GOYA Research Group Seminar. 13th June 2025

ZONAL FUNCTION NETWORKS: SUPER-RESOLUTION APPROACH

ANDREI MARTÍNEZ-FINKHELSTEIN, HRUSHIKESH N. MHASKAR, AND JUAN ANTONIO VILLEGAS

Abstract

Given an even function $f : \mathbb{S}^2 \longrightarrow \mathbb{R}$ defined on the unit sphere, we assume that $f : \mathbb{S}^2 \longrightarrow \mathbb{R}$ has a specific form (or it is well-approximated by such an expression), called zonal function network (ZF-Network or ZFN):

$$f(\mathbf{x}) = \sum_{i=1}^{K} d_i \, G(\mathbf{x}, y_i^*), \quad G(\mathbf{x}, \mathbf{y}) = |\mathbf{x} \cdot \mathbf{y}|^{2\gamma+1},$$

where $\gamma > -1/2$ is known and $2\gamma + 1$ is not an even integer. We aim to determine the values of d_i's and y_i^*'s from a sample $\{(\mathbf{x}_j, f(\mathbf{x}_j))\}_{j=1}^{M}$.

There are several approaches to this problem, but this talk explores approximation techniques using spherical harmonics and localized kernels, emphasizing their theoretical foundations and practical implementations. We will use the known sample and cubature formulas to define a function which indicates the location of the centers $\{y_i^*, i = 1, \ldots, K\}$ and the value of the coefficients $\{d_i, i = 1, \ldots, K\}$. Finally, we will present some numerical examples to illustrate the performance of the proposed method.

Keywords: Orthogonal Polynomials, Approximation Theory, ZF Networks, Spherical Harmonics, ultraspherical polynomials, localized kernels.
AMS Classification: 33C45, 33C50, 42C05.

Bibliography

[1] H.N. Mhaskar and R. O'Dowd. Learning on manifolds without manifold learning. *Neural Networks*, 181, 2025.
[2] H. N. Mhaskar. Kernel-based analysis of massive data. *Frontiers in Applied Mathematics and Statistics*, 6:30, 2020.
[3] H.N. Mhaskar. Function approximation with zonal function networks with activation functions analogous to the rectified linear unit functions. *Journal of Complexity*, 51, 1–19, 2019.
[4] Q.T. Gia and H.N. Mhaskar. Localized Linear Polynomial Operators and Quadrature Formulas on the Sphere. *SIAM J. Numer. Anal.*, 47, 440–466, 2008.

Juan Antonio Villegas,
Instituto de Matemáticas (IMAG) and Departamento de Matemática Aplicada,
Universidad de Granada.
`jantoniovr@ugr.es`

Andrei Martínez-Finkhelstein,
Department of Mathematics,
Baylor University.
`A_Martinez-Finkhelstein@baylor.edu`

Hrushikesh N. Mhaskar,
Distinguished Research Professor,
Claremont Graduate University.
`Hrushikesh.Mhaskar@cgu.edu`

Universidad de Granada; IMAG-María de Maeztu, grant "IMAG CEX2020–001105–M"; GOYA: "Grupo de Ortogonalidad Y Aplicaciones" and "Departamento de Matemática Aplicada".