Boundary Harnack principles and degenerate equations on singular sets

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Papers

- [TTV 2023] Susanna Terracini, Giorgio Tortone and Stefano Vita, In preparation,
- [TTV 2022] Susanna Terracini, Giorgio Tortone and Stefano Vita, Higher order boundary Harnack principle via degenerate equations, preprint 2022
- [STV 2021] Yannick Sire, Susanna Terracini and Stefano Vita, Liouville type theorems and local behaviour of solutions to degenerate or singular problems part I: even solutions. *Commun. Partial Differ. Equations 46, No. 2, 310-361 (2021)* (https://arxiv.org/abs/1903.02143)
- [STV 2021] Yannick Sire, Susanna Terracini and Stefano Vita, Liouville type theorems and local behaviour of solutions to degenerate or singular problems part II: odd solutions. *Mathematics in Engeneering, 2021, 3 (1): 50 pp.,* (https://arxiv.org/abs/2003.09023),

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The equation

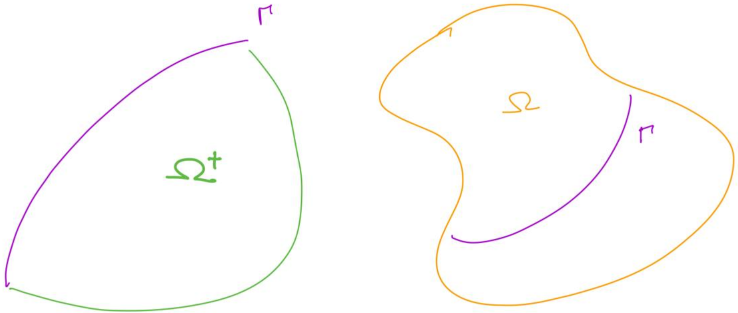
Let us consider a regular hyper-surface Γ embedded in $\mathbb{R}^n$ with $n \geq 2$ and a weight ρ vanishing on it with non zero gradient. Our goal is to obtain higher order local Schauder estimates up to Γ for solutions to singular/degenerate elliptic equations of type:

$$-\mathrm{div}\left(\rho^a A \nabla w\right) = \rho^a f + \mathrm{div}\left(\rho^a F\right) \quad \text{in } \Omega$$

where Ω is a bounded domain and the exponent satisfy

$$a > -1.$$

So that ρ^a is locally integrable. The hypersurface Γ can be either contained in the boundary of Ω or in its interior. In the first case, we shall use the notation Ω^+ to emphasise that Ω lies on one side of Γ.


Γ
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Energy Solutions

In either case, solutions have to be intended in the energy sense, as elements of the weighted Sobolev space $H^{1,a}(\Omega) = H^1(\Omega, \rho^a \mathrm{d}z)$ which satisfy

$$\int_\Omega \rho^a A\nabla w \cdot \nabla\phi = \int_\Omega \rho^a (f\phi - F\cdot\nabla\phi),$$

for any test function belonging to $H^{1,a}(\Omega)$. In other words, solutions are critical points of the energy functional on $H^{1,a}(\Omega)$:

$$\int_\Omega \frac{1}{2}\rho^a A\nabla w \cdot \nabla w - \int_\Omega \rho^a (fw - F\cdot\nabla w)$$

which is well defined under suitable assumptions on the right hand sides f, F.

Natural boundary condition

Note that elements of the energy space $H^1(\Omega, \rho^a \mathrm{d}z)$ do have a trace on Γ only when $a \in (-1, 1)$ (the case of Muckenhoupt weights) whereas in the superdegenerate case $a \geq 1$ the space $C_c^\infty(\Omega)$ is dense in $H^{1,a}(\Omega)$, so traces on Γ are meaningless.
Formally, if it exists, the conormal derivate satisfies

$$\lim_{\rho(x) \to 0^+} \rho^a (A \nabla w + F) \cdot \nu = 0,$$

where ν is the outward unit normal vector on Γ. Thus we are associating with the equation its natural boundary condition at Γ.

Natural boundary conditions in the superdegenerate case

When, $a \geq 1$ all solutions of

$$-\mathrm{div}\,(\rho^a A \nabla w) = \rho^a f + \mathrm{div}\,(\rho^a F) \quad \text{in } \Omega \setminus \Gamma$$

belonging to the energy space $H^1(\Omega, \rho^a \mathrm{d}z)$ are indeed solutions in the energy sense. The boundary condition is automatically satisfied. It is worthwhile noticing that, when $a \geq 1$, then the energy solutions solutions can be discontinuous at Γ.

Motivations

Such equations arise in the context of fractional powers of elliptic operators (when $a \in (-1, 1)$, Caffarelli-Silvestre, Chang-Gonzalez, etc..), the analysis of edge operators (Mazzeo, Graham-Zworski, etc..) and appear also in the study of ratios $w = v/u$ of solutions to elliptic equations of type

$$-\mathrm{div}\,(A\nabla u) = g\ , \quad -\mathrm{div}\,(A\nabla v) = f \quad \text{in } \Omega^+$$

such that $u \equiv v \equiv 0$ on $\Gamma \subset \partial\Omega^+$. Indeed, one easily sees that the ratio $w = v/u$ satisfies equation

$$-\mathrm{div}\left(u^2 A\nabla\left(\frac{v}{u}\right)\right) = \text{r.h.s.}$$

for a suitable right hand side depending on u, v, f, g.

Ratios of harmonic functions

The Schauder regularity of the ratio v/u is usually referred as higher order boundary Harnack principle and has been studied with a different approach, which actually totally neglects the link with the degenerate equation, also in the parabolic case (De Silva-Savin (2015), Banjeree-Garofalo (2016), Kukuljan (2022)).

When A is analytic, Logunov-Malinnikova (2015-16) proved that if u and v are harmonic and the zero set of v includes that of u ($Z(u) \subset Z(v)$), then v/u is analytic. Lin-Lin (2021) obtained C^α regularity for Lipschitz A.

In these works, a very different point of view from ours is adopted, but the question already arises as to whether it is possible to approach them through the study of the associated superdegenerate equation.

Degenerate and singular operators on an hyperplane

Let $z = (x, y) \in \Omega \subset \mathbb{R}^{n+1}$, with $x \in \mathbb{R}^n$ and $y \in \mathbb{R}$, $n \geq 1$, $a \in \mathbb{R}$. We are concerned with qualitative properties of solutions to a class of problems involving the operator in divergence form given by

$$\mathcal{L}_a u := \mathrm{div}(|y|^a A(x, y)\nabla u) \ ,$$

where the matrix A is symmetric, continuous and satisfy the uniform ellipticity condition:

$$\lambda_1|\xi|^2 \leq A(x, y)\xi \cdot \xi \leq \lambda_2|\xi|^2 \ ,$$

for all $\xi \in \mathbb{R}^{n+1}$, for every $(x, y) \in \Omega$ and some ellipticity constants $0 < \lambda_1 \leq \lambda_2$.
We denote by $\Sigma := \{y = 0\} \subset \mathbb{R}^{n+1}$ the characteristic manifold.

Uniformly elliptic operators I

Theorem (E. De Giorgi (1957); J. Nash (1957); J. Moser (1960))

Let A be a symmetric matrix with measurable coefficients, satisfying the uniform ellipticity condition:

$$\lambda_1|\xi|^2 \leq A(x)\xi \cdot \xi \leq \lambda_2|\xi|^2 \,,$$

for all $\xi \in \mathbb{R}^n$, for every $x \in \Omega$ and some ellipticity constants $0 < \lambda_1 \leq \lambda_2$. Let u be a weak solutions to the equation

$$-\mathrm{div}(A(x)\nabla u) = 0$$

in B_1. Fixed $r \in (0,1)$, $\beta > 1$, there are constants c and $\alpha \in (0,1)$ depending only on r, β, λ_1, λ_2 and the dimension n such that

$$\|u\|_{C^{0,\alpha}(B_r)} \leq c\|u\|_{L^\beta(B_1)}.$$

Remark: when A is continuous then we can take any $\alpha \in (0,1)$.

Uniformly elliptic operators I

Theorem (E. De Giorgi (1957); J. Nash (1957); J. Moser (1960))

Let A be a symmetric matrix with measurable coefficients, satisfying the uniform ellipticity condition:

$$\lambda_1|\xi|^2 \le A(x)\xi\cdot\xi \le \lambda_2|\xi|^2 \,,$$

for all $\xi \in \mathbb{R}^n$, for every $x \in \Omega$ and some ellipticity constants $0 < \lambda_1 \le \lambda_2$. Let u be a weak solutions to the equation

$$-\mathrm{div}(A(x)\nabla u) = f + \mathrm{div}(F)$$

in B_1. Fixed $r \in (0,1)$, $\beta > 1$, there are constants c and $\alpha \in (0,1)$ depending only on r, β, the dimension n, λ_1, λ_2, $p_1 > n/2$, $p_2 > n$ such that

$$\|u\|_{C^{0,\alpha}(B_r)} \le c\left(\|u\|_{L^\beta(B_1)} + \|f\|_{L^{p_1}(B_1)} + \|F\|_{L^{p_2}(B_1)}\right).$$

Remark: when A is continuous then c and α depend on r, p_1 and p_2 only.

Uniformly elliptic operators II

Theorem (Schauder $C^{1,\alpha}$-estimates, Campanato (1963-65), M. Giaquinta and E. Giusti (1984))

Under the assumptions of uniform ellipticity of the De Giorgi-Nash-Moser Theorem, assume in addition that

$$A \in \mathcal{C}^{0,\alpha}(B_1)$$

Let u be a weak solutions to the equation

$$-\mathrm{div}(A(x)\nabla u) = f + \mathrm{div}(F)$$

in B_1. Fixed $r \in (0,1)$, $\beta > 1$, there is a constant c, depending only on r, β, the dimension n, λ_1, α and $p > n/(1-\alpha)$ such that

$$\|u\|_{C^{1,\alpha}(B_r)} \leq c \left(\|u\|_{L^\beta(B_1)} + \|f\|_{L^p(B_1)} + \|F\|_{C^{0,\alpha}(B_1)} \right).$$

Singular or degenerate operators

Our class of elliptic operators may be degenerate or singular, in the sense that the eigenvalues of A may vanish or become infinite over Σ, and this happens respectively when $a > 0$ and $a < 0$.
Such behaviour affects the regularity of solutions: indeed

$$u(x,y) = |y|^{-a} y$$

is $\mathcal{L}_a$ harmonic, when $A \equiv \mathbb{I}$ and lacks smoothness whenever a is not an integer.
We frame our operators in the class of weighted singular/degenerate elliptic ones

$$\mathcal{L}_\rho u := \mathrm{div}(\rho(z)A(z)\nabla u) = r.h.s. \,,$$

and we focus on the features of the weights w.

Muckenhoupt weights

We recall that a function $\rho \in L^1_{\mathrm{loc}}(\mathbb{R}^{n+1})$ is said to be an A_p weight if the following holds

A_p weights

$$\sup_{B\subset\mathbb{R}^{n+1}} \left(\frac{1}{|B|}\int_B \rho\right)\left(\frac{1}{|B|}\int_B \rho^{-1/(p-1)}\right)^{p-1} < \infty$$

where B is any ball. The weights $\rho(y) = |y|^a$ belong to the Muckenhoupt class A_p when $a \in (-1, p-1)$. In particular, they belong to the Muckenhoupt class A_2 when $a \in (-1, 1)$, and have studied in a seminal paper by Fabes, Kenig and Serapioni.

A_2 weights

The fundamental properties of A_2 weights include:

Weighted Sobolev embeddings:

$$\left(\frac{1}{\rho(B)}\int_B |\varphi|^{2k}\rho(x)dx\right)^{1/2k} \le C\mathrm{diam}(B)\left(\frac{1}{\rho(B)}\int_B |\nabla\varphi|^2\rho(x)dx\right)^{1/2}$$

for all $\varphi \in C_0^\infty(B)$.

Poincaré inequality:

$$\frac{1}{\rho(B)}\int_B |\varphi-\varphi_B|^2\rho(x)dx \le C\mathrm{diam}(B)^2\frac{1}{w(B)}\int_B |\nabla\varphi|^2\rho(x)dx$$

for all $\varphi \in C^1(\overline{B})$, where $\varphi_B = (1/\rho(B))\int_B \varphi\rho(x)dx$.

These properties entail the validity of Maximum principle, unique continuation priciple, Harnack inequality and eventually the Hölder continuity of all energy solutions.

The superdegenerate case

We are going to deal with energy solutions and we will solve completely the problem in the energy space for all values $a \in (-1, \infty)$ and this range is wider than the A_2 one, i.e. $(-1, 1)$. Moreover, in the super degenerate range $a \geq 1$, the natural boundary condition can not be removed for Hölder regularity (also for continuity). In fact, we have the following counterexample:

Example

When $a \geq 1$, the jump function

$$\overline{u}(z) = \begin{cases} 1 & \text{in } B_1^+ \\ -1 & \text{in } B_1^-, \end{cases}$$

belongs to the energy space, but it is not an energy $\mathcal{L}_a$-harmonic function. Even more, replacing the constant 1 (say) in B_1^+ by 0, one produces also an energy $\mathcal{L}_a$-harmonic function for which the unique continuation principle does not hold.

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Regularized operators for approximation

In order to better understand the regularity of solutions to degenerate and singular problems involving the operator L_a, but also the local behaviour near their nodal set and the geometric structure of the nodal set itself, we introduce a family of regularized operators. For $a \in \mathbb{R}$ fixed, let us consider the family in $\varepsilon \geq 0$ of functions $\rho_\varepsilon^a(y) : \Omega \to \mathbb{R}_+$ defined by

$$\rho_\varepsilon^a(y) := \begin{cases} (\varepsilon^2 + y^2)^{a/2} \min\{\varepsilon^{-a}, 1\} & \text{if } a \geq 0, \\ (\varepsilon^2 + y^2)^{a/2} \max\{\varepsilon^{-a}, 1\} & \text{if } a \leq 0, \end{cases}$$

and that of the associated operators

$$\mathcal{L}_{\rho_\varepsilon^a} u = \mathrm{div}\left(\rho_\varepsilon^a(y) \nabla u\right).$$

Inhomogenous equations

We are going to follow a perturbative method, actually allowing us to deal with more general equations with right hands in possibly divergence form, and to deal with an entire class of (possibly) regularised problems in the form ($\varepsilon \geq 0$):

$$\begin{cases} -\mathrm{div}((\varepsilon^2+y^2)^{a/2}A(x,y)\nabla u) = (\varepsilon^2+y^2)^{a/2}f(x,y) \\ \qquad\qquad + \mathrm{div}\left((\varepsilon^2+y^2)^{a/2}F(x,y)\right), & \text{in } B_1^+ \\ (\varepsilon^2+y^2)^{a/2}(A\nabla u_\varepsilon + F_\varepsilon)\cdot \vec{e}_y = 0 & \text{on } \Sigma. \end{cases}$$

and derive both $C^{0,\alpha}$ and $C^{1,\alpha}$ estimates which are uniform with respect to the parameter $\varepsilon \in [0,1]$ (we shall refer to this fact as a ε-*stable property*).

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Theorem (STV 2021. Schauder estimates up to the characteristic hyperplane Σ)

Let $a > -1$, $p > n + 1 + a^+$, $\alpha \in (0, 1 - \frac{n+1+a^+}{p}]$, $F, A \in C^{0,\alpha}(B_1^+)$, $f \in L^p(B_1^+, y^a \mathrm{d}z)$. Then, any energy solution to

$$-\mathrm{div}\,(y^a A \nabla u) = y^a f + \mathrm{div}\,(y^a F) \qquad \text{in } B_1^+,$$

belongs to $C^{1,\alpha}(B_r^+)$ for any $r \in (0,1)$. Moreover, fixed $r \in (0,1)$ and $\beta > 1$, then there exists a positive constant depending only on $n, a, p, \alpha, r, \beta$ and $\|A\|_{C^{0,\alpha}(B_1^+)}$ such that, for any energy solution there holds

$$\|u\|_{C^{1,\alpha}(B_r^+)} \le c \left(\|u\|_{L^\beta(B_1^+, y^a \mathrm{d}z)} + \|f\|_{L^p(B_1^+, y^a \mathrm{d}z)} + \|F\|_{C^{0,\alpha}(B_1^+)} \right),$$

and the estimate is ε-stable in the sense stated above. Moreover, the solutions satisfy the following boundary condition

$$(A \nabla u + F) \cdot \vec{e}_y = 0 \qquad \text{on } \Sigma.$$

We remark also that the estimate above holds for any $\alpha \in (0,1)$ if $p = \infty$.

Theorem (STV 2021. Higher order Schauder estimates up to the characteristic hyperplane Σ)

Let $a > -1$, $k \in \mathbb{N} \setminus \{0\}$, $\alpha \in (0,1)$, $F, A \in C^{k,\alpha}(B_1^+)$, $f \in C^{k-1,\alpha}(B_1^+)$. Then, any energy solution to

$$-\mathrm{div}\,(y^a A\nabla u) = y^a f + \mathrm{div}\,(y^a F) \qquad \text{in } B_1^+,$$

belongs to $C^{k+1,\alpha}(B_r^+)$ for any $r \in (0,1)$. Moreover, fixed $r \in (0,1)$ and $\beta > 1$, then there exists a positive constant depending only on $n, a, k, \alpha, r, \beta$ and $\|A\|_{C^{k,\alpha}(B_1^+)}$ such that, for any energy solution there holds

$$\|u\|_{C^{k+1,\alpha}(B_r^+)} \le c\left(\|u\|_{L^\beta(B_1^+, y^a\mathrm{d}z)} + \|f\|_{C^{k-1,\alpha}(B_1^+)} + \|F\|_{C^{k,\alpha}(B_1^+)}\right).$$

Moreover, the solutions satisfy the boundary condition as before.

This result is somewhat surprising, in view of the lack of regularity of the coefficients of the operators and can be attributed to the joint regularising effect of the equation and the Neumann boundary condition in the half ball, associated with evenness. We stress that odd solutions may indeed lack regularity, as shown by the example $u(x,y) = |y|^{-a}y$

Singular forcings

When $a > 0$ we also will provide Schauder estimates for solutions to the singularly forced equation

$$-\mathrm{div}\left(y^a A\nabla u\right) = y^{a-1} f \qquad \text{in } B_1^+,$$

Corollary (STV 2021. Schauder estimates up to the characteristic hyperplane Σ with a singular forcing term)

Let $a > 0$, $k \in \mathbb{N}$, $\alpha \in (0,1)$, $f, A \in C^{k,\alpha}(B_1^+)$. Then, any energy solution to the singular equation belongs to $C^{k+1,\alpha}(B_r^+)$ for any $r \in (0,1)$. Moreover, fixed $r \in (0,1)$ and $\beta > 1$, then there exists a positive constant depending only on $n, a, k, \alpha, r, \beta$ and $\|A\|_{C^{k,\alpha}(B_1^+)}$ such that, for any energy solution there holds

$$\|u\|_{C^{k+1,\alpha}(B_r^+)} \leq c\left(\|u\|_{L^\beta(B_1^+, y^a \mathrm{d}z)} + \|f\|_{C^{k,\alpha}(B_1^+)}\right).$$

Moreover, the solutions satisfy the following boundary condition

$$aA\nabla u \cdot \vec{e}_y + f = 0 \qquad \text{on } \Sigma.$$

Scheme of the proof

1. As already said, we first regularize the problem by introducing a parameter ε such that the operator becomes uniformly elliptic when $\varepsilon > 0$;
2. by means of appropriate Liouville-type theorems, which may be of independent interest, we then obtain uniform estimates in $\varepsilon \geq 0$ in Hölder spaces $C^{0,\alpha}$ and $C^{1,\alpha}$ for energy.
3. we prove that all solution to the singular/degenerate equation can be obtained as limits of solutions to a sequence of regularized problems;

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Let us denote

$$L_A = \mathrm{div}\,(A\nabla\cdot)\,.$$

Our Theorem finds a remarkable application to higher order boundary Harnack inequalities for ratios of L_A-harmonic functions vanishing on a common part of the boundary. Let us consider the upper side of a regular hyper-surface Γ embedded in $\mathbb{R}^n$, and localize the problem on a ball centered in 0 which lies on Γ. Thus

$$\Omega_\varphi^+ \cap B_1 = \{y > \varphi(x)\} \cap B_1 \qquad \text{with} \qquad \Gamma \cap B_1 = \{y = \varphi(x)\} \cap B_1,$$

with $\varphi(0) = 0$ and $\nabla_x\varphi(0) = 0$. In other words, we are locally describing the upper side of the manifold as the epigraph of a function φ and the manifold as its graph. For $z \in \Gamma$, let us denote by $\nu^+(z)$ the outward unit normal vector to Ω_φ^+.

Now, consider two functions u, v solving

$$\begin{cases} -L_A v = f & \text{in } \Omega_\varphi^+ \cap B_1, \\ -L_A u = g & \text{in } \Omega_\varphi^+ \cap B_1, \\ u > 0 & \text{in } \Omega_\varphi^+ \cap B_1, \\ u = v = 0, \qquad \partial_{\nu^+} u < 0 & \text{on } \Gamma \cap B_1. \end{cases}$$

It can be easily proven that the ratio $w = v/u$ is an energy solution to

$$-\mathrm{div}\left(u^2 A \nabla w\right) = u(f - gw) \quad \text{in } \Omega_\varphi^+ \cap B_1.$$

Note the presence of u as a factor in the right hand side (instead of u^2).

Theorem (TTV 2022. One side higher order boundary Harnack principle)

Let $k \in \mathbb{N}$ *and* $\alpha \in (0,1)$. *Let us consider two functions* u, v *as above. Let us assume that* $A, f, g \in C^{k,\alpha}(\Omega_\varphi^+ \cap B_1)$ *and* $\varphi \in C^{k+1,\alpha}(B_1')$. *Then,* $w = v/u$ *belongs to* $C^{k+1,\alpha}_{\mathrm{loc}}(\overline{\Omega_\varphi^+} \cap B_1)$ *and satisfies the following boundary condition*

$$2(\nabla u \cdot \nu^+)A\nabla w \cdot \nu^+ + f - gw = 0 \qquad \text{on } \Gamma \cap B_1.$$

Moreover, the following estimate holds true

$$\left\| \frac{v}{u} \right\|_{C^{k+1,\alpha}(\Omega_\varphi^+ \cap B_{1/2})} \leq C \left(\|v\|_{L^2(\Omega_\varphi^+ \cap B_1)} + \|f\|_{C^{k,\alpha}(\Omega_\varphi^+ \cap B_1)} \right)$$

with a positive constant C *depending on* n, α, k, $\|A\|_{C^{k,\alpha}(\Omega_\varphi^+ \cap B_1)}$, $\|\varphi\|_{C^{k+1,\alpha}(B_1')}$, $\|g\|_{C^{k,\alpha}(\Omega_\varphi^+ \cap B_1)}$, $\|u\|_{L^2(\Omega_\varphi^+ \cap B_1)}$. *Finally, normalizing* $u(\vec{e}_n/2) = 1$ *and if* $v > 0$ *in* $\Omega_\varphi^+ \cap B_1$, *then one can express the estimate above as*

$$\left\| \frac{v}{u} \right\|_{C^{k+1,\alpha}(\Omega_\varphi^+ \cap B_{1/2})} \leq C \left(\left| \frac{v}{u}(\vec{e}_n/2) \right| + \|f\|_{C^{k,\alpha}(\Omega_\varphi^+ \cap B_1)} \right)$$

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Schauder estimates across the common nodal set

Another interesting application of our results and methods concerns the regularity of ratios of L_A harmonic functions across their common nodal set. This is connected with extending Logunov and Malinnikova's theorem stating analiticity of ratios of harmonic functions sharing the same nodal set. While the approach there strongly relies on division lemmas and analiticity estimates, the authors wonder whether an alternative one could be conducted through the analysis of the associated superdegenerate equation fullfilled by the ratio $w = v/u$. This is our next goal.

Let $n \geq 2$ and $u \in H^1(B_1)$ be a weak solution to

$$L_A u = 0 \qquad \text{in } B_1,$$

where $A(z) = (a_{ij}(z))_{ij}$ is a symmetric uniformly elliptic matrix with α-Hölder continuous coefficients for some $\alpha \in (0, 1)$.

The nodal set of u

By standard Schauder theory, any weak solution is actually of class $C^{1,\alpha}_{\mathrm{loc}}(B_1)$. Thus the nodal set $Z(u) = u^{-1}(\{0\})$ of u splits into a regular part $R(u)$ and a singular part $S(u)$ defined as

$$R(u) = \{z \in Z(u) \ : \ |\nabla u| \neq 0\}, \qquad S(u) = \{z \in Z(u) \ : \ |\nabla u| = 0\};$$

where $R(u)$ is in fact locally a $(n-1)$-dimensional hyper-surface of class $C^{1,\alpha}$ and $S(u)$ has Haudorff dimension at most $n-2$. Let us remark here that, when $A \in C^{k,\alpha}(B_1)$ for certain $k \geq 1$, $\alpha \in (0,1)$, then any weak solution is actually of class $C^{k+1,\alpha}_{\mathrm{loc}}(B_1)$ and the regular part $R(u)$ is locally a $(n-1)$-dimensional hyper-surface of class $C^{k+1,\alpha}$.
Given a second solution v to $L_A v = 0$ in B_1 with $Z(u) \subseteq Z(v)$, it is not difficult to prove that the ratio $w = v/u$ is in fact an energy solution to the degenerate elliptic equation

$$\mathrm{div}(u^2 A \nabla w) = 0 \qquad \text{in } B_1,$$

in the sense of weak solution belonging to the weighted Sobolev space $H^1(B_1, u^2 \mathrm{d}z)$.

Schauder estimates across the regular part

Though, generally speaking, solutions to

$$\operatorname{div}(u^2 A\nabla w) = 0 \qquad \text{in } B_1,$$

need not to be continuous at the zero set of the weight u, the ratio $w = v/u$ is Hölder continuous, when A is locally Lipschitz, thanks to the result of Lin-Lin.

Theorem (Schauder estimates for the ratio across the regular part of the nodal set $R(u)$)

Let $A \in C^{k,\alpha}(B_1)$, for some $k \in \mathbb{N}$ and $\alpha \in (0,1)$, and (u, v) a pair of L_A-harmonic functions such that $S(u) \cap B_1 = \emptyset$ and $Z(u) \subseteq Z(v)$. Then $w = v/u \in C^{k+1,\alpha}_{\mathrm{loc}}(B_1)$ and

$$A\nabla w \cdot \nu = 0 \qquad \text{on } R(u) \cap B_1,$$

where ν is the unit normal vector on $R(u)$.

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The mother Liouville theorem

Theorem (STV 2021)

Let $a \in (-1, +\infty)$, $\varepsilon \geq 0$ *and* w *be a solution to*

$$\begin{cases} -\mathrm{div}(\rho_\varepsilon^a(y)\nabla w) = 0 & \text{in } \mathbb{R}^{n+1}_+ \\ \rho_\varepsilon^a \partial_y w = 0 & \text{in } \mathbb{R}^n \times \{0\}, \end{cases}$$

and let us suppose that for some $\gamma \in [0, 1)$, $C > 0$ *it holds*

$$|w(z)| \leq C(1 + |z|^\gamma)$$

for every z. *Then* w *is constant.*

The baby Liouville theorem

Corollary (STV 2021)

Let $a \in (-1, +\infty)$, $\varepsilon \geq 0$ *and* w *be a solution to*

$$\begin{cases} -\mathrm{div}(\rho_\varepsilon^a(y)\nabla w) = 0 & \text{in } \mathbb{R}^{n+1}_+ \\ \rho_\varepsilon^a \partial_y w = 0 & \text{in } \mathbb{R}^n \times \{0\}, \end{cases}$$

and let us suppose that for some $\gamma \in [0,1)$, $C > 0$ *it holds*

$$|\nabla w(z)| \leq C(1 + |z|^\gamma)$$

for every z. *Then* w *is a linear function, depending on* x *only.*

The division Lemma by Murdoch as a Liouville Theorem

The division lemma by Murdoch states that if Q is a harmonic polynomial and P is a polynomial such that $Z(Q) \subseteq Z(P)$, then $P = QR$ for some polynomial R. In other words:

Theorem (Liouville theorem for ratio of entire harmonic functions sharing the same zero set)

Let u be a harmonic polynomial and v be an entire harmonic function in $\mathbb{R}^n$ such that $Z(u) \subseteq Z(v)$. Suppose that there exists $\gamma \geq 0$ and $C > 0$ such that

$$\left|\frac{v}{u}\right|(z) \leq C(1 + |z|)^\gamma \quad \text{in } \mathbb{R}^n.$$

Then the ratio v/u is a polynomial of degree at most $\lfloor \gamma \rfloor$.

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A Liouville Theorem for the weighted equation

Theorem (TTV 2022: Liouville theorem for the weighted equation)

Let u be a harmonic polynomial and w be an entire continuous solution to.

$$-\mathrm{div}(u^2 \nabla w) = 0 \quad \text{in } \mathbb{R}^n.$$

Suppose that there exists $\gamma \geq 0$ and $C > 0$ such that

$$|w(z)| \leq C(1 + |z|)^\gamma \quad \text{in } \mathbb{R}^n.$$

Then w is a polynomial of degree at most $\lfloor \gamma \rfloor$.

As a byproduct, we can extend the Liouville theorem by Lin-Lin in which the existence of a least growth at infinity is deduced by the iteration of a Harnack type inequality for the ratio.

Variable coefficient cases: Almgren's frequency

Let N denote the Almgren frequency function: (here B is the geodesic ball in the metric associated with $A(z)$)

$$N(z_0,u,r) = r\frac{\displaystyle\int_{B_r(z_0)} A(z)\nabla u\cdot\nabla u}{\displaystyle\int_{\partial B_r(z_0)} \mu u^2}, \quad \text{for } r\in(0,1-|z_0|),$$

and

$$\mu(z) = \frac{A(z)z\cdot z}{|z|^2} \in [\lambda,\Lambda] \quad \text{for every } z\in B_1.$$

Is is known that, when A is Lipschitz then N is absolutely continuous ($e^{cr}N(r)$ is monotone). We define the set of normalized L_A-harmonic functions with bounded frequency:

$$\mathcal{S}_{N_0} = \left\{u\in H^1(B_1)\ :\ L_A u = 0 \text{ in } B_1,\ N(0,u,1)\le N_0,\ \|u\|_{L^2(B_1)} = 1\right\},$$

Uniform Hölder estimates for the ratio in $\mathcal{C}_{N_0}$

Theorem (TTV 2023)

Let $n \geq 2$, $0 < \lambda \leq \Lambda$, $L, N_0 > 0$ and $\alpha \in (0,1)$. Then there exists a positive constant C depending only on $n, \lambda, \Lambda, L, N_0$, and α such that for any $A \in \mathcal{A}_{\lambda,\Lambda,L}$ and any pair $(u, v) \in \mathcal{S}_{N_0}$ with $Z(u) \subseteq Z(v)$ and $v/u \in C^{0,\alpha}_{\mathrm{loc}}(B_1)$

$$\left\| \frac{v}{u} \right\|_{C^{0,\alpha}(B_{1/2})} \leq C \|v\|_{L^2(B_1)}.$$

This extends to any space dimension $n \geq 2$ the uniform Harnack estimates by Logunov and Malinnikova in case of real analytic variable coefficients. Note that, in dimension $n = 2$, a bound on the number of nodal domains of an entire harmonic function is equivalent to a bound on its frequency at infinity.

The uniformity part is the hardest.

Uniform gradient estimates for the ratio in $\mathcal{S}_{N_0}$ in dimension $n = 2$

Theorem (TTV 2022-23)

Let $n = 2$ and let us consider a pair $(u, v) \in \mathcal{C}_{N_0}$ in B_1. Then the ratio $w = v/u$ belongs to $C^{1,1/N_0-}_{\mathrm{loc}}(B_1)$ and satisfies the following condition

$$\begin{cases} A\nabla w \cdot \nu = 0 & \text{on } R(u) \cap B_1 \\ \nabla w = 0 & \text{on } S(u) \cap B_1. \end{cases}$$

Moreover, fixed $0 < \beta < 1/N_0$, the following estimate holds true

$$\left\| \frac{v}{u} \right\|_{C^{1,\beta}(B_{1/2})} \leq C\|v\|_{L^2(B_1)}.$$

for any $(u, v) \in \mathcal{C}_{N_0}$ with a positive constant which depends only on λ, Λ, L, N_0 and β.

Our estimate is consistent with the L^∞ bound given by Mangoubi for gradients of ratios of harmonic functions.

Powers of L_A-harmonic weights

Now, let $u \in \mathcal{C}_{N_0}$ in B_1 (for example u can be an harmonic polynomial) and let us consider continuous solutions to

$$-\mathrm{div}(|u|^a \nabla w) = 0 \quad \text{in } B_1.$$

with $a > -1$. We seek Hölder and Schauder estimates for w which are uniform with respect to the choice of $u \in \mathcal{C}_{N_0}$ in the bounded frequency class. To this purpose, we need to extend the Liouville theorems. We are able to achieve this only in dimension $n = 2$.

A Liouville Theorem for the weighted equation

Theorem (TTV 2023: Liouville theorem for the weighted equation)

Let u be a harmonic polynomial and w be an entire continuous and nonconstant solution to.

$$-\mathrm{div}(|u|^a \nabla w) = 0 \quad \text{in } \mathbb{R}^2.$$

With $a > -1$. Suppose that there exists $\gamma \in [0, 2)$ and $C > 0$ such that

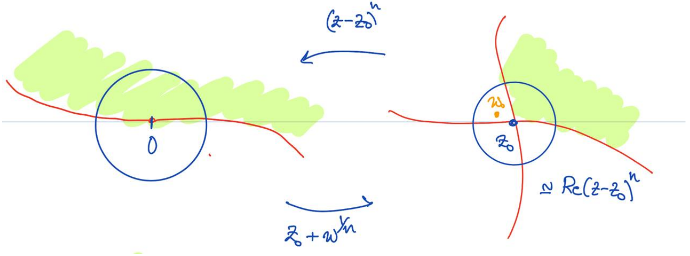
$$|w(z)| \leq C(1 + |z|)^\gamma \quad \text{in } \mathbb{R}^2.$$

Then u and w are affine functions with $\nabla u \cdot \nabla w \equiv 0$.

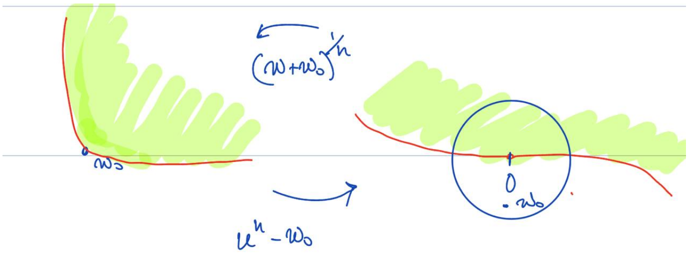
The proof uses conformal invariance and unique continuation principle for continuous solutions. The results yields Schauder estimates which are uniform with respect to $u \in \mathcal{C}_{N_0}$.

Remark: If u is an harmonic polynomial and w is its harmonic conjugate ($\nabla u \cdot \nabla w \equiv 0$) then w solves the weighted equation for every power a.

Idea: exploit conformal invariance to reduce to the case of a half-space.
First we smoothen the corners one by one:

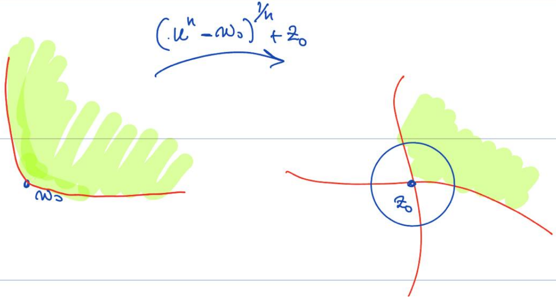


Idea: exploit conformal invariance and reduce to the case of a half-space. First we smoothen the corners one by one:



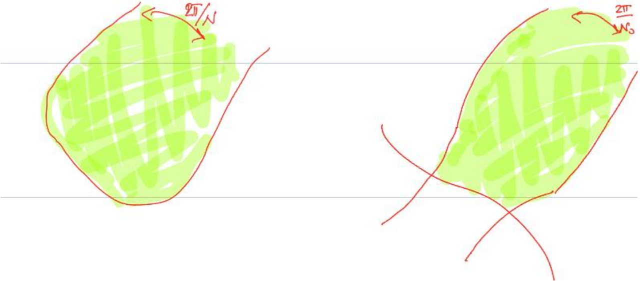
Liouville theorem in two dimensions

Idea: exploit conformal invariance and reduce to the case of a half-space. First we smoothen all the corners:



Liouville theorem in two dimensions

Then we reduce to the case of the half-space and we apply the previous theorem:

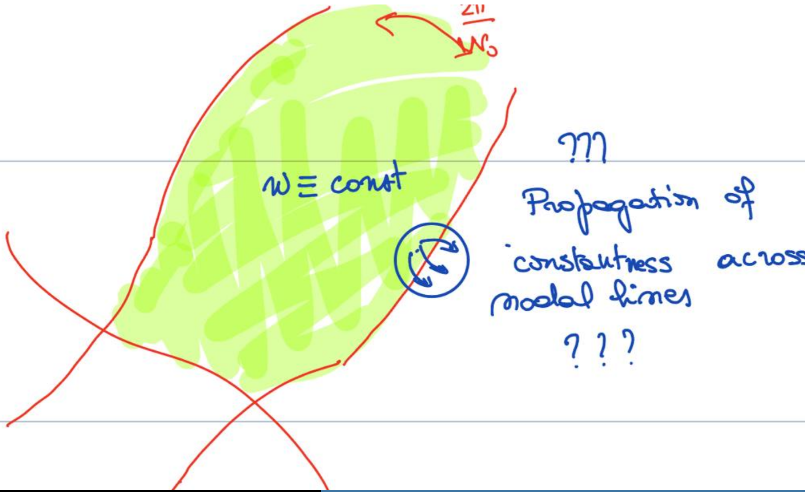


Liouville theorem in two dimensions

Then we reduce to the case of the half-space and we apply the previous theorem:

Liouville theorem in two dimensions

We need to propagate constantness across the regular part of the nodal set.



Theorem (Strong trance unique continuation on the regular part of the zero set)

Let $n \geq 2$, $a > -1$ and w be weak solutions in $H^{1,a}(B_1^+)$ to

$$\begin{cases} \mathrm{div}\,(y^a A \nabla w) = 0 & \text{in } B_1^+ \\ y^a A \nabla w \cdot \vec{e}_n = 0 & \text{in } B_1', \end{cases}$$

and A symmetric, uniformly elliptic, with Lipschitz continuous entries and such that $A(x,0)\vec{e}_n = m(x)\vec{e}_n$. If $u(x,0) = O(|x|^k)$ as $|x| \to 0^+$ for any $k \in \mathbb{N}$, then, u is trivial in B_1^+.

Theorem (Liouville theorem for general powers $a > -1$)

Let $n \geq 2$, $a > -1$ and u be a harmonic polynomial. Let w be an entire (and continuous) energy solution) to

$$\mathrm{div}\left(|u|^a \nabla w\right) = 0 \qquad \text{in } \mathbb{R}^n.$$

Suppose that there exist $\gamma \geq 0$ and $C > 0$ such that

$$|w(z)| \leq C(1 + |z|)^\gamma \qquad \text{in } \mathbb{R}^n.$$

Then the following points hold true:

i) *if $\deg(u) = 1$, then w is a polynomial of degree at most $\lfloor\gamma\rfloor$ and symmetric with respect to the hyperplane $Z(u)$;*

ii) *if $\deg(u) \geq 2$, $n = 2$ and $\gamma < \deg(u)$, then w is constant.*