The Nirenberg problem for the disk

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Prescribing the Gaussian curvature via a conformal metric

A classical problem in geometry is the prescription of the Gaussian curvature on a compact Riemannian surface Σ under a conformal change of the metric.

Denote by g the original metric and $\tilde{g} = e^u g$. The curvature then transforms according to the law:

$$-\Delta u + 2K_g(x) = 2K(x)e^u,$$

where $\Delta = \Delta_g$ is the Laplace-Beltrami operator and K_g, K stand for the Gaussian curvatures with respect to g and $\tilde{g}$, respectively.

The Nirenberg problem

The case of the unit sphere $\mathbb{S}^2$ is particularly interesting and complicated:

$$-\Delta u + 2 = 2K(x)e^u. \qquad \text{(N)}$$

This is a very classical problem and many contributions have been made: [Moser '73], [Chang-Yang '87, '88], [Chang-Gursky-Yang '93], [Ji '04]

The Nirenberg problem for the disk

The case $\Sigma = \mathbb{D}^2$ is the natural analogue of the Nirenberg problem:

$$\begin{cases} -\Delta u = 2K(x)e^u & \text{in } \mathbb{D}^2, \\ \frac{\partial u}{\partial \nu} + 2 = 2h(x)e^{u/2} & \text{on } \partial\mathbb{D}^2. \end{cases} \tag{ND}$$

If u is a solution, $(\mathbb{D}^2, e^u\, dx)$ has Gaussian curvature $K(x)$ and the geodesic curvature of its boundary is $h(x)$.

Also here, the noncompact action of the conformal group of the disk gives rise to bubbling of solutions.

This problem has not been studied in detail so far.

We are interested in the case in which both curvatures are different from 0.

What is the interplay between K and h for blowing-up and existence?

The Gauss-Bonnet formula

$$\int_{\mathbb{D}^2} K(x)e^u + \int_{\partial\mathbb{D}^2} h(x)e^{u/2} = 2\pi.$$

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Kazdan-Warner identities ([Hamza '90])

$$2\int_{\partial\mathbb{D}^2} h_\tau e^{u/2} + \int_{\mathbb{D}^2} \langle \nabla K(x), T(x)\rangle e^u = 0, \text{ where } T(x) = ix;$$

$$4\int_{\partial\mathbb{D}^2} h_\tau e^{u/2} x_2 - \int_{\mathbb{D}^2} e^u \langle \nabla K(x), F(x)\rangle = 0, \text{ where } F(x) = 1 - x^2.$$

Previous results on blow-up analysis

The only case studied is $K = 0$ ([Guo-Liu '06], [Da Lio-Martinazzi-Riviére '15]).

Under certain assumptions one can prove that there is a unique singular point located at the boundary, but nothing is known about the localization of such point.

A blow-up analysis ([Jevnikar-López Soriano-Medina-R. '21])

Let u_n be an unbounded sequence of solutions to:

$$\begin{cases} -\Delta u_n = 2K_n(x)e^{u_n} & \text{in } \mathbb{D}^2, \\ \frac{\partial u_n}{\partial \nu} + 2 = 2h_n(x)e^{u_n/2} & \text{on } \partial\mathbb{D}^2, \end{cases}$$

with $K_n \to K$ and $h_n \to h$ in C^2 sense. Assume also that:

$$\int_{\mathbb{D}^2} e^{u_n} + \int_{\partial\mathbb{D}^2} e^{u_n/2} < C. \tag{m}$$

Then, there exists $p \in \partial\mathbb{D}^2$ such that:

- $u_n \to -\infty$ outside p.
- Either $K(p) > 0$ or $h(p) > \sqrt{-K(p)}$ (otherwise there are no limit solutions in the half-plane, [Li-Zhu '95]).

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- $h_n e^{u_n/2} \rightharpoonup 2\pi\beta\delta_p$ and $K_n e^{u_n} \rightharpoonup 2\pi(1-\beta)\delta_p$, where

$$\beta = \frac{h(p)}{\sqrt{h^2(p) + K(p)}}.$$

By plugging this information in the first K-W condition, we obtain:

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$$2h_\tau(p) + \frac{K_\tau(p)}{h(p) + \sqrt{h^2(p) + K(p)}} = 0.$$

In order to make use of the second K-W condition, a more delicate analysis is needed.

Asymptotic expansion

Indeed, we prove that there exists $a_n \in \mathbb{D}^2$, $a_n \to p$ so that

$$u_n(z) = u_{a_n}(z) + o(1) \text{ in } C^{0,\alpha},\ \alpha \in (0, 1/2),$$

$$u_{a_n}(z) := 2\log\left\{\frac{2\hat{\phi}_n(1-|a_n|^2)}{\hat{\phi}_n^2|1-\bar{a}_n z|^2 + \hat{k}_n|z-a_n|^2}\right\},$$

with

$$\hat{\phi}_n := \phi_n\left(\frac{a_n}{|a_n|}\right), \qquad \hat{k}_n := K_n\left(\frac{a_n}{|a_n|}\right),$$

where

$$\phi_n(z) := h_n(z) + \sqrt{h_n(z)^2 + K_n(z)}.$$

The conditions on the singular point

Let us recall:

- Either $K(p) > 0$ or $h(p) > \sqrt{-K(p)}$.
- 1^{st} K-W condition $\Rightarrow 2h_\tau(p) + \dfrac{K_\tau(p)}{h(p) + \sqrt{h^2(p) + K(p)}} = 0.$

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The two last conditions are equivalent to $\nabla\Phi(p) = 0$, where

$$\Phi(x) = H(x) + \sqrt{H^2(x) + K(x)}.$$

Here H denotes the harmonic extension of h.

If $h = 0$, then $\Phi = \sqrt{K(x)}$. If, instead, $K = 0$, then $\Phi = 2H$.

Existence of bubbling solutions

The existence of bubbling solutions has indeed been proved, see [Battaglia-Medina-Pistoia '20]. Indeed, they consider:

$$\begin{cases} -\Delta u = 2K_\varepsilon(x)e^u & \text{in } \mathbb{D}^2, \\ \frac{\partial u}{\partial \nu} + 2 = 2h_\varepsilon(x)e^{u/2} & \text{on } \partial\mathbb{D}^2. \end{cases}$$

They take $K(x)$, $h(x)$, so that $\nabla\Phi$ vanishes at a point $p \in \partial\mathbb{D}^2$, and

$$K_\varepsilon = K(x) + \varepsilon G(x), \; h_\varepsilon(x) = h(x) + \varepsilon I(x).$$

Under some assumptions they prove the existence of solutions u_ε (for ε small) which blow-up around the point p when $\varepsilon \to 0$. The proof uses singular perturbation methods.

Remarks on condition (m)

- If condition (m) does not hold, the result is not true.

  Indeed, if $K < 0$ a different blow-up phenomena can occur, with diverging area and length ([López Soriano-Malchiodi-R. '21]).

- If $K(x) > 0$ and $h(x) > 0$, then (m) trivially holds.

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Some aid from Geometry

- If $K_n(x) \geq 0$ and $h(x) > 0$, then the length $\int_{\partial \mathbb{D}^2} e^{u_n/2}$ is bounded.
  By the isoperimetric inequality, also the area is bounded.
- If $K(x) > 0$ and $h_n(x) \geq 0$, then the area $\int_{\mathbb{D}^2} e^{u_n}$ is bounded.
  By the Toponogov length bound, also the length is bounded.

Previous results on existence

There are very few results on existence of solutions.

- There are solutions for constants K, h, if $K > 0$ or $h > \sqrt{-K}$.
- The case $K = 0$ has been studied ([Chang-Liu '96], [Gehrig, PhD thesis, '20]).
- The case $h = 0$, $\partial_\nu K(x) = 0$ for all $x \in \partial\mathbb{D}^2$ ([Chang-Yang '88]).
- The case of symmetric functions $K \geq 0$, $h \geq 0$, not both of them equal to 0 ([Cruz Blázquez-R. '18]).

An existence result ([R. '21, preprint])

Let $K \in C^2(\overline{\mathbb{D}^2})$, $h \in C^2(\partial\mathbb{D}^2)$, with

$$K \geq 0,\ h > 0 \text{ or } K > 0,\ h \geq 0,$$

and define $\Phi(x) = H(x) + \sqrt{H^2(x) + K(x)} > 0$, where H denotes the harmonic extension of h. Assume that

$$\nabla\Phi(x) \neq 0 \ \forall\, x \in \partial\mathbb{D}^2.$$

Then the associated Leray-Schauder degree is equal to $\pm\, deg_B(\nabla\Phi, \mathbb{D}^2, 0)$.

In particular, if it is different from 0, there exists a solution.

If $K = 0$ and H has isolated critical points we recover the result of [Chang-Liu '96]. Their proof is completely different.

First step: reduction to the case $h = 0$

We define, for $s \in [0, 1]$,

$$h_s(x) = sh(x) \Rightarrow H_s(x) = sH(x),\ K_s(x) = sK(x) + (1 - s)\Phi^2(x).$$

With this definition, $\Phi_s(x) = H_s(x) + \sqrt{H_s^2(x) + K_s(x)} = \Phi(x)$.

By the previous blow-up analysis, the set of solutions for the problem:

$$\begin{cases} -\Delta u = 2K_s(x)e^u & \text{in } \mathbb{D}^2, \\ \frac{\partial u}{\partial \nu} + 2 = 2h_s(x)e^{u/2} & \text{on } \partial\mathbb{D}^2. \end{cases}$$

form a compact set. Hence, the Leray-Schauder degree (in apropriately large balls) is independent of s.

The reduced problem

Then, we can consider:

$$\begin{cases} -\Delta u = 2K(x)e^u & \text{in } \mathbb{D}^2, \\ \frac{\partial u}{\partial \nu} + 2 = 0 & \text{on } \partial\mathbb{D}^2. \end{cases}$$

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A mean field equation

The Gauss-Bonnet formula implies that $\int_{\mathbb{D}^2} Ke^u = 2\pi$. Hence we can rewrite the equation as:

$$\begin{cases} -\Delta u = 4\pi \frac{K(x)e^u}{\int_{\mathbb{D}^2} K(x)e^u} & \text{in } \mathbb{D}^2, \\ \frac{\partial u}{\partial \nu} + 2 = 0 & \text{on } \partial\mathbb{D}^2. \end{cases}$$

The variational formulation

This is the Euler-Lagrange equation of the functional:

$$J(u) = \frac{1}{2} \int_{\mathbb{D}^2} |\nabla u|^2 + 2 \int_{\partial \mathbb{D}^2} u - 4\pi \log \left(\int_{\mathbb{D}^2} K e^u \right).$$

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A barycenter constraint

The functional J becomes coercive when restricted to M_a ($a \in \mathbb{D}^2$),

$$M_a = \{u \in X : \int_{\mathbb{D}^2} e^u (x - a) = 0\}.$$

This is inspired by [Aubin, '79].

A Lagrange multiplier

The minimizer gives rise to a solution u_a of the problem:

$$\begin{cases} -\Delta u_a = 2\Big(K(x) + \lambda(a)\cdot(x-a)\Big)e^{u_a} & \text{in } \mathbb{D}^2, \\ \dfrac{\partial u_a}{\partial \nu} + 2 = 0 & \text{on } \partial\mathbb{D}^2, \end{cases} \tag{1}$$

where $\lambda(a) \in \mathbb{R}^2$ is a Lagrange multiplier.

Question

Can we find $a \in \mathbb{D}^2$ so that $\lambda(a) = 0$?

The Brouwer degree

If we take $a_n \to p \in \partial \mathbb{D}^2$, the solutions so obtained are blowing-up. By the blow-up analysis,

$$\lambda(a_n) + \nabla K(p) \to 0.$$

Hence λ can be extended continuously to the points $p \in \partial \mathbb{D}^2$ by $\lambda(p) = -\nabla K(p)$. Moreover,

$$deg_B(\lambda, \mathbb{D}^2, 0) = deg_B(-\nabla K, \mathbb{D}^2, 0) = deg_B(\nabla K, \mathbb{D}^2, 0) \neq 0,$$

which implies that the equation $\lambda(a) = 0$ is solvable.

Final comments

Of course all above is a heuristic argument. There are two main obstacles in giving a rigorous proof:

1. First, the minimizer u_a needs not be unique, and it is not clear that we can make a continuous choice $\lambda(a)$.

   We overcome this difficulty by passing, via an homotopy, to a case in which K is close to the constant 1.

2. Second, we need to compute the Leray-Schauder degree, and see that it coincides the Brouwer degree of ∇K.

Open problems

- The case of K, h nonpositive. A first result has been given in [López Soriano-Malchiodi-R. '21] for $K < 0$ and $\chi(\Sigma) \leq 0$.

  The case of the disk is part of an ongoing project (jointly with López-Soriano and Reyes-Sánchez).

- The higher dimension case: prescribing scalar and mean curvature.

  This has been treated in [Cruz Blázquez-Malchiodi-R. '22], for general manifolds of dimension 3, when the conformal Escobar invariant is negative or 0.

  The higher dimensional case becomes more intricate, and the behavior around blow-up points can be complicated.

Thank you for your attention!

And thank you very much for everything,

Azahara and Mar!