# A Gradient Bound and a Liouville Theorem for Nonlinear Poisson Equations

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## Introduction

In this note we prove the following theorem.

THEOREM I. *Let $F \in C^2(\mathbb{R})$ be a non-negative function and $u \in C^3(\mathbb{R}^n)$ a solution in all of $\mathbb{R}^n$ of the equation*

$$\Delta u = f(u),$$

*where $f = F'$ is the first derivative of $F$. If $u$ is bounded on $\mathbb{R}^n$ and there exists $x_0 \in \mathbb{R}^n$ such that $F(u(x_0)) = 0$, then $u$ is constant.*

The point of this result is that no convexity is assumed about $F$. Indeed, it is well known by a theorem of J. Serrin [5] that, when $F$ is convex, then any bounded entire solution of $\Delta u = f(u)$ is constant. On the other hand, if we drop the assumption $F(u(x_0)) = 0$, it is very easy, as we show in a moment, to give examples of bounded entire solutions which are not constant. Here and in the following, by entire solution we mean solution in the whole $\mathbb{R}^n$.

A nonlinear Poisson equation to which our result applies is $\Delta u = \sin u$ (it suffices to choose $F(t) = 1 - \cos t$) and, in this case, Theorem I could be formulated as follows: any bounded nonconstant entire solution of $\Delta u = \sin u$ lies entirely in some strip $2k\pi < u < 2(k+1)\pi$ with $k \in \mathbb{Z}$. If we observe that $u(x_1, \cdots, x_n) = 4 \arctan \exp\{x_1\}$ is a bounded entire solution of $\Delta u = \sin u$ such that $\inf u = 0$ and $\sup u = 2\pi$, then it is clear that in some sense our result is optimal.

In the physical literature some "entire" solutions of $\Delta u = \sin u$ with oscillation in $\mathbb{R}^n$ equal to $4\pi, 6\pi, \cdots$ are constructed, for example (see G. Leibbrandt [2]), in $\mathbb{R}^2$ with polar coordinates

$$u(\rho, \theta) = 4 \arctan\left[\frac{\exp\{\rho\cos\theta\} - \exp\{\rho\sin\theta\}}{1 - \exp\{\rho(\cos\theta + \sin\theta)\}}\right].$$

These solutions appear to contradict our theorem but actually they are not entire solutions in our sense. In fact, even if formally we have $\Delta u = \sin u$ in the whole $\mathbb{R}^n$, $u$ is not defined as single-valued function on the straight line $\cos\theta + \sin\theta = 0$.

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Note that, even if we consider arctan as a multiple-valued function, $u$ is singular at the origin.

The proof of Theorem I is an immediate consequence of the following gradient bound.

THEOREM II. *Let $F \in C^2(\mathbb{R})$ be a non-negative function and $u \in C^3(\mathbb{R}^n)$ be a bounded entire solution of the equation $\Delta u = f(u)$. Then $|Du|^2(x) \leq 2F(u(x))$ for every $x \in \mathbb{R}^n$.*

A proof of Theorem II, under some additional conditions on $F$ and, in particular, under the assumption $F(u(x)) \neq 0$ for every $x \in \mathbb{R}^n$, was obtained in a joint paper with S. Mortola [3] by using the gradient bounds proved by L. A. Peletier and J. Serrin [4]. Attempts to eliminate the condition $F(u(x)) \neq 0$ in order to obtain the Liouville theorem (Theorem I) remained unsuccessful until B. Kawohl pointed out to us the work of L. E. Payne, R. P. Sperb, I. Stakgold and others (see Sperb's book [6]) about the so-called $P$-functions for the equation $\Delta u = f(u)$; the simplest example is just $|Du|^2(x) - 2F(u(x))$. Thus, Theorem II can be regarded as a theorem about $P$-functions in unbounded domains without boundary conditions on $u$ except the boundedness, while, to our knowledge, the results of the authors quoted above are mainly concerned with bounded domains and given boundary conditions (Dirichlet's, Neumann's, Robin's, etc.). However, the proof of Theorem II we give here does not depend on known results about the $P$-functions: we essentially apply to a $P$-function the classical technique of Bernstein for obtaining global gradient bounds via the maximum principle.

## 1. Proof of the Theorems

Let us begin by proving Theorem II. Since $u$ is bounded, we have $\inf_{\mathbb{R}^n}|Du|^2 = 0$. Fix $\delta > 0$; since the statement of Theorem II is translation-invariant, we may choose the origin in $\mathbb{R}^n$ so that

$$|Du|^2(0) < \delta. \tag{1}$$

Define on $\mathbb{R}^n$

$$P(x) = |Du|^2(x) - 2F(u(x)).$$

It is easy to deduce from the inequality

$$|D_i u|(x_0) \leq \frac{n}{d} \sup_Q |u| + \tfrac{1}{2} d \sup_Q |\Delta u|, \qquad i = 1, \cdots, n,$$

which holds for any cube $Q$ with edge $d$ and center $x_0$ (e.g. see D. Gilbarg and N. S. Trudinger [1], Section 3.4), that $|Du|$ is bounded on $\mathbb{R}^n$; thus $P$ also is bounded on $\mathbb{R}^n$.

Since $P$ is of class $C^2$, following R. Sperb [6], Chapter 5, we obtain that, for every $i = 1, \cdots, n$,

$$D_i P = 2 \sum_{j=1}^{n} D_j u D_{ij} u - 2f(u) D_i u;$$

hence

$$D_i P + 2f(u) D_i u = 2 \sum_{j=1}^{n} D_j u D_{ij} u \leqq 2|Du| \left[ \sum_{j=1}^{n} (D_{ij} u)^2 \right]^{1/2}. \tag{2}$$

Consequently, using $\Delta u = f(u)$, we find

$$\begin{aligned} \Delta P &= 2 \sum_{i,j=1}^{n} (D_{ij} u)^2 + 2 \sum_{j=1}^{n} D_j u D_j(\Delta u) - 2f'(u)|Du|^2 - 2f^2(u) \\ &= 2 \sum_{i,j=1}^{n} (D_{ij} u)^2 - 2f^2(u), \end{aligned}$$

so that, by (2),

$$\begin{aligned} |Du|^2 \Delta P &\geqq \tfrac{1}{2} \sum_{i=1}^{n} (D_i P + 2f(u) D_i u)^2 - 2f^2(u)|Du|^2 \\ &= \tfrac{1}{2}|DP|^2 + 2f(u) Du \cdot DP. \end{aligned} \tag{3}$$

Now, let us fix $\varepsilon > 0$, $\rho_0 > 0$ and suppose we have constructed a function $\eta = \eta_{\varepsilon, \rho_0}\colon [\rho_0, +\infty[ \to \mathbb{R}$ of class $C^2$ with the following properties:

$$\eta(\rho_0) = 1, \quad \eta > 0, \quad \eta' < 0, \quad \lim_{\rho \to +\infty} \eta(\rho) = 0, \tag{4}$$

$$\lim_{\varepsilon \to 0^+} \eta_{\varepsilon, \rho_0}(\rho) = 1 \quad \text{for all} \quad \rho \geqq \rho_0, \tag{5}$$

$$\frac{\eta^2}{\eta'^2} \left( \frac{2\eta'^2}{\eta} - \frac{M}{\varepsilon} \eta' - \eta'' - \frac{(n-1)\eta'}{\rho} \right) \leqq \frac{\varepsilon}{L} \quad \text{for all} \quad \rho \geqq \rho_0, \tag{6}$$

where

$$M = \sup_{\mathbb{R}^n} 2|f(u)||Du|, \qquad L = \sup_{\mathbb{R}^n} 2|Du|^2. \tag{7}$$

Of course, we may assume $M > 0, L > 0$.

Letting $v(x) = \eta(|x|)P(x)$ for $|x| \geqq \rho_0$, we shall prove that

$$(8) \quad v(x) \leqq \max\left\{\varepsilon, \max_{|x|=\rho_0} v(x)\right\} = \max\left\{\varepsilon, \max_{|x|=\rho_0} P(x)\right\} \quad \text{for all} \quad x\colon |x| \geqq \rho_0.$$

This is obvious if $\sup_{|x| \geqq \rho_0} v(x) \leqq 0$, so we may suppose $v > 0$ somewhere. Since

$$\lim_{|x| \to +\infty} v(x) = 0,$$

because $P$ is bounded and (4) holds, it suffices to prove that $v(\bar{x}) \leqq \varepsilon$ in any interior maximum point $\bar{x}$ (if there are any), that is for any $\bar{x}$ such that $|\bar{x}| > \rho_0$ and $v(\bar{x}) = \sup_{|x| \geqq \rho_0} v(x) > 0$. With $\Delta\eta$ and $D\eta$ having the obvious meaning, we find that, at $\bar{x}$, $Dv = \eta DP + PD\eta = 0$ and $\eta > 0$, and hence $DP = -PD\eta/\eta$ and also, by (3), that, at $\bar{x}$,

$$|Du|^2 \Delta v \geqq |Du|^2 P \Delta\eta + 2|Du|^2 DP \cdot D\eta + \tfrac{1}{2}\eta |DP|^2 + 2\eta f(u) Du \cdot DP$$
$$= \left[|Du|^2 \Delta\eta - \frac{2|Du|^2|D\eta|^2}{\eta} - 2f(u) Du \cdot D\eta\right] P + \frac{P^2|D\eta|^2}{2\eta}.$$

Now $\Delta v(\bar{x}) \leqq 0$, since $\bar{x}$ is an interior maximum, and $P(\bar{x}) > 0$, because $\eta(|\bar{x}|) > 0$ and $v(\bar{x}) > 0$; thus, at $\bar{x}$,

$$(9) \qquad \frac{P|D\eta|^2}{2\eta} \leqq \frac{2|Du|^2|D\eta|^2}{\eta} + 2f(u) Du \cdot D\eta - |Du|^2 \Delta\eta.$$

Let us remark that, if $|Du|^2(\bar{x}) \leqq \varepsilon$, then, recalling that $\eta \leqq 1$ by (4), and $F \geqq 0$, we have

$$v(x) \leqq v(\bar{x}) \leqq P(\bar{x}) \leqq |Du|^2(\bar{x}) \leqq \varepsilon \quad \text{for all} \quad x\colon |x| \geqq \rho_0$$

and therefore (8) holds. In the other case $|Du|^2(\bar{x}) > \varepsilon$, we obtain from the inequality $\eta' < 0$, (7) and (9), that, at $\bar{x}$,

$$v = \eta P \leqq \frac{2\eta^2}{|D\eta|^2} |Du|^2 \left[\frac{2|D\eta|^2}{\eta} + \frac{M|D\eta|}{|Du|^2} - \Delta\eta\right]$$
$$\leqq L \frac{\eta^2}{|D\eta|^2} \left[\frac{2|D\eta|^2}{\eta} + \frac{M}{\varepsilon}|D\eta| - \Delta\eta\right]$$

and finally, observing that $|D\eta| = -\eta'$, $\Delta\eta = \eta'' + (n-1)\eta'/\rho$ and recalling (6), we conclude that $v(\bar{x}) \leqq \varepsilon$; thus (8) is proved.

Now, recalling (5) and letting $\varepsilon \to 0+$ in (8) we obtain

$$P(x) \leqq \max\left\{0, \max_{|x|=\rho_0} P(x)\right\} \quad \text{for all} \quad x\colon |x| \geqq \rho_0;$$

letting $\rho_0 \to 0+$ in this last inequality we infer that

$$P(x) \leqq \max\{0, P(0)\} \quad \text{for all} \quad x \in \mathbb{R}^n.$$

Furthermore, by (1),

$$P(0) = |Du|^2(0) - 2F(u(0)) \leqq |Du|^2(0) < \delta;$$

hence, $\delta > 0$ being arbitrary, we find that $P \leqq 0$ as asserted in Theorem II.

It remains to construct $\eta = \eta_{\varepsilon,\rho_0}$ so that (4), (5), (6) hold. Let

$$g_\varepsilon(t) = \int_t^1 \frac{e^{-\varepsilon/Ls}}{s^2}\, ds \quad \text{for} \quad 0 \leqq t \leqq 1$$

and

$$h_{\varepsilon,\rho_0}(t) = \int_{\rho_0}^t \frac{e^{-(M/\varepsilon)s}}{s^{n-1}}\, ds \quad \text{for} \quad t \geqq \rho_0$$

and define

$$\eta_{\varepsilon,\rho_0}(\rho) = g_\varepsilon^{-1}\left(c h_{\varepsilon,\rho_0}(\rho)\right) \quad \text{for} \quad \rho \geqq \rho_0,$$

where $c = g_\varepsilon(0)/h_{\varepsilon,\rho_0}(+\infty)$ and $g_\varepsilon^{-1}$ denotes the inverse function of $g_\varepsilon$. It is trivial to check that $\eta$ is well defined and (4) holds. For (5) it suffices to remark that the $g_\varepsilon^{-1}$ converge as $\varepsilon \to 0+$, uniformly on any compact interval of $\mathbb{R}_+$, to $\gamma(t) = 1/(1+t)$, while the $h_\varepsilon$ converge pointwise to 0. For (6) it suffices to differentiate, to take the logarithm, and to differentiate again with respect to $\rho$ the following equality which implicitly defines $\eta$:

$$\int_{\eta(\rho)}^1 \frac{e^{-\varepsilon/Ls}}{s^2}\, ds = \int_{\rho_0}^{\rho} \frac{e^{-(M/\varepsilon)s}}{s^{n-1}}\, ds.$$

Thus Theorem II is completely proved.

The proof of Theorem I is now straightforward. Let $a = u(x_0)$ and $A = \{x \in \mathbb{R}^n\colon u(x) = a\}$. As $A$ is nonvoid and closed, it suffices to prove that $A$ is open. Let $x_1 \in A$. Since $F \geqq 0$ and $F(a) = 0$, we have $F'(a) = 0$ and $F''(a) \geqq 0$; hence there exists $k \geqq 0$ such that

$$F(t) \leqq k(t-a)^2 \quad \text{for all} \quad t\colon |t-a| \leqq \delta,$$

provided that $\delta$ is small enough. Now, if $w \in \mathbb{R}^n$ and $|w| = 1$ and if we define

$$\phi(t) = u(x_1 + tw) - u(x_1) \quad \text{for} \quad |t| < \delta,$$

Theorem II implies that

$$|\phi'(t)|^2 \leqq k|\phi(t)|^2$$

and since $\phi(0) = 0$ it follows that $\phi \equiv 0$. Hence $u$ is constant in the ball $B(x_1, \delta)$ and Theorem I is proved.

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