Capillary hypersurfaces

Guofang Wang

University of Freiburg

June 28, 2023 IMAG, Granada

The talk based on the jonit work with

Chao Xia, Julian Scheuer, Liangjun Weng, Xiaoyan Jia, Xuwen Zhang and Xinqun Mei

on hypersurfaces in a space form with boundary about

1. Minkowski formulas
2. Alexandrov-Fenchel inequalities (Isoperimetric ineqality)
3. Karcher-Heintze inequality
4. Geometric curvature flow

Minkowski formulas for closed hypersrufaces

Let Σ be a closed hypersurfaces in $\mathbb{R}^{n+1}$.

1. Minkowski formula:

$$\int_\Sigma \langle x,\nu\rangle H_1 = \int_\Sigma 1, \quad \int_\Sigma \langle x,\nu\rangle H_{k+1} = \int_\Sigma H_k.$$

Proof: $\operatorname{div} x = n+1$ implies

$$\operatorname{div}_\Sigma x^T = \operatorname{div}_\Sigma (x - \langle x,\nu\rangle \nu) = n - nH_1$$

Applications: The classification of stable CMC hypersrufaces (Barbosa-Do Carmo, Barbosa-Do Carmo-Eschenberg)

Test function: $1 - \langle x,\nu\rangle H_1$, since $\int_\Sigma (1 - \langle x,\nu\rangle H_1) = 0$

2. Karcher-Heintze-Ros inequality for embedded hypersurfaces

$$\int_\Sigma \frac{1}{H_1} \geq (n+1)|\widehat{\Sigma}|,$$

equality iff Σ is a sphere, where $\widehat{\Sigma}$ is the domain enclosed by Σ.

Application: If $H_1 = const.$, then

$$|\Sigma| = \int_\Sigma \langle x, \nu \rangle H_1 = H_1 \int_\Sigma \langle x, \nu \rangle = H_1 (n+1)|\widehat{\Sigma}|,$$

i.e,

$$\int_\Sigma \frac{1}{H_1} = (n+1)|\widehat{\Sigma}|,$$

and hence Σ is a sphere. (Ros, Montiel-Ros)

3. Alexandrov-Fenchel inequalities and isoperimetric inequality

$$\frac{\int_\Sigma H_l}{\omega_n} \geq \left(\frac{\int_\Sigma H_k}{\omega_n}\right)^{\frac{n-l}{n-k}}, \ (l > k), \qquad \frac{|\Sigma|}{\omega_n} \geq \left(\frac{(n+1)|\widehat{\Sigma}|}{\omega_n}\right)^{\frac{n}{n+1}}$$

Equality iff Σ is a sphere.

There are many applications.

Guan-Li, Huisken, Chang-Y. Wang, Qiu, Agostiniani-Fogagnolo-Mazzieri for non-convex

Open Problem: Is Minkowski inequality true for mean convex hypersurfaces

$$\frac{\int_\Sigma H_1}{\omega_n} \geq \left(\frac{|\Sigma|}{\omega_n}\right)^{\frac{n-1}{n}} ?$$

One flow approach to prove AF inequality

The Minkowski formulas can be used to prove AF (Guan-Li):

1. $$\partial_t x = \left(\frac{H_k}{H_{k+1}} - \langle x, \nu \rangle \right) \nu = f\nu$$

2. Variational formula (W_k is the quermassintegral)

$$\frac{d}{dt} W_{k+1} = \frac{d}{dt} \int_\Sigma H_k = \int H_{k+1} f.$$

$$\begin{aligned}
\frac{d}{dt} W_{k+1} &= \int H_{k+1} \left(\frac{H_k}{H_{k+1}} - \langle x, \nu \rangle \right) = \int (H_k - H_{k+1} \langle x, \nu \rangle) = 0. \\
\frac{d}{dt} W_k &= \int H_k \left(\frac{H_k}{H_{k+1}} - \langle x, \nu \rangle \right) \geq \int (H_{k-1} - H_k \langle x, \nu \rangle) = 0.
\end{aligned}$$

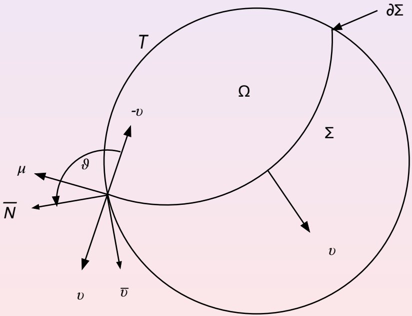
If the flow converges to a sphere, then we obtain the AF inequality!

Free boundary or capillary hypersurfaces in $\mathbb{B}^{n+1}$

We consider hypersurfaces Σ in $\mathbb{B}^{n+1}$ with free boundary $\partial\Sigma \subset \partial\mathbb{B}^{n+1} = \mathbb{S}^n$

$$\theta = \frac{\pi}{2}, \quad \text{or} \quad \mu = \bar{N}.$$

General $\theta \in (0, \pi)$, it is called a capillary hypersurface.



Two geometries, Σ in $\mathbb{R}^{n+1}$ and $\partial\Sigma$ on $\mathbb{S}^n$.

A new Minkowski formula

An old approach: from $\mathrm{div}_\Sigma x^T = n - nH_1$ (Ros-Vergasta)

$$\int_\Sigma (n - nH_1) = \int_{\partial\Sigma} \langle x^T, \mu\rangle = \int_{\partial\Sigma} 1 = |\partial\Sigma|.$$

Not good in applications.

Instead we considered

$$X_a = \langle x, a\rangle x - \frac{1}{2}(|x|^2 + 1)a, \qquad \text{for } a \in \mathbb{R}^{n+1}.$$

$\frac{1}{2}\left[\bar\nabla_i (X_a)_j + \bar\nabla_j (X_a)_i\right] = \langle x, a\rangle \delta_{ij} \Rightarrow$

$$\mathrm{div}_\Sigma X_a^T = n(\langle x, a\rangle - H_1\langle X_a, \nu\rangle), \qquad \langle X_a, x\rangle|_{\partial\mathbb{B}} = 0.$$

New Minkowski formula (W.-Xia Math. Ann 2019)

$$\int_\Sigma \{\langle x, a\rangle - H_1\langle X_a, \nu\rangle\} dA = 0,$$

Stable CMC capillary hypersurface are unique

Application: Classification of stable CMC free boundary hypersurfaces

W.-Xia ($n \geq 2$)

Any immersed stable free boundary (and also capillary) CMC hypersurface is a spherical cap.

Ros-Vergasta, Nunes ($n = 2$)

Heintze-Karcher-Ros type inequality (W.-Xia)

Let $x : \Sigma \to \mathbb{B}^{n+1}$ be an embedded free boundary hypersurfaces with Σ lies in a half ball $\mathbb{B}^{n+1}_+ = \{\langle x, a\rangle > 0\}$. If $H_1 > 0$, then

$$\int_\Sigma \frac{\langle x, a\rangle}{H_1} \geq (n+1) \int_\Omega \langle x, a\rangle,$$

equality iff Σ is a spherical cap.

Theorem (Scheuer-W.-Xia JDG 2022)

$$W_n(\Sigma) \geq (f_n \circ f_k^{-1})(W_k(\Sigma)),$$

where $f_k = f_k(r)$ *is the strictly increasing real function*

$$f_k = W_k(C_r).$$

Equality holds if and only if Σ *is a spherical cap or a flat disk* C_r.

$$\frac{d}{dt}x = \left\{ \frac{\langle x, a\rangle}{\frac{H_n}{H_{n-1}}} - \langle X_a, \nu\rangle \right\}\nu$$

The flow preserves W_n and increases W_k for $k < n$.

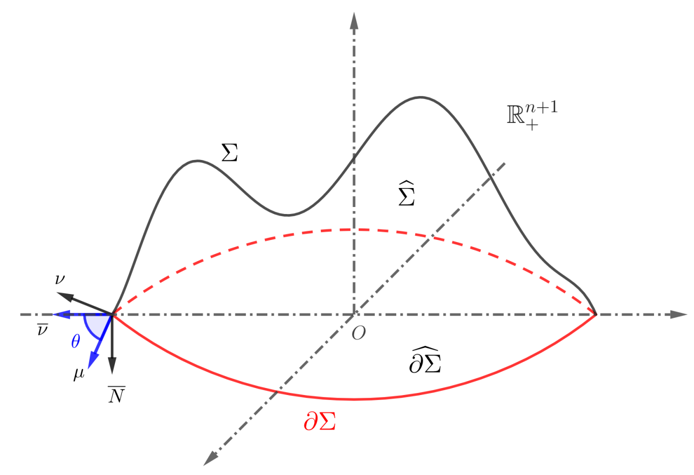
Recent work: W.-Weng, Weng-Xia, Hu-Wei-Yang-Zhou

$$
\begin{aligned}
\partial_t W_k(\Sigma_t) &= \int_{\Sigma_t} \frac{H_k H_{n-1}}{H_n}\langle x, a\rangle - H_k\langle X_a, \nu\rangle \\
&\geq \int_{\Sigma_t} H_{k-1}\langle x, e\rangle - H_k\langle X_a, \nu\rangle \\
&= 0,
\end{aligned}
$$

If $k = n$, we have equality

$$
\begin{aligned}
\partial_t W_n(\Sigma_t) &= \int_{\Sigma_t} \frac{H_n H_{n-1}}{H_n}\langle x, e\rangle - H_n\langle X_e, \nu\rangle \\
&= \int_{\Sigma_t} H_{n-1}\langle x, e\rangle - H_n\langle X_e, \nu\rangle \\
&= 0
\end{aligned}
$$

Capillary hypersurface in $\mathbb{R}^{n+1}_+$



Capillary hypersurface in $\mathbb{R}^{n+1}_+$

Capillary area functional

$$|\Sigma| - \cos\theta |\widehat{\partial\Sigma}|$$

Capillary isoperimetric inequality

$$\frac{|\Sigma| - \cos\theta|\widehat{\partial\Sigma}|}{|\mathbb{S}^n_\theta| - \cos\theta|\widehat{\partial\mathbb{S}^n_\theta}|} \geq \left(\frac{|\widehat{\Sigma}|}{|\mathbb{B}^{n+1}_\theta|}\right)^{\frac{n}{n+1}}$$

$\mathbb{S}^n_\theta = \{x \in \mathbb{S}^n | x_{n+1} \geq \cos\theta\}$, $\mathbb{B}^{n+1}_\theta = \{x \in \mathbb{B}^{n+1} | x_{n+1} \geq \cos\theta\}$

Question:

1. What is the capillary Alexandrov-Fenchel inequalities?
2. What is the quermassintegral for capillary hypersurfaces in $\mathbb{R}^{n+1}_+$?

Quermassintegral for capillary hypersurfaces

$$\mathcal{W}_{0,\theta}(\widehat{\Sigma}) := |\widehat{\Sigma}|, \qquad \mathcal{W}_{1,\theta}(\widehat{\Sigma}) := \frac{1}{n+1}(|\Sigma| - \cos\theta|\widehat{\partial\Sigma}|),$$

$$\mathcal{W}_{k+1,\theta}(\widehat{\Sigma}) \quad := \quad \frac{1}{n+1}\left(\int_\Sigma H_k dA - \frac{\cos\theta\sin^k\theta}{n}\int_{\partial\Sigma} H_{k-1}^{\partial\Sigma} ds\right),$$

$$\mathcal{W}_{n+1,\theta}(\widehat{\Sigma}) = \frac{1}{n+1}\int_\Sigma H_n dA - \cos\theta\sin^n\theta\frac{\omega_n}{n(n+1)}.$$

Variational formula:

$$\frac{d}{dt}\mathcal{W}_{k,\theta}(\widehat{\Sigma_t}) = \frac{n+1-k}{n+1}\int_{\Sigma_t} fH_k dA_t$$

for $(\partial_t x)^\perp = f\nu$.

Minkowski formula

Minkowski formula:

$$\int_\Sigma H_{k-1}(1+\cos\theta\langle \nu, e\rangle)dA = \int_\Sigma H_k\langle x,\nu\rangle dA$$

A crucial vector filed: $x+\cos\theta(\langle \nu, e\rangle x - \langle x,\nu\rangle e)$.

Set $P_e := \langle \nu, e\rangle x - \langle x,\nu\rangle e$

$$\begin{aligned} & \nabla_i\left((x^T+\cos\theta P_e^T)_j\right)+\nabla_j\left((x^T+\cos\theta P_e^T)_i\right) \\ = \; & (1+\cos\theta\langle \nu, e\rangle)g_{ij}-h_{ij}\langle x,\nu\rangle \end{aligned}$$

$$\mathrm{div}\,(x^T+\cos\theta P_e) = n(1+\cos\theta\langle \nu, e\rangle)-nH_1\langle x,\nu\rangle$$

$x^T+\cos\theta P_e\perp\mu$ along $\partial\Sigma$.

Theorem (W.-Weng-Xia Math Ann. (2023))

For $n \geq 2$, let $\Sigma \subset \overline{\mathbb{R}}^{n+1}_+$ be a convex capillary hypersurface with a contact angle $\theta \in (0, \frac{\pi}{2}]$, then there holds

$$\frac{\mathcal{W}_{n,\theta}(\widehat{\Sigma})}{\mathbf{b}_\theta} \geq \left(\frac{\mathcal{W}_{k,\theta}(\widehat{\Sigma})}{\mathbf{b}_\theta} \right)^{\frac{1}{n+1-k}}, \quad \forall\, 0 \leq k < n,$$

with equality if and only if Σ is a spherical cap.

$$(\partial_t x)^\perp = \left[(1 + \cos\theta \langle \nu, e \rangle) \frac{H_{l-1}}{H_l} - \langle x, \nu \rangle \right] \nu.$$

W.-Weng-Xia for star-shaped and k-convex, ie. a boundary version of Guan-Li.

Hu-Wei-Yang-Zhou for convex.

But still open, when $\theta > \frac{\pi}{2}$.

Minkowski inequality for capillary hypersurfaces

W.-Weng-Xia

Star-shaped and mean convex capillary hypersurface with a contact angle $\theta \in (0, \pi)$ satisfies the following Minkowski inequality

$$\int_\Sigma H\,dA - \sin\theta\cos\theta|\partial\Sigma| \geq n(n+1)^{\frac{1}{n}}|\mathbb{S}^n_\theta|^{\frac{1}{n}}\left(|\Sigma| - \cos\theta|\widehat{\partial\Sigma}|\right)^{\frac{n-1}{n}},$$

with equality if and only if Σ is a capillary spherical cap.

Open problem: Can one remove the condition of star-shaped?

Heintze-Karcher inequality for capillary hypersurface

The Minkowski type formula capillary hypersurfaces in $\mathbb{R}^{n+1}_+$

$$\int_\Sigma (1 - \cos\theta \langle \nu, e_{n+1} \rangle) = \int_\Sigma H \langle x, \nu \rangle.$$

It is natural to ask if the Heintze-Karcher inequality holds true

$$\int_\Sigma \frac{1 - \cos\theta \langle \nu, e_{n+1} \rangle}{H} \geq (n+1)|\widehat{\Sigma}|.$$

Theorem Jia-Xia-Zhang ($\theta \in (0, \pi/2]$)

$$\int_\Sigma \frac{1}{H} - \cos\theta \frac{(\int_\Sigma \langle \nu, e_{n+1} \rangle)^2}{\int_\Sigma H \langle \nu, e_{n+1} \rangle} \geq (n+1)|\widehat{\Sigma}|$$

Equality holds if and only if Σ is a spherical cap.

(Proof uses the Reilly type formula)

Theorem (Jia-W.-Xia-Zhang)

For $\theta \in (0, \pi)$, it holds

$$\int_{\Sigma} \frac{1 - \cos\theta \langle \nu, e_{n+1} \rangle}{H} \geq (n+1)|\widehat{\Sigma}|.$$

Equality holds if and only if Σ is a spherical cap.

Idea: Consider parallel hypersurfaces

$$x + t(\nu - \cos\theta \langle \nu, e_{n+1} \rangle)$$

which are still capillary hypersurfaces.

A similar result holds for capillary hypersurfaces in the unit ball.

Capillary Minkowski Problem

Observation:

$$A(\Sigma) = |\Sigma| - \cos\theta|\widehat{\partial\Sigma}| = \int_{\Sigma}(1 - \cos\theta\langle\nu(x), E_{n+1}\rangle)d\mathcal{H}^n(x),$$

It induces capillary area measure on Borel sets in
$\mathcal{C}_\theta = \{\xi \in \mathbb{R}^{n+1}_+ | |\xi + \cos\theta E_{n+1}| = 1\}$ $(\tilde{\nu} = \nu - \cos\theta E_{n+1})$

$$m_\theta(B) := A(\tilde{\nu}^{-1}(B)) = \int_{\tilde{\nu}^{-1}(B)}(1 - \cos\theta\langle\nu(x), E_{n+1}\rangle)d\mathcal{H}^n(x)$$

Capillary Minkowski Problem: Given a measure on $\mathcal{C}_\theta$, is there a convex capillary hypersurface such that its capillary area measure $m_\theta = m$?

Capillary Minkowski Problem (Xinqun Mei-W.-Liangjun Weng)

Let $\theta \in (0, \frac{\pi}{2}]$, $f \in C^2(\mathcal{C}_\theta)$ be a positive function with

$$\int_{\mathcal{C}_\theta} \frac{\langle \xi, E_\alpha \rangle}{f(\xi)} d\mathcal{H}^n(\xi) = 0, \qquad \forall 1 \leq \alpha \leq n,$$

Then there exists a $C^{3,\gamma}$ $(\gamma \in (0,1))$ strictly convex, capillary hypersurface $\Sigma \subset \bar{\mathbb{R}}^{n+1}_+$ such that its Gauss-Kroneckor curvature K satisfying

$$K(\tilde{\nu}^{-1}(\xi))) = f(\xi),$$

for all $\xi \in \mathcal{C}_\theta$. Moreover, Σ is unique up to a horizontal translation in $\bar{\mathbb{R}}^{n+1}_+$.

Thank You very much!