bar{x})\ (\leq 3k), \quad \text{for } p \geq p^\star$$



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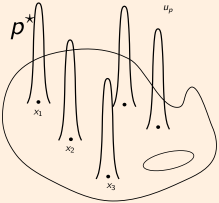

In particular if $\bar{x}$ is non-degenerate then

$$\boxed{m(u_p) = k + m(\bar{x}), \quad \text{for } p \geq p^\star}$$

and u_p is non-degenerate for $p \geq p^\star$

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Back to the proof of the uniqueness, for $\Omega \subset \mathbb{R}^2$ convex

From this general Morse index formula then one deduces the following:

Key Theorem

If $\mathbf{N = 2}$ and $\boldsymbol{\Omega}$ **convex** then $\exists\ p^\star = p^\star(\Omega) > 1$ such that

$$\mathbf{m(u_p) = 1}$$

for **any solution** u_p of $(LE)_p$ when $\mathbf{p \geq p^\star}$

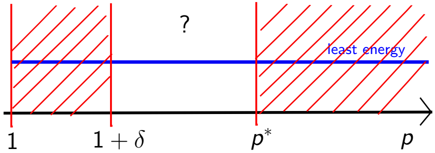
Uniqueness of positive solutions in convex domains

$$\begin{cases} -\Delta u = u^p & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \\ u > 0 & \text{in } \Omega \end{cases} \qquad (LE)_p$$

Theorem [De Marchis, Grossi, I., Pacella]

Let $\Omega \subset \mathbb{R}^2$ be a smooth bounded and convex domain, then there exists $p^\star = p^\star(\Omega) > 1$ such that

$$(LE)_p \text{ has a unique solution for any } p \geq p^\star$$



Uniqueness for the fractional Dirichlet problem

$$\begin{cases} (-\Delta)^s u + \lambda u = u^p & \text{in } \Omega \\ u = 0 & \text{on } \mathbb{R}^N \setminus \Omega \\ u > 0 & \text{in } \Omega \end{cases} \qquad (*)_p$$

where

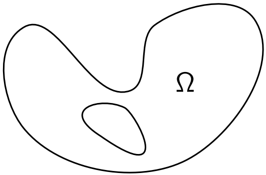
$s \in (0,1)$

$\Omega \subset \mathbb{R}^N$, $N \geq 2$, smooth bounded

$1 < p < \frac{N+2s}{N-2s}$

$\lambda > -\lambda_1^s(\Omega)$

$$(-\Delta)^s u(x) := C_{N,s} \; p.v. \int_{\mathbb{R}^N} \frac{u(x) - u(y)}{|x - y|^{N+2s}} \, dy$$



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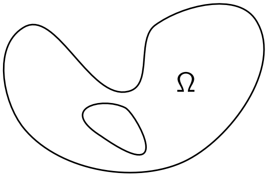

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- existence

  Servadei, & Valdinoci, J. Math. Anal. Appl. 2012, DCDS 2013

- multiplicity results

  may be deduced for instance for $\lambda = 0$ and $p = \frac{N+2s}{N-2s} - \varepsilon$ from the one bubble solutions in

  Davila, Lopez Rios & Sire, Rev. Mat. Iberoam. 2017

Uniqueness for the fractional Dirichlet problem

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NO RESULTS IN THE LITERATURE

Some uniqueness results for fractional problems in $\mathbb{R}^N$

For ground states of fractional Schrödinger equations in $\mathbb{R}^N$:

- Fall & Valdinoci, Comm. Math. Phys. 2014
- Frank & Lenzmann, Acta Math. 2013
- Frank, Lenzmann & Silvestre, Comm. Pure Appl. Math. 2016

For radial ground states of the fractional plasma equation in $\mathbb{R}^N$ and of the fractional critical and supercritical Lane-Emden equation in $\mathbb{R}^N$:

- Chan, del Mar Gonzalez, Huang, Mainini & Volzone, Calc. Var. 2020

For radial ground states of the critical fractional Henon equation in $\mathbb{R}^N$:

- Alarcon, Barrios & Quaas, DCDS 2023

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NO RESULTS IN THE LITERATURE

Even for $\lambda = 0$ and $\Omega = B$ uniqueness is still an open problem!

Jarohs & Weth, DCDS 2014 radial and decreasing

Main issues:

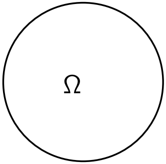
no ODE techniques

Ao, Chan, DelaTorre, Fontelos, del Mar Gonzalez & Wei, J. Math. St. 2020

no Hopf Lemma for sign-changing solutions

no Courant's nodal theorem

no monotonicity formulas in bounded domains



Our first result: uniqueness in the *asymptotically local* case ($s \sim 1$)

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Theorem [Dieb, I., Saldaña]

If uniqueness and nondegeneracy hold for

$$\begin{cases} -\Delta u + \lambda u = u^p & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \\ u > 0 & \text{in } \Omega \end{cases} \tag{L}$$

then there exists $\sigma = \sigma(\Omega, \lambda, p) \in (0, 1)$ such that

uniqueness and nondegeneracy hold for problem (F), if $s \in (\sigma, 1]$

Some corollaries

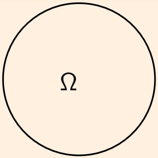
$$\begin{cases} (-\Delta)^s u + \lambda u = u^p & \text{in } \Omega \\ u = 0 & \text{on } \mathbb{R}^N \setminus \Omega \\ u > 0 & \text{in } \Omega \end{cases} \qquad (F)$$

Corollary

If $\Omega = B$, $\lambda > -\lambda_1(B)$ and $1 < p < p_c := \begin{cases} \frac{N+2}{N-2} & \text{if } N \geq 3 \\ +\infty & \text{if } N = 2 \end{cases}$

then there exists $\sigma = \sigma(N, \lambda, p) \in (0, 1)$ such that

uniqueness and nondegeneracy hold for problem (F), if $s \in (\sigma, 1]$



local:

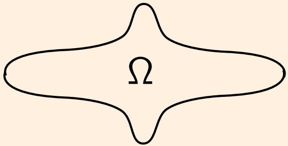
- Gidas, Ni & Nirenberg, Comm. Math. Phys. 1979
- Kwong & Li, TAMS 1992
- Ni & Nussbaum, Comm. Pure Appl. Math. 1985
- Srikanth, Diff. Int. Eqs. 1993
- Zhang, Comm. Part. PDEs, 1992

Some corollaries

$$\begin{cases} (-\Delta)^s u + \lambda u = u^p & \text{in } \Omega \\ u = 0 & \text{on } \mathbb{R}^N \setminus \Omega \\ u > 0 & \text{in } \Omega \end{cases} \qquad (F)$$

Corollary

If $\Omega \subset \mathbb{R}^N$ symmetric and convex with respect to N orthogonal directions, $\lambda = 0$ and



$p > 1$ if $N = 2$

$p \in (\frac{N+2}{N-2} - \varepsilon, \frac{N+2}{N-2})$, $\varepsilon > 0$ small, if $N \geq 3$

then there exists $\sigma = \sigma(\Omega, p) \in (0, 1)$ such that

uniqueness and nondegeneracy hold for problem (F), if $s \in (\sigma, 1]$

local: $N = 2$, $\forall p > 1$ — Dancer, JDE 1988 ($\Omega \sim B$); Damascelli, Grossi & Pacella, AHIP 1999

$N \geq 3$, $p \in (\frac{N+2}{N-2} - \varepsilon, \frac{N+2}{N-2})$ — Grossi, ADE 2000

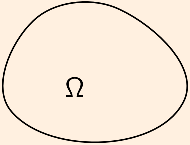
Some corollaries

$$\begin{cases} (-\Delta)^s u + \lambda u = u^p & \text{in } \Omega \\ u = 0 & \text{on } \mathbb{R}^N \setminus \Omega \\ u > 0 & \text{in } \Omega \end{cases} \qquad (F)$$

Corollary

If $\Omega \subset \mathbb{R}^2$ convex, $\lambda = 0$ and $p > 1$
then there exists $\sigma = \sigma(\Omega, p) \in (0, 1)$ such that

problem (F) admits a unique least energy solution
if $s \in (\sigma, 1]$, and it is nondegenerate



local: C.-S. Lin, Manuscr. Math 1994

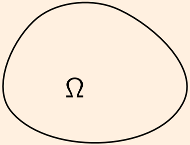
Some corollaries

$$\begin{cases} (-\Delta)^s u + \lambda u = u^p & \text{in } \Omega \\ u = 0 & \text{on } \mathbb{R}^N \setminus \Omega \\ u > 0 & \text{in } \Omega \end{cases} \qquad (F)$$

Corollary

Let $\Omega \subset \mathbb{R}^2$ convex and $\lambda = 0$. There exists $p^* = p^*(\Omega) > 1$ such that for any $p > p^*$ there is $\sigma = \sigma(\Omega, p) \in (0, 1)$ such that

problem (F) admits a unique solution if $s \in (\sigma, 1]$, and it is nondegenerate



local: De Marchis, Grossi, I. & Pacella, J. Math. Pures Appl. 2019

Idea of the proof (uniqueness)

Theorem [Dieb, I., Saldaña]

Assume that uniqueness and nondegeneracy hold for

$$\begin{cases} -\Delta u + \lambda u = u^p & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \\ u > 0 & \text{in } \Omega \end{cases} \qquad (L)$$

then there exists $\sigma = \sigma(\Omega, \lambda, p) \in (0,1)$ such that for $s \in (\sigma, 1]$ uniqueness and nondegeneracy hold for

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PROOF It follows by a standard contradiction argument, ... with some delicate points

Assume by contradiction that (F) admits 2 nontrivial solutions u_n and v_n

$$u_n \neq v_n \qquad \text{for} \qquad s = s_n \to 1^-$$

- we show that, up to a subsequence

$$u_n \to u \quad \text{in } L^\infty(\Omega) \quad \text{as } s_n \to 1$$
$$v_n \to v \quad \text{in } L^\infty(\Omega) \quad \text{as } s_n \to 1$$

where u, v are nontrivial positive solutions of (L)

- by assumption $u = v = u_*$
  where u_* is the unique positive solution of (L)
  hence
$$V_n := \int_0^1 p(tu_n + (1-t)v_n)^{p-1} dt \;\to pu_*^{p-1}$$
- Define
$$w_n := \frac{u_n - v_n}{\|u_n - v_n\|_{L^\infty}} \neq 0$$
  which solve
$$\begin{cases} (-\Delta)^{s_n} w_n = V_n w_n - \lambda w_n & \text{in } \Omega \\ w_n = 0 & \text{on } \mathbb{R}^N \setminus \Omega \\ \|w_n\|_{L^\infty} = 1 & \text{in } \Omega \end{cases}$$
- we can prove that
$$w_n \to w \neq 0 \quad \text{in } L^\infty(\Omega)$$
  which solves
$$\begin{cases} -\Delta w + \lambda w = pu_*^{p-1} w & \text{in } \Omega \\ w = 0 & \text{on } \partial\Omega \end{cases}$$
  against the nondegeneracy assumption on u_* □

- by assumption $\quad u = v = u_*$
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then there exists $\sigma = \sigma(\Omega, \lambda, p) \in (0,1)$ such that for $s \in (\sigma, 1]$ uniqueness and nondegeneracy hold for

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$$\left.\begin{array}{lll} u_n \to u & \text{in } L^\infty(\Omega) & \text{as } s_n \to 1 \\ v_n \to v & \text{in } L^\infty(\Omega) & \text{as } s_n \to 1 \end{array}\right\} \longleftarrow$$

where u, v are nontrivial positive solutions of (L)

SUBTLE ISSUE in the proof: the nonlocal-to-local transition

$$\begin{cases} (-\Delta)^s u + \lambda u = u^p & \text{in } \Omega \\ u = 0 & \text{on } \mathbb{R}^N \setminus \Omega \\ u > 0 & \text{in } \Omega \end{cases} \qquad (F)$$

Let u_s be a solution of (F). Then up to a subsequence

$$u_s \to u \quad \text{in } L^\infty(\Omega) \quad \text{as } s \to 1^-$$

where u is a (positive) solution of (L)

we need *uniform-in-s a-priori* bounds, for $s \sim 1$:

There exists $\sigma \in (0,1)$ and $C = C(\lambda, p, \Omega, \sigma) > 0$ (independent on s) such that

$$\|u_s\|_{L^\infty(\mathbb{R}^N)} \leq C$$

for any $s \in (\sigma, 1)$ and any solution u_s of (F)

Proof It relies on a blow-up argument.
Since the blow-up parameter is the Laplacian's exponent s, we need to control (as $s \to 1$) the constants which appear from regularity estimates.

Uniqueness for fractional *asymptotically linear* problems ($p \sim 1$)

$$
\begin{cases}
(-\Delta)^s u + \lambda u = u^p & \text{in } \Omega \\
u = 0 & \text{on } \mathbb{R}^N \setminus \Omega \\
u > 0 & \text{in } \Omega
\end{cases}
\qquad (F)
$$

Theorem [Dieb, I., Saldaña]

Let $s \in (0,1)$. There exists $\delta = \delta(\Omega, \lambda, s)$ such that uniqueness and nondegeneracy hold for problem (F), if $p \in (1, 1+\delta)$

THE END

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Asymptotic characterization of positive solutions:

- De Marchis, I. & Pacella, J. Europ. Math. Soc. 2015
- De Marchis, I. & Pacella, J. Fix Point Th. Appl. 2017
- De Marchis, Grossi, I. & Pacella, Arch. der Math. 2018

Uniqueness of positive solutions in convex domains & Morse index for 1-peak solutions:

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Local uniqueness, non-degeneracy and Morse index for k-peaks, solutions, $\forall k \geq 1$

- Grossi, I., Luo & Yan, J. Math. Pures Appl. 2022
- I., Luo & Yan, preprint

The fractional case

- Dieb, I. & Saldana, Nonlinear Analysis, to appear

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Thank you!

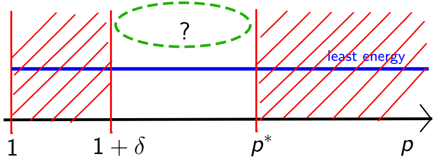
Some open problems:

$$\begin{cases} -\Delta u + \lambda u = u^p & \text{in } \Omega \subset \mathbb{R}^N \text{ convex} \\ u = 0 & \text{on } \partial\Omega \\ u > 0 & \text{in } \Omega \end{cases} \qquad (L)$$

- $N \geq 3$

- the case $\lambda \neq 0$ in $N = 2$

- $p \in (1+\delta, p^\star)$, when $\lambda = 0$, $N = 2$



no quantitative information about δ and $p^\star$

Some open problems:

$$\begin{cases} (-\Delta)^s u + \lambda u = u^p & \text{in } \Omega \\ u = 0 & \text{on } \mathbb{R}^N \setminus \Omega \\ u > 0 & \text{in } \Omega \end{cases} \qquad (F)$$

- $\Omega = B$, for any $s \in (0,1)$