**International Doctoral Summer School in**
**Conformal Geometry and Non-local Operators**
Conference abstracts

IMAG, Granada, June 26-30th, 2023

**Speaker: Virginia Agostiniani (U. di Trento)**
Title: A PDE proof of the Riemannian Penrose inequality.
Abstract: We present a simple proof of the Riemannian Penrose inequality which builds on the analysis of the level sets of p-harmonic functions.

**Speaker: Renato G. Bettiol (CUNY)**
Title: Bifurcating minimal surfaces in ellipsoids of revolution.
Abstract: In this talk, I will discuss nonuniqueness results for embedded minimal spheres and tori in 3-dimensional ellipsoids with continuous symmetries. These families of minimal surfaces are obtained as bifurcating branches issuing from highly symmetric minimal surfaces. This talk is based on joint work with P. Piccione.

**Speaker: Esther Cabezas-Rivas (Universidad de Valencia)**
Title: Characterization of the subdifferential and minimizers for the anisotropic $p$-capacity.
Abstract: We obtain existence and uniqueness of minimizers for the $p$-capacity functional defined with respect to a centrally symmetric anisotropy for $1 < p < \infty$, including the case of a crystalline norm in $\mathbb{R}^N$. The result is obtained by a characterization of the corresponding subdifferential and it applies for unbounded domains of the form $\mathbb{R}^N \setminus \overline{\Omega}$ under mild regularity assumptions (Lipschitz-continuous boundary) and no convexity requirements on the bounded domain $\Omega$. If we further assume an interior ball condition (where the Wulff shape plays the role of a ball), then the minimizer is shown to be Lipschitz continuous. This is a joint work with Salvador Moll and Marcos Solera.

**Speaker: Jeffrey Case (Penn State)**
Title: A Gauss-Bonnet formula for renormalized areas.
Abstract: We use the extrinsic $Q$-curvature from Robin's talk to express the renormalized area of a minimal submanifold of a Poincaré-Einstein space as a linear combination of the Euler characteristic and the integral of a scalar conformal submanifold invariant. For two-dimensional submanifolds, this recovers a formula of Alexakis-Mazzeo. For four-dimensional submanifolds, this generalizes a formula of Tyrrell. In higher dimensions, we require a conjectural submanifold analogue of Alexakis' decomposition; we prove this decomposition in dimensions two and four. This is joint work with Robin Graham, Tzu-Mo Kuo, Aaron Tyrrell, and Andrew Waldron.

**Speaker: Eleonora Cinti (U. Bologna)**
Title: Optimal regularity for isoperimetric sets with density.
Abstract: In this talk, I will present a recent result which establishes optimal regularity for isoperimetric sets with densities, under mild Hölder regularity assumptions on the density functions. Our main Theorem improves some previous results and allows to reach the optimal regularity class $C^{1,\frac{\alpha}{2-\alpha}}$ in any dimension. This is a joint work with L. Beck and C. Seis.

**Speaker: Francesca Colasuonno (U. Bologna)**

Title: Critical double phase problems
Abstract: In this talk, we will discuss Brezis-Nirenberg type Dirichlet problems governed by the double phase operator $-\mathrm{div}(|\nabla u|^{p-2}\nabla u + a(x)|\nabla u|^{q-2}\nabla u)$ and involving a critical nonlinear term of the form $|u|^{p^*-2}u + b(x)|u|^{q^*-2}u$. We prove new compactness and existence results in Musielak-Orlicz Sobolev spaces via variational techniques, and also nonexistence results of Pohožaev type. This is joint work with Kanishka Perera.

**Speaker: Francesca De Marchis (La Sapienza U. di Roma)**
Title: On critical points of the Moser-Trudinger functional.
Abstract: Since the fundamental work by Trudinger from 1967 it is known that in two dimensions Sobolev functions in $H^1$ satisfy embedding properties of exponential type. In 1971 Moser then obtained a sharp form of the embedding, controlling the integrability of $F(u) := \int e^{u^2}$ in terms of the Sobolev norm of $u$. On a closed Riemannian surface, $F(u)$ is unbounded above for $\|u\|_{H^1} > 4\pi$. We are however able to find critical points of F constrained to any sphere $\{\|u\|_{H^1} = \beta\}$, with $\beta > 0$ arbitrary. The proof combines min-max theory, a monotonicity argument by Struwe, blow-up analysis and compactness estimates. This is joint work with A. Malchiodi, L. Martinazzi and P.D. Thizy.

**Speaker: José Espinar (Universidad de Granada)**
Title: Min-Oo conjecture for fully nonlinear conformally invariant equations
Abstract: In this talk we show rigidity results for super-solutions to fully nonlinear elliptic conformally invariant equations on subdomains of the standard n-sphere $S^n$ under suitable conditions along the boundary. We emphasize that our results do not assume concavity assumption on the fully nonlinear equations we will work with.

This proves rigidity for compact connected locally conformally flat manifolds $(M, g)$ with boundary such that the eigenvalues of the Schouten tensor satisfy a fully nonlinear elliptic inequality and whose boundary is isometric to a geodesic sphere $\partial D(r)$, where $D(r)$ denotes a geodesic ball of radius $r \in (0, \pi/2]$ in $S^n$, and totally umbilical with mean curvature bounded bellow by the mean curvature of this geodesic sphere. Under the above conditions, $(M, g)$ must be isometric to the closed geodesic ball $\overline{D(r)}$. In particular, we recover the solution by F.M. Spiegel to the Min-Oo conjecture for locally conformally flat manifolds.

As a side product, in dimension 2 our methods provide a new proof to Toponogov's Theorem about rigidity of compact surfaces carrying a shortest simple geodesic. Roughly speaking, Toponogov's Theorem is equivalent to a rigidity theorem for spherical caps in the Hyperbolic three-space $H^3$. In fact, we extend it to obtain rigidity for super-solutions to certain Monge-Ampère equations.

**Speaker: Pierpaolo Esposito (Roma Tre U.)**
Title: Exponential PDEs in high dimensions.
Abstract: For a quasilinear equation involving the $n-$Laplacian and an exponential nonlinearity, I will discuss quantization issues for blow-up masses in the non-compact situation, where the exponential nonlinearity concentrates as a sum of Dirac measures. A fundamental tool is provided here by some Harnack inequality of $\sup + \inf$ type, a question of independent interest that we prove in the quasilinear context through a new and simple blow-up approach.

**Speaker: Yuxin Ge (U. Toulouse)**

Title: Compactness of asymptotically hyperbolic Einstein manifolds in dimension 4 and applications

Abstract: Given a closed Riemannian manifold of dimension 3 $(M^3, [h])$, when will we fill in an asymptotically hyperbolic Einstein manifold of dimension 4 $(X^4, g_+)$ such that $r^2 g_+|_M = h$ on the boundary $M = \partial X$ for some defining function $r$ on $X^4$? This problem is motivated by the correspondence AdS/CFT in quantum gravity proposed by Maldacena in 1998 et comes also from the study of the structure of asymptotically hyperbolic Einstein manifolds.

In this talk, I discuss the compactness issue of asymptotically hyperbolic Einstein manifolds in dimension 4, that is, how the compactness on conformal infinity leads to the compactness of the compactification of such manifolds under the suitable conditions on the topology and on some conformal invariants. As application, I discuss the uniqueness problem and non-existence result. It is based on the works with Alice Chang.

**Speaker: Rod Gover (U. Auckland)**

Title: Some linear and non-linear (conformal) Dirichlet-Neumann maps

Abstract: Recall that the conformal-Robin operator and Yamabe operators leads to a conformally invariant first order Dirichlet-to-Neumann operator on conformal manifolds with boundary. We discuss the construction of canonical invariant differential operators along a hypersurface in a conformal manifold that are higher order analogues of the conformal-Robin operator and so facilitate the construction of higher odd order Laplacian powers on the boundary of conformal manifolds, and in particular on conformally compact manifolds including Poincare-Einstein geometries.

Using these tools, we then show how, in certain cases, the scalar field scattering map on conformally compact manifolds can also be recovered by an explicit conformal Dirichlet-to-Neumann map. Extending these ideas, we study the non-linear Dirichlet-to-Neumann map for the even-dimensional Poincaré–Einstein problem. In particular, we describe its range in terms of a natural rank two tensor along the boundary. In low dimensions we give this explicitly, while in higher dimensions we can provide a tractor formula.

**Speaker: Robin Graham (U. of Washington)**

Title: Extrinsic GJMS Operators for Submanifolds.

Abstract: The GJMS operators are a family of conformally covariant differential operators on Riemannian manifolds with principal part a power of the Laplacian. This talk will describe a construction and properties of an analogous family of operators associated to a submanifold of a Riemannian manifold. These are differential operators on the submanifold which depend on the extrinsic geometry of the submanifold and whose principal part is a power of the Laplcian in the induced metric. This is joint work with Jeffrey Case and Tzu-Mo Kuo.

**Speaker: Massimo Grossi (La Sapienza U. di Roma)**

Title: The number of critical points of the Robin function in domains with a small hole.

Abstract: In many nonlinear elliptic problems depending on a parameter (critical Sobolev exponent, Liouville equation of the plane) like

$$\begin{cases} -\Delta u = f_\epsilon(u) & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

the solution $u_\epsilon$ concentrates at a point $P \in \Omega$, i.e.

$$u_\epsilon(x) \to C_0 G(x,P) \quad \text{as } \epsilon \to 0 \text{ far away from } P,$$

where $G(x,y)$ is the Green function of $-\Delta$.

It is very important to characterize $P$ and in various situations it turns out that $P$ is a critical point of the Robin function $R(x)$. Recall that $R(x) = H(x,x)$ where $H(x,y)$ is the regular part of the Green function. In this talk we will calculate the number of critical points of $R$ when $\Omega$ is a domain with a small hole. We will see that the location of the hole plays a role in the results.

Results in collaboration with F. Gladiali, P. Luo and S. Yan (to appear in JEMS).

**Speaker: Matt Gursky (Notre Dame)**
Title: The ASD deformation complex and a conjecture of Singer.
Abstract: In this talk I will give a brief overview of the deformation complex for anti-self-dual 4-manifolds, and formulate a conjecture (attributed to Singer) about the vanishing of one of the cohomology groups. I will then report on some recent progress on resolving this conjecture, and its connection to a nonlinear PDE from spectral geometry. This is joint work with Rod Gover.

**Speaker: Isabella Ianni (La Sapienza U. di Roma)**
Title: Uniqueness results for local and non-local Dirichlet problems.
Abstract: We present our contributions on the question of the uniqueness for the positive solutions of semilinear Dirichlet problems with a power nonlinearity. This problem arose from the famous symmetry result by Gidas, Ni, Nirenberg (1979), which implies uniqueness for the Lane-Emden problem when the domain is a ball. A conjecture on the uniqueness in any convex domain was then formulated during the eighties, but only partial answers have been given so far. In this talk we describe recent results for the Lane-Emden problem obtained in dim=2. We also discuss some uniqueness results in the nonlocal case. This is from joint works with Francesca De Marchis, Massimo Grossi, Filomena Pacella (University Sapienza of Roma, Italy), Peng Luo and Shusen Yan (Wuhan Normal University, China), Alberto Saldana (UNAM, Mexico), Abdelrazek Dieb (University Ibn Khaldoun of Tiaret, Algeria).

**Speaker: Ernst Kuwert (Freiburg)**
Title: Curvature varifolds with free boundary.
Abstract: We consider Willmore surfaces in a domain with a free boundary condition, and also briefly in 3-manifolds. The problem to obtain mass bounds leads to a varifold setting (joint work with Marius Mueller).

**Speaker: Yueh-Ju Lin (Wichita)**
Title: Conformally covariant polydifferential operators associated with CVIs and its applications.
Abstract: Conformally variational Riemannian invariants (CVIs) such as scalar curvature and $Q$-curvature are homogeneous scalar invariants which appear as the conformal gradient of a Riemannian functional. In this talk, I will talk about constructing a formally self-adjoint, conformally covariant multilinear operator associated with a given CVI. This construction recovers the relationship between GJMS operator and higher order Q-curvature $Q_{2k}$. I will also discuss a complete classification of tangential bidifferential operators in terms of Laplacian of the ambient space. This result is a

curved analogue of such operators on spheres classified by Ovsienko-Redou and Clerc. As an application, we construct a large class of formally self-adjoint conformally covariant differential operators. At the end of the talk, I will present a family of sharp, fully nonlinear Sobolev inequalities involving the Paneitz operator and Ovsienko-Redou operator. This talk is based on joint works with Jeffrey Case and Wei Yuan.

**Speaker: Ali Maalaoui (Clark U.)**
Title: Compactness of Dirac-Einstein Structures in Dimensions 3 and 4.
Abstract: Let $M$ be a compact spin manifold. We consider then the functional

$$E(g,\psi) = \int_M R_g + \langle D_g\psi, \psi\rangle - \lambda|\psi|^2 \; dv_g,$$

where $g$ is a Riemannian metric and $\psi$ is a spinor on M. The critical points of such a functional are called Einstein-Dirac structures. In this talk, I will discuss the compactness of sets of such structures. I will show that under fairly mild conditions the set of such structures is compact in dimension 3 but in dimension 4, there is compactness up to bubbling. The bubbling profile and the local structure under perturbation of the metric are also discussed. This is a joint work with V. Martino.

**Speaker: Vittorio Martino (U. Bologna)**
Title: On the conformal Dirac-Einstein equations
Abstract: We will introduce the Dirac-Einstein equations on a spin manifold, starting from motivations coming from Physics. After restricting the related functional on a conformal class, we will show the classification of the Palais-Smale sequences and some results regarding the existence of solutions.

**Speaker: Stephen McKeown (UT Dallas)**
Title: A fourth-order Escobar-Yamabe problem on a half-ball.
Abstract: The celebrated Yamabe problem asks us to make a conformal change on a compact Riemannian manifold such that the scalar curvature becomes constant. The (Type-II) Escobar-Yamabe problem is to make a conformal change on a compact Riemannian manifold with boundary so that the scalar curvature vanishes and the boundary mean curvature is constant. Both of these problems have been generalized to higher order: the Q-Yamabe problem is a Yamabe-type problem for a fourth-order curvature invariant, while the (Q,T) Yamabe-Escobar problem adds a third-order boundary curvature. In the critical dimension n = 4, Q and T appear as integrands in the Gauss-Bonnet formula, giving the problem a particularly nice interpretation. We review this story, then introduce a form of the Gauss-Bonnet theorem on a four-manifold with corners for which the curvatures have nice conformal transformation properties. A natural question of Escobar-Yamabe type is: can one make a conformal change so that all of the Gauss-Bonnet integrands vanish except on the corner, and are there constant? We answer this in the affirmative for the half-ball in four-space. This is joint work with Jeffrey Case, Tzu-Mo Kuo, Yueh-Ju Lin, Cheikh Ndiaye, Andrew Waldron, and Paul Yang.

**Speaker: Angela Pistoia (Sapienza U. di Roma)**
Title: Conformal metrics with prescribed curvatures.
Abstract: I will present some recent results concerning the classical geometric problem of prescribing the scalar and boundary mean curvatures via conformal deformation of the metric on a $n$-dimensional compact Riemannian manifold.

**Speaker: Jie Qing (UC Santa Cruz)**
Title: $p$-Laplace equations, $p$-superharmonic functions, and their applications in conformal geometry.
Abstract: In this talk we will report our recent work in introducing the $p$-Laplace equations for the intermediate Schouten curvature in conformal geometry. These $p$-Laplace equations provide more tools for the study of geometry and topology of manifolds. First, the positivity of the intermediate Schouten curvature yields the vanishing of Betti numbers on locally conformally flat manifolds as consequences of the Böchner formula for harmonic forms studied by Guan-Lin-Wang in 2005. Secondly and more interestingly, when the intermediate Schouten curvature is nonnegative, these $p$-Laplace equations facilitate the geometric applications of the study of singularities of $p$-superharmonic functions by the nonlinear potential theory. This leads to the vanishing of homotopy groups that is inspired by and extends the work of Schoen-Yau in 1988. Our results on the asymptotic of $p$-superharmonic functions at singularities extend Arsove-Huber type theorems in higher dimensions.

**Speaker: Jesse Ratzkin (Würzburg U.)**
Title: Compactness of singular constant $Q$-curvature metric on punctured spheres.
Abstract: The $Q$-curvatures of various orders of a Riemannian metric generalize its scalar curvature, except that they transform under a conformal change according to a higher order differential equation (or, more generally, a nonlocal operator). Much recent work concentrates on recreating Yamabe's program (and all the theory growing out of it) in this more general setting. As with scalar curvature, conformal invariance forces one to consider singular solutions, even if one is at first only interested in smooth solutions. In this talk I will describe an analog of the singular Yamabe problem and discuss some properties of the moduli space of singular solutions on a punctured sphere. In particular, I will describe geometric properties that allow one to extract convergent subsequences from a family of solutions. A Pohozaev-type invariant is a key tool, and is also of independent interest. This is joint work with Joào Henrique Andrade, João Marcos do Ò and Juncheng Wei.

**Speaker: David Ruiz (U. de Granada)**
Title: Steady solutions for the 2D Euler equations and overdetermined elliptic problems.

An important open question is to show whether compactly supported solutions to the stationary 2D Euler equations have circular streamlines. In this communication we will consider this question also for weak solutions, and we will show its relation with overdetermined elliptic problems. We will be interested in both rigidity results and the construction of nonsymmetric solutions.

For the latter, we will show an example of a $C^2$ function $f(u)$ and a bounded domain $\Omega$ different from a ball such that the overdetermined elliptic problem:

$$\begin{cases} -\Delta u = f(u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \\ \partial_\nu u = \text{constant} & \text{on } \partial\Omega, \end{cases}$$

admits a sign-changing solution. This construction is of independent interest and is valid also in dimensions 3 and 4.

**Speaker: Ben Sharp (Leeds U.)**

Title: Degree-one $\alpha$-harmonic maps with low energy.

Abstract: In the celebrated works of Sacks-Uhlenbeck, a sub-critical functional $E_\alpha$ is utilised in order to prove existence of non-trivial harmonic maps from a surface to a Riemannian manifold. For $\alpha > 1$ this functional satisfies the Palais-Smale condition and all critical points, $\alpha$-harmonic maps, are smooth up to some uniform regularity scale which depends on $\alpha$. As $\alpha \to 1$, $E_\alpha$ becomes the Dirichlet energy and the regularity scale potentially degenerates leading to the formation of harmonic spheres. This is the well-known bubbling phenomenon.

We will classify low-energy degree-one $\alpha$-harmonic maps from non-spherical surfaces $\Sigma$ to the round sphere via their bubble scales and centres. In particular for $\alpha$ close to 1 and $E_\alpha$ sufficiently small, these maps (which are guaranteed to exist) must blow a bubble centred at a critical point $a_c$ of a geometric function $\mathcal{J}$ and at scale $\sqrt{|\mathcal{J}(a_c)|^{-1}(\alpha-1)}$. Up to a constant, $\mathcal{J}$ is the sum of the squares of any $L^2$-orthonormal basis of holomorphic one-forms on the domain.

The talk is based on an ongoing joint work with Tobias Lamm and Mario Micallef. We may also discuss links to a joint work with Andrea Malchiodi and Melanie Rupflin concerning $H$-surfaces.

**Speaker: Susanna Terracini (U. di Torino)**

Title: Boundary Harnack principles and degenerate equations on singular sets.

Abstract: The ratio $v/u$ of two solutions to a second order elliptic equation in divergence form solves a degenerate elliptic equation if $u$ and $v$ share the zero set; that is, $Z(u) \subseteq Z(v)$. The coefficients of the degenerate equation vanish on the nodal set as $u^2$. Developing a Schauder theory for such equations, we prove $C^{k,\alpha}$-regularity of the ratio from one side of the regular part of the nodal set in the spirit of the higher order boundary Harnack principle established by De Silva and Savin in [4]. Then, by a gluing lemma, the estimates extend across the regular part of the nodal set. Eventually, using conformal mapping in dimension $n = 2$, we provide local gradient estimates for the ratio which hold also across the singular part of the nodal set and depends on the highest value attained by the Almgren frequency function.

## References

[1] S. Terracini, G. Tortone and S. Vita, *Higher order boundary Harnack principle on nodal domains via degenerate equations*, preprint, 2022.

[2] Y. Sire, S. Terracini, S. Vita. *Liouville type theorems and regularity of solutions to degenerate or singular problems part I: even solutions*. Comm. Partial Differential Equations, 46-2 (2021), 310-361.

[3] Y. Sire, S. Terracini, S. Vita. *Liouville type theorems and regularity of solutions to degenerate or singular problems part II: odd solutions*. Mathematics in Engineering, 3-1 (2021), 1-50.

[4] D. De Silva, O. Savin. *A note on higher regularity boundary Harnack inequality*. DCDS-A, 35(12), (2015) 6155-6163.

**Speaker: Guofang Wang (Freiburg)**

Title: Geometric inequalities for capillary hypersurfaces.
Abstract: In this talk we first introduce suitable geometric integrals for capillary hypersurfaces, then study isoperimetric problems between these quantities and establish corresponding geometric inequalities by using suitable geometric curvature flows.

**Speaker: Yi Wang (Johns Hopkins University)**
Title: Yamabe flow of asymptotically flat metrics.
Abstract: In this talk, we will discuss the behavior of the Yamabe flow on an asymptotically flat (AF) manifold. We will first show the long-time existence of the Yamabe flow starting from an AF manifold and discuss the uniform estimates on manifolds with positive Yamabe constant. This would allow us to prove global weighted convergence along the Yamabe flow on such manifolds. We will also talk about the case when the Yamabe constant is nonpositive. This is joint work with Eric Chen and Gilles Carron.

**Speaker: Juncheng Wei (UBC Vancouver)**
Title: Optimal Beckner Inequality for Axially Symmetric Functions on $S^4$ and $S^6$
Abstract: We prove that for $n = 4$ and $n = 6$ axially symmetric solutions to the $Q$-curvature type problem

$$\alpha P_n u + c_n\left(1 - \frac{e^{nu}}{\int_{\mathbb{S}^n} e^{nu}}\right) = 0 \qquad \text{on } \mathbb{S}^n$$

must be constants, provided that $\frac{1}{2} \leq \alpha < 1$. This result is sharp and also closes some gaps in existing literature. The improvement is based on two types of new estimates: one is a better estimate of the semi-norm $\lfloor G \rfloor^2$, the other one is a family of refined estimates on Gegenbauer coefficients, such as pointwise decaying and cancellations properties. (Joint work with Gui, Li and Ye.)

**Speaker: Paul Yang (Princeton U.)**
Title: Variation of the Einstein-Hilbert action for pseudohermitian manifolds.
Abstract: We compute the first and second variation of the Einstein-Hilbert action and show that the critical points are pseudo-Einstein structures. We then compute the second variation on the standard CR sphere. The result differs depending on whether the deformation is in the embedded direction.