Min-Oo conjecture for fully nonlinear conformally invariant equations

Joint work with E. Barbosa and M. P. Cavalcante

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Introduction

Introduction:

In 1979, Schoen and Yau, using minimal surface techniques, proved:

Positive Mass Theorem (Schoen-Yau): Let (M^n, g), $n \leq 7$, be an asymptotically flat manifold with non-negative scalar curvature, then the ADM mass is non-negative. Furthermore, if the ADM mass is zero, then M must be isometric to the Euclidean space.

- **Witten** gave a different proof that works for any dimension under the assumption that M is spin.
- **Schoen-Yau** announced the validity of the **Positive Mass Theorem** with no dimensional restrictions.

Introduction:

The Positive Mass Theorem implies the following rigidity result:

Theorem (Miao): Suppose that g is a smooth metric on the unit ball $B^n \subset \mathbb{R}^n$ with the following properties:

- The scalar curvature of g is non-negative,
- $g_{|\partial B^n}$ agrees with the standard metric on ∂B^n,
- the mean curvature of ∂B^n with respect to g is at least $n-1$.

Then, g is isometric to the standard metric on B^n.

Similar results for asymptotically hyperbolic manifolds (**Min-Oo, Anderson-Dahl, Chrusciel-Herzlich....**)

Introduction:

In 1995, Min-Oo, inspired by the Positive Mass Theorem, conjectured:

Min-Oo Conjecture: Let (M^n, g) be a compact manifold with boundary $\partial M = \Sigma$ so that $R(g) \geq n(n-1)$, Σ is isometric to $\mathbb{S}^{n-1}$ and totally geodesic, then (M, g) is isometric to the hemisphere $\mathbb{S}^n_+$.

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- **False in general:** **Brendle-Marques-Neves** showed the existence of such non-trivial metric in the hemisphere, however such metric is not conformal to the standard one.
- **True:** For locally conformally flat manifolds. **Hang-Wang, Spiegel**.

Introduction (Schouten Tensor):

On a Riemannian manifold (M^n, g), $n > 2$, we have

$$\mathrm{Riem} = \mathbf{W_g} + \mathrm{Sch}_{\mathbf{g}} \odot \mathbf{g},$$

where Riem is the Riemann curvature tensor, W_g is the Weyl tensor, $\odot$ is the Kulkarni-Nomizu product, and

$$\mathrm{Sch}_{\mathbf{g}} := \frac{\mathbf{1}}{\mathbf{n-2}} \left(\mathrm{Ric}_{\mathbf{g}} - \frac{\mathbf{R(g)}}{\mathbf{2(n-1)}} \mathbf{g} \right)$$

is the **Schouten tensor**.

Remark: $\mathbf{Trace}(\mathrm{Sch}_g) = C(n)\, R(g)$.

Introduction:

We can rewrite:

Spiegel, Hang-Wang Theorem: *Let (M, g) be a loc. conformally flat manifold so that*

$$\frac{2}{n}\mathrm{Trace}(\mathrm{Sch}_g) \geq 1 \ \textit{in } M$$

and ∂M is isometric to $\mathbb{S}^{n-1}$ and totally geodesic. Then, M is the hemisphere $\mathbb{S}^n_+$.

(Recall that $\mathrm{Sch}_{g_0} = \frac{1}{2}g_0$, g_0 standard metric of $\mathbb{S}^n$)

Introduction:

We focus on a more general type of equation, a rich subject in the last few years: conformally invariant equations. More precisely:

Given a smooth functional $f(x_1, \ldots, x_n)$*, does there exist a conformal metric* $g = e^{2\rho} g_0$ *in* $\mathbb{S}^n_+$ *such that the eigenvalues* λ_i *of its Schouten tensor satisfy*

$$f(\lambda_1, \ldots, \lambda_n) \geq b \text{ in } \mathbb{S}^n_+,$$

imposing restrictions along the boundary?

Min-Oo Problem: $f(x_1, \ldots, x_n) = \sum_{i=1}^n x_i$.

Introduction (Elliptic Functionals):

We define the curvature function $f : \mathbb{R}^n \to \mathbb{R}$, considered as a function on the eigenvalues of the Schouten tensor, is elliptic for conformal metrics in the following way. Set

$$\Gamma_1 := \{(x_1, \cdots, x_n) : \sum_{i=1}^{n} x_i > 0, i = 1, 2, \cdots, n\}$$

and

$$\Gamma_n := \{(x_1, \cdots, x_n) : x_i > 0, i = 1, 2, \cdots, n\}$$

Introduction (Elliptic Functionals):

Let $\Gamma \subset \mathbb{R}^n$ be a symmetric open convex cone and $f \in C^1(\Gamma) \cap C^0(\overline{\Gamma})$. Then, (f, Γ) is **an elliptic datum** if

1. $\Gamma_n \subset \Gamma \subset \Gamma_1$,
2. f is symmetric,
3. $f > 0$ in Γ,
4. $f|_{\partial\Gamma} = 0$,
5. for all $x \in \Gamma$ it holds $\nabla f(x) \in \Gamma_n$,
6. f is homogeneous of degree 1,
7. $f(1, \ldots, 1) = 2$.

In this talk:

A **rigidity** result for non-degenerate fully nonlinear Min-Oo type problems

Theorem (Barbosa, Cavalcante & –): *Let* $g = e^{2\rho} g_0$ *be a conformal metric in the hemisphere* $\mathbb{S}^n_+$*, and let* (f, Γ) *be an elliptic datum. Assume that*

i) $f(\lambda(p)) \geq 1$*, for* $\lambda(p) \in \Gamma$ *and* $p \in \mathbb{S}^n_+$*,*

ii) The boundary $\partial\mathbb{S}^n_+$ *with respect to* g *is isometric to* $\partial\mathbb{S}^n_+$*.*

Then $g = \Phi^* g_0$*, where* $\Phi \in \mathrm{Conf}(\mathbb{S}^n)$ *so that* $\Phi(\mathbb{S}^n_+) = \mathbb{S}^n_+$*.*

The objective...

So far, this is a PDE problem. Here, we show a geometric point of view...

$$\begin{cases} f\left(-\nabla^2\rho + d\rho\otimes d\rho + \frac{1}{2}(1-|\nabla\rho|^2)g_0\right) \geq 1 \\ \rho = 0 \end{cases}$$

The objective...

We use a **bridge** between the theories of:

- **Conformal metrics:** $g = e^{2\rho} g_0$ on $\Omega \subset \mathbb{S}^n$.
- **Hypersurfaces:** $M^n \subset \mathbb{H}^{n+1}$ with regular hyperbolic Gauss map.

E.-Gálvez-Mira, E.-Bonini-Qing, E.-Abantos:

There is a global correspondence between properly immersed horospherically concave hypersurfaces $\phi : M^n \to \mathbb{H}^{n+1}$ and complete conformal metrics $e^{2\rho} g_{\mathbb{S}^n}$ on domains Ω of the sphere $\mathbb{S}^n$.

Hypersurfaces in $\mathbb{H}^{n+1}$

Hypersurfaces in $\mathbb{H}^{n+1}$: Ball Model

Let $\phi : M^n \to \mathbb{H}^{n+1} \equiv (\mathbb{B}^{n+1}, g_{-1})$ immersed and oriented, η its unit normal.

The **hyperbolic Gauss map**

$$G : M^n \to \mathbb{S}^n$$

of ϕ is defined as follows: for every $p \in M^n$, $G(p) \in \mathbb{S}^n$ is the point at infinity of the unique horosphere $\mathcal{H}_p$ in $\mathbb{H}^{n+1}$ passing through $\phi(p)$ and with the inner unit normal the same as $-\eta(p)$ at $\phi(p)$.

Hypersurfaces in $\mathbb{H}^{n+1}$: Ball Model

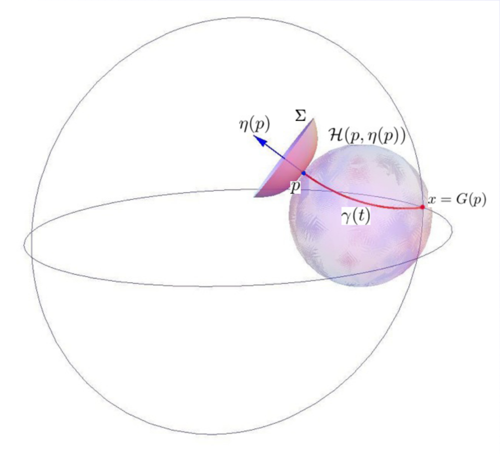


Figure 1: Hyperbolic Gauss Map

Hypersurfaces in $\mathbb{H}^{n+1}$: Ball Model

Remark: If $\phi(M^n) = \mathcal{H}$ is a horosphere and η its outward orientation, then its hyperbolic Gauss map G is constant, i.e., $G(M^n) = x \in \mathbb{S}^n$.

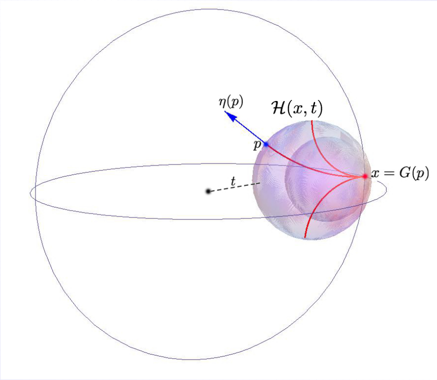


Figure 2: Horospheres

Hypersurfaces in $\mathbb{H}^{n+1}$: Hyperboloid Model

Let $\phi : M^n \to \mathbb{H}^{n+1} \subset \mathbb{L}^{n+2}$ immersed and oriented, $\eta : M^n \to \mathbb{S}_1^{n+1} \subset \mathbb{L}^{n+2}$ its unit normal, then, we define the **associated light cone map** as

$$\psi := \phi - \eta : M^n \to \mathbb{N}_+^{n+1} \subset \mathbb{L}^{n+2}$$

If we write $\psi = (\psi_0, \ldots, \psi_{n+1})$, consider the map G (**the hyperbolic Gauss map**) given by:

$$G = \frac{1}{\psi_0}(\psi_1, \ldots, \psi_{n+1}) : M \to \mathbb{S}^n$$

So, if we label $e^\rho := \psi_0$ (**the hyperbolic support function**), we get

$$\psi = e^\rho(1, G).$$

Hypersurfaces in $\mathbb{H}^{n+1}$: Hyperboloid Model

If $\phi(M^n) = \mathcal{H}$ is a horosphere

$$\mathcal{H} = \left\{ y \in \mathbb{H}^{n+1} \ : \ \langle y, a \rangle = 1 \right\} \text{ where } \langle a, a \rangle = 0,$$

and $\eta(y) = y - a$ its outward orientation, then its associated light cone map ψ is constant, i.e.,

$$\psi = v = e^{\rho}(1, x) \in \mathbb{N}_+^{n+1},$$

where

- $\rho := dist(0, \phi(M^n))$
- $x \in \mathbb{S}^n$

Properties of the horospherical metric

(A) Since $\psi = e^{\rho}(1, G)$, it satisfies

$$g = \langle d\psi, d\psi \rangle = e^{2\rho} \langle dG, dG \rangle_{\mathbb{S}^n}.$$

Hence, g is a Riemannian metric iff G is a local diffeomorphism.
g is the **horospherical metric**.

(B) In particular, the horospherical metric g is conformally flat.

We need a geometric property to ensure that g is a Riemannian metric

Horospherically concave hypersurfaces in $\mathbb{H}^{n+1}$

Let $M^n \subset \mathbb{H}^{n+1}$ be immersed and oriented. The following conditions are equivalent:

1. M^n lies around any point p **strictly** at the concave side of the tangent horosphere at p whose normal points into the concave side of the tangent horosphere.

2. $\kappa_i > -1$ hold simultaneously at every point.

$$g = \langle d\psi(e_i), d\psi(e_j) \rangle = (1 + \kappa_i)(1 + \kappa_j)\delta_{ij}$$

3. $G : M^n \to \mathbb{S}^n$ is a **local diffeomorphism**.

4. Its horospherical metric g is **Riemannian**.

The bridge principle

The bridge

Let $\phi : U \subset (\mathbb{S}^n, g_0) \to \mathbb{H}^{n+1}$ be a locally horospherical hypersurface with $G(x) = x$ and hyperbolic support function e^ρ. Then, it holds

$$\begin{aligned}
\phi &= \frac{e^\rho}{2}\left(1 + e^{-2\rho}\left(1 + ||\nabla^{g_0}\rho||^2_{g_0}\right)\right)(1, x) + e^{-\rho}(0, -x + \nabla^{g_0}\rho) \\
I(e_i, e_j) &= \frac{e^{-2\rho}}{4}\left(g(e_i, e_j) - 2\mathrm{Sch}_g(e_i, e_j)\right)^2 \\
II &= I - \frac{1}{2}g + \mathrm{Sch}_g
\end{aligned}$$

Here: $g = e^{2\rho}g_0 \equiv$ horospherical metric and $e_1, \ldots, e_n \in T_x\mathbb{S}^n$ orthonormal frame w.r.t. g_0, such that $\nabla^{g_0}_{e_i} e_j = 0$.

From hypersurfaces in $\mathbb{H}^{n+1}$ to conformal metrics

Let $\phi : M^n \to \mathbb{H}^{n+1}$ be horospherically concave with $G(x) = x$.

Then, its horospherical metric $g = e^{2\rho} g_0$ is a conformal metric on $G(M^n) \subset \mathbb{S}^n$ whose Schouten tensor eigenvalues are

$$\lambda_i = \frac{1}{2} - \frac{1}{1 + \kappa_i}.$$

Moreover, it holds (when we view $\mathbb{H}^{n+1} \subset \mathbb{L}^{n+2}$)

$$\phi = \frac{e^{\rho}}{2} \left(1 + e^{-2\rho} \left(1 + ||\nabla^{g_0} \rho||^2_{g_0}\right)\right) (1, x) + e^{-\rho}(0, -x + \nabla^{g_0} \rho).$$

From conformal metrics to hypersurfaces in $\mathbb{H}^{n+1}$

Let $g = e^{2\rho} g_0$ denote a conformal metric having Schouten tensor eigenvalues $\{\lambda_1, \ldots, \lambda_n\}$. Assume

$$\lambda_i < 1/2 \qquad \text{(or equivalently, if } g/2 - \mathrm{Sch}_g > 0\text{)}.$$

Then the map $\phi : \mathbb{S}^n \to \mathbb{H}^{n+1} \subset \mathbb{L}^{n+2}$ given by

$$\phi = \frac{e^{\rho}}{2}\left(1 + e^{-2\rho}\left(1 + ||\nabla^{g_0}\rho||^2_{g_0}\right)\right)(1, x) + e^{-\rho}(0, -x + \nabla^{g_0}\rho)$$

is a horospherical ovaloid with horospherical metric g, such that

$$\lambda_i = \frac{1}{2} - \frac{1}{1 + \kappa_i}.$$

Remark: the condition in blue can always be attained by a dilation.

Geodesic Flow and Dilatations

Let $\phi : \Omega \subset \mathbb{S}^n \to \mathbb{H}^{n+1}$ be an oriented horospherical concave hypersurface so that $G(x) = x$.

Let $\{\phi_t\}_{t\in\mathbb{R}}$ denote the **geodesic flow of ϕ** in hyperbolic space $\mathbb{H}^{n+1}$, that is,

$$\phi_t(x) := \exp_{\phi(x)}(-t\eta(x)) = \phi(x)\cosh t - \eta(x)\sinh t,$$

where exp denotes the exponential map for the hyperbolic metric.

The hyperbolic Gauss maps G_t **remain invariant under this flow** and the horospherical metric of ϕ_t is $g_t := e^{2t}g$, where g is the horospherical metric of ϕ.

Geodesic Flow and Dilatations

Conversely, given a conformal metric $g := e^{2\rho} g_{\mathbb{S}^n}$ on $\mathbb{S}^n$, one considers a family of rescaled metric $g_t = e^{2t} g$.

- Choosing t_0 large so that $e^{-t_0}\mathrm{Sch_g} < \frac{1}{2}$, it follows that the one parameter family of hypersurfaces

$$\phi_t = \frac{e^{\rho+t}}{2}\left(1 + e^{-2\rho-2t}\left(1 + |\nabla\rho|^2\right)\right)(1, x) + e^{-\rho-t}(0, -x + \nabla\rho)$$

  for $t > t_0$ consists of immersed, horospherically convex hypersurfaces with hyperbolic Gauss map $G_t(x) = x$ the identity.

- The eigenvalues of the Schouten tensor change as

$$\lambda_i^t = e^{-t}\lambda_i.$$

Isometries and Conformal Diffeomorphism

$T \in \mathrm{Iso}(\mathbb{H}^{n+1})$ one-to-one $\Phi \in \mathrm{Conf}(\mathbb{S}^n)$

Let $\Sigma \subset \mathbb{H}^{n+1}$ be horospherically concave with horospherical metric g, then the horospherical metric $\tilde{g}$ associated to $\tilde{\Sigma} = T(\Sigma)$ is given by $\tilde{g} = \Phi^* g$. Viceversa, given a conformal metric g on a subdomain of the sphere with associated hypersurface Σ, given by the representation formula under the appropriated conditions, the associated horospherically concave hypersurface $\tilde{\Sigma}$ associated to the conformal metric $\tilde{g} = \Phi^* g$ is given by $\tilde{\Sigma} = T(\Sigma)$.

Fully nonlinear
Min-Oo Conjecture

Min-Oo Conjecture

Theorem (Barbosa, Cavalcante & –): *Let* $g = e^{2\rho} g_0$ *be a conformal metric in the hemisphere* $\mathbb{S}^n_+$, *and let* (f, Γ) *be an elliptic datum. Assume that*

i) $f(\lambda(p)) \geq 1$, *for* $\lambda(p) \in \Gamma$ *and* $p \in \mathbb{S}^n_+$,

ii) The boundary $\partial\mathbb{S}^n_+$ *with respect to* g *is isometric to* $\partial\mathbb{S}^n_+$.

Then $g = g_0$ *is the standard metric.*

Ideas involved in the proof

Idea of the proof...

Claim A: *Given a conformal metric on* $\overline{\mathbb{S}^n_+}$ *super-solution to an elliptic datum* (f, Γ), *there exists* $t_0 > 0$ *so that the dilated metric* $g_t := e^{2t} g$, $t \geq t_0$, *satisfies:*

- g_t *is a super-solution to the elliptic problem*

$$\begin{cases} f_t(\lambda^t(p)) \geq e^{-t}, \quad \lambda^t(p) \in \Gamma_t, \; p \in \mathbb{S}^n_+, \\ \rho_t := \rho + t = t, \quad \text{on } \partial\mathbb{S}^n_+. \end{cases}$$

where $f_t(\lambda^t(p)) = f(e^{-t}\lambda(p))$ *and* $\Gamma_t = e^{-t}\Gamma$.

- (P1) $\rho = 0$ *along* $\partial\mathbb{S}^n_+$.

- (P2) $|\mathrm{Sch}_{g_t}| < 1/2$.

Claim A

(P1) By Obata Theorem, up to a conformal diffeomorphism, we can assume that $\rho = 0$ along the boundary.

(P2) This follows from

$$\lambda_i^t = e^{-t}\lambda_i,\ i = 1, \ldots, n.$$

Idea of the proof...

Claim B: *Given a conformal metric on* $\overline{\mathbb{S}^n_+}$, *there exists a horospherically concave* **embedded** *hypersurface*

$$\phi_t(\overline{\mathbb{S}^n_+}) =: \Sigma_t \subset \mathbb{H}^{n+1}$$

with compact boundary $\partial\Sigma_t$ *such that* Σ_t *and* $\partial\Sigma_t$ *are topologically* $\mathbb{S}^n_+$ *and* $\partial\mathbb{S}^n_+ = \mathbb{S}^{n-1}$ *respectively.*

Idea of the proof...

Theorem (Abanto-E.): *Let* $\Omega \subset \mathbb{S}^n$ *be an open domain and* $\partial\Omega = \mathcal{V}_1 \cup \mathcal{V}_2$, $\mathcal{V}_1 \cap \mathcal{V}_2 = \emptyset$. *Let* $\rho \in C^{2,\alpha}(\Omega \cup \mathcal{V}_1)$ *be such that* $\sigma = e^{-\rho} \in C^{2,\alpha}(\Omega \cup \mathcal{V}_1)$ *satisfies:*

1. σ^2 *can be extended to a* $C^{1,1}$ *function on* $\overline{\Omega}$.
2. $|\nabla\sigma|^2$ *can be extended to a Lipschitz function on* $\overline{\Omega}$.

Then, there is $t_0 > 0$ *such that for all* $t > t_0$ *the map* $\phi_t : \Omega \cup \mathcal{V}_1 \to \mathbb{H}^{n+1}$ *associated to* $\rho_t = \rho + t$ *is an embedded horospherically concave hypersurface.*

$$\Phi_\epsilon(x) = x - \frac{\epsilon}{1 + \epsilon\left[\sigma^2(x) + |\nabla\sigma|^2(x)\right]}\left(2\sigma^2(x)\,x + \nabla\sigma^2(x)\right) \in \mathbb{B}^{n+1}$$

$$\epsilon = e^{-2t}$$

Idea of the proof...

Claim C: *Let* $\gamma : \mathbb{R} \to \mathbb{H}^{n+1}$ *be the complete geodesic (parametrized by arc-length) joining the south and north poles. Let* $\mathcal{C}_t$ *be the cylinder in* $\mathbb{H}^{n+1}$ *of axis* γ *and radius* t*. Then,* $\partial\Sigma_t$ *lies outside the interior of* $\mathcal{C}_t$ *and* $\partial\Sigma_t \cap \mathcal{C}_t \subset P$*, that is, at points at the boundary where* Σ_t *is orthogonal to* P*.*

A direct calculation shows that if $x \in \partial\mathbb{S}^n_+$ then:

$$d_{\mathbb{H}^{n+1}}(\phi_t(x), \gamma(s)) \geq \operatorname{arc}\cosh\left(\cosh(\rho_t) + \frac{e^{\rho_t}}{2}\left(\frac{\partial \rho_t}{\partial \nu}\right)^2\right)$$
$$\geq t$$

Claim C

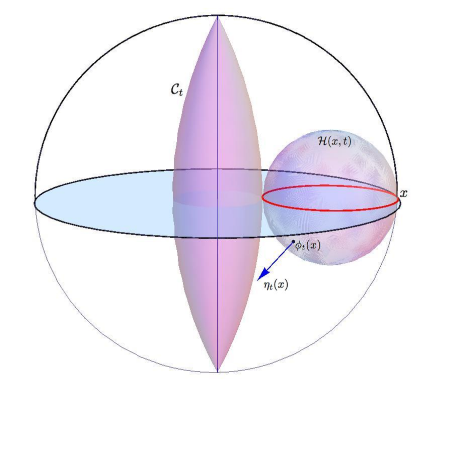


Figure 3: The boundary $\partial\Sigma_t$ lies in the envelope by horospheres at distance t from the origin

Finishing the proof....

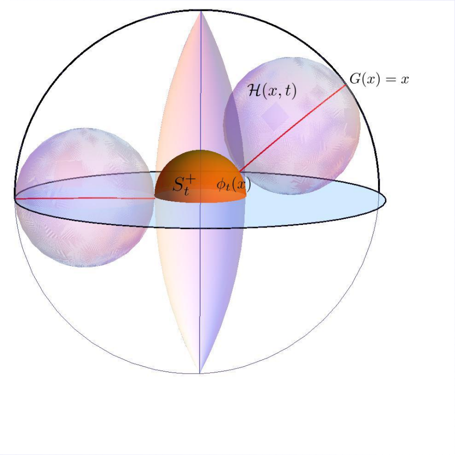


Figure 4: Consider the half-sphere $S_t^+ = S_t \cap \overline{P^+}$ and observe that S_t^+ is orthogonal to P along the boundary ∂S_t^+ and its horospherical metric is given by $\tilde{g}_t = e^{2t} g_0$ and $\tilde{\lambda}_t := e^{-t}/2$.

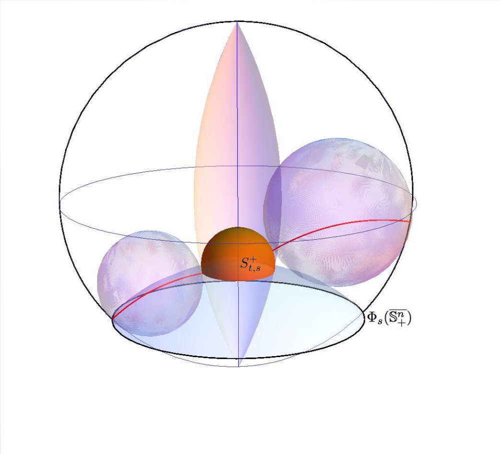
Finishing the proof....

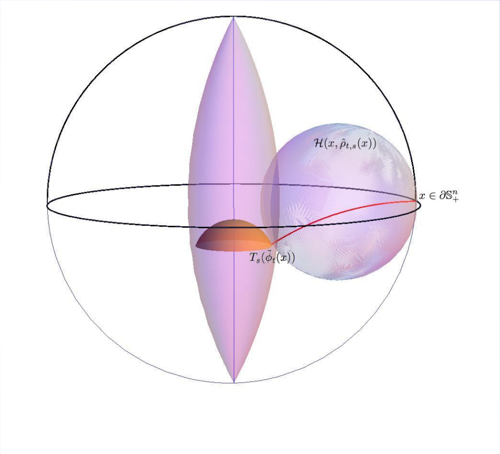
- Let $T_s : \mathbb{H}^{n+1} \to \mathbb{H}^{n+1}$ be the hyperbolic translation at distance s along γ so that $T_s((1, \mathbf{0})) = \gamma(s)$, an isometry of $\mathbb{H}^{n+1}$. It is clear that $T_s(S_t^+ \setminus \partial S_t^+) \cap \partial\Sigma_t = \emptyset$, for all $s \in \mathbb{R}$.

- Let $\Phi_s \in \mathrm{Conf}(\mathbb{S}^n)$ be the unique conformal diffeomorphism associated to T_s. Set $S_{t,s} := T_s(S_t)$ for all $s \in \mathbb{R}$, then the horospherical metric associated to $S_{t,s}$ is given by $\tilde{g}_{t,s} = e^{2t}\Phi_s^* g_0$ in $\mathbb{S}^n$ and denote by $\tilde{\rho}_{t,s} \in C^\infty(\mathbb{S}^n)$ the horospherical support function associated to $S_{t,s}$, i.e, $\tilde{g}_{t,s} = e^{2\tilde{\rho}_{t,s}} g_0$.

- Let $\hat{g}_{t,s}$ be the restriction of $\tilde{g}_{t,s}$ to $\overline{\mathbb{S}^n_+}$, i.e., ${\tilde{g}_{t,s}}_{|\mathbb{S}^n_+} = \hat{g}_{t,s}$, and $\hat{\rho}_{t,s}$ the restriction of $\tilde{\rho}_{t,s}$ to $\overline{\mathbb{S}^n_+}$.

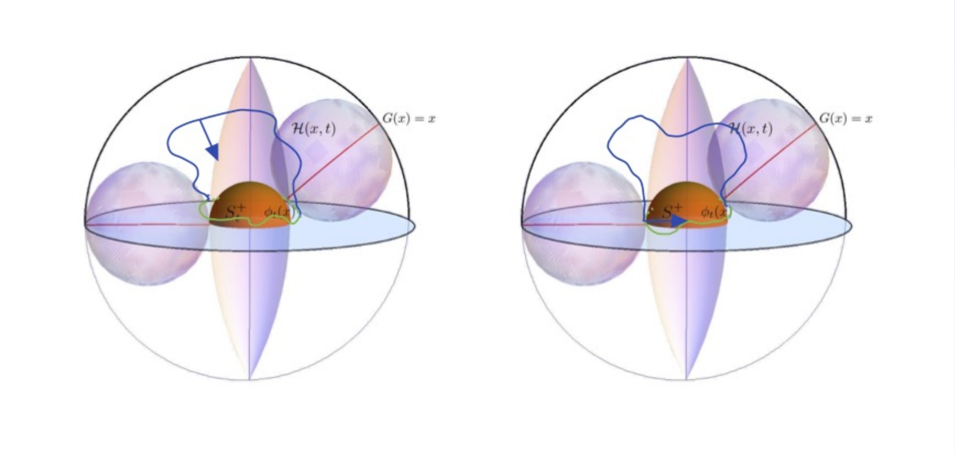
Finishing the proof....





Finishing the proof....

Claim F: $\rho_t \geq \hat{\rho}_{t,s_0}$ *in* $\overline{\mathbb{S}^n_+}$.



Finishing the proof....

Note that, since the elliptic data is homogeneous of degree one, we have that g_t satisfies

$$f(\lambda_{g_t}(p)) = f(e^{-t}\lambda_g(p)) \geq e^{-t} \text{ for all } p \in \mathbb{S}^n_+$$

and the horospherical metric of S^+_{t,s_0} satisfies

$$f(\lambda_{\hat{g}_{t,s_0}}(p)) = f(e^{-t}\lambda_{g_0}(p)) = e^{-t} f(1/2, \ldots, 1/2) = e^{-t} \text{ for all } p \in \mathbb{S}^n_+,$$

that is

$$f(\lambda_{g_t}(p)) \geq f(\lambda_{\hat{g}_{t,s_0}}(p)) \text{ for all } p \in \mathbb{S}^n_+$$

and

$$\rho_t \geq \hat{\rho}_{t,s_0} \text{ in } \overline{\mathbb{S}^n_+} \text{ and } \rho_t(\bar{x}) = \hat{\rho}_{t,s_0}(\bar{x})$$

By Li-Li's Maximum Principle, we get

$$\rho_t \equiv \hat{\rho}_{t,s_0} \text{ in } \overline{\mathbb{S}^n_+}$$

Thanks you!