A Gauss–Bonnet Formula for Renormalized Areas

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Renormalized Volumes and Areas

Theorem (Graham (2000))

Let (X^{n+1}, g_+), *n odd, be an asymptotically hyperbolic manifold such that* $g_+ = r^{-2}\left(dr^2 + g_r\right)$ *with*

$$g_r = g_{(0)} + r^2 g_{(2)} + \cdots + r^{n-1} g_{(n-1)} + r^n g_{(n)} + o(r^n)$$

and $\operatorname{tr}_{g_{(0)}} g_{(n)} = 0$. *Then*

$$\int_{\{r>\varepsilon\}} \mathrm{dV}_{g_+} = c_0 r^{-n} + c_2 r^{2-n} + \cdots + c_{n-1} r^{-1} + \mathcal{V} + o(1).$$

Moreover, the renormalized volume $\mathcal{V}$ *depends only on* g_+.

Renormalized Volumes and Areas

Theorem (Chang–Qing–Yang (2006))

Let (X^{n+1}, g_+) be an even-dimensional Poincaré–Einstein manifold. There is a scalar conformal invariant $\mathcal{X}_{n+1}$ of weight $-n-1$ such that

$$(-1)^{(k+1)/2}\mathcal{V} = \frac{(2\pi)^{(n+1)/2}}{n!!}\chi(X) + \int_X \mathcal{X}_{n+1}.$$

Remarks

1. $\mathcal{X}_2$, $\mathcal{X}_4$, and $\mathcal{X}_6$ are explicitly known.
2. Existence due to Alexakis' decomposition (2006–12).
3. Key ingredients are factorization of GJMS operators for Einstein manifolds and the (Q_{n+1}-flat) scattering compactification, or adapted metric.

The Extrinsic GJMS Operator and Extrinsic Q-curvature

Theorem (C.–Graham–Kuo (2023))

Let $i\colon Y^{k+1} \to (X^{n+1}, [g])$, $k < n$ odd, be a conformal immersion. For each $h \in i^*[g]$, there is an operator $P_{k+1}\colon C^\infty(Y) \to C^\infty(Y)$ and a scalar $Q_{k+1} \in C^\infty(Y)$ such that

1. *P_{k+1} and Q_{k+1} are natural;*
2. *$P_{k+1} = (-\overline{\Delta})^{(k+1)/2} + \mathrm{l.o.t.}$;*
3. *P_{k+1} is formally self-adjoint with $P_{k+1}(1) = 0$;*
4. *if $u \in C^\infty(Y)$, then*

$$e^{(k+1)u} Q_{k+1}^{e^{2u}h} = Q_{k+1}^h + P_{k+1}^h(u);$$

5. *if $i(Y)$ is minimal and $h = i^*g$ for $\mathrm{Ric}_g = n\lambda g$, then*

$$P_{k+1} = \prod_{j=0}^{\frac{k-1}{2}} \left(-\overline{\Delta} + \frac{(k+2j+1)(k-2j-1)}{4} \lambda \right).$$

The Gauss–Bonnet Formula in General Dimension

Theorem (C.–Graham–Kuo–Tyrrell–Waldron (WIP))

Let $k \in \mathbb{N}$ be odd. Suppose that there is a constant $c_{n,k}$ and a scalar conformal submanifold invariant $\mathcal{W}_{k+1}$ such that

$$Q_{k+1} = c_{n,k}\overline{\mathrm{Pf}} + \mathcal{W}_{k+1} + \overline{\nabla}_\alpha V^\alpha$$

for every Riemannian embedding $i \colon Y^{k+1} \to (X^{n+1}, g)$, $k < n$. If $i \colon Y^{k+1} \to (X^{n+1}, g_+)$ is a regular minimal immersion into a Poincaré–Einstein space, then

$$(-1)^{(k+1)/2}\mathcal{A} = \frac{(2\pi)^{(k+1)/2}}{k!!}\chi(Y) + \frac{1}{k!}\int_Y \mathcal{W}_{k+1}.$$

The Gauss–Bonnet Formula in Low Dimensions

Corollary (Alexakis–Mazzeo (2010), CGKTW (WIP))

Let $i\colon Y^2 \to (X^{n+1}, g_+)$, $n > 2$, be a regular minimal immersion into a Poincaré–Einstein space. Then

$$-\mathcal{A} = 2\pi\chi(Y) + \frac{1}{2}\int_Y \left(|\mathring{L}|^2 - W_{\alpha\beta}{}^{\alpha\beta} \right) \mathrm{dA}.$$

Corollary (CGKTW (WIP), cf. Tyrrell (2022))

Let $i\colon Y^4 \to (X^{n+1}, g_+)$, $n > 4$, be a regular minimal immersion into a Poincaré–Einstein space. Then

$$\mathcal{A} = \frac{4\pi^2}{3}\chi(Y) - \frac{1}{6}\int_Y \left(\frac{1}{4}|\overline{W}|^2 - 3\mathcal{I} - 2|\mathsf{F}|^2 + 2\mathsf{G}^2 \right) \mathrm{dA}.$$

The Gauss–Bonnet Formula in Low Dimensions

Corollary (CGKTW (WIP), cf. Tyrrell (2022))

Let $i\colon Y^4 \to (X^{n+1}, g_+)$, $n > 4$, *be minimal. Then*

$$\mathcal{A} = \frac{4\pi^2}{3}\chi(Y) - \frac{1}{6}\int_Y \left(\frac{1}{4}|\overline{W}|^2 - 3\mathcal{I} - 2|\mathrm{F}|^2 + 2\mathrm{G}^2 \right) \mathrm{dA}.$$

Some conformal scalar submanifold invariants

$$\mathrm{F} := \frac{1}{k-1}\left(\mathring{L}_{\alpha\gamma\alpha'}\mathring{L}_{\beta}{}^{\gamma\alpha'} - W_{\alpha\gamma\beta}{}^{\gamma} - \mathrm{G}h \right), \quad \mathrm{G} := \operatorname{tr}_h \mathrm{F} = \frac{1}{2k}\left(|\mathring{L}|^2 - W_{\alpha\beta}{}^{\alpha\beta} \right)$$

$$\begin{aligned} \mathcal{I} := &-\frac{2k-5}{k}\overline{\Delta}\mathrm{G} + \frac{k-5}{k}\overline{\nabla}^{\alpha}\overline{\nabla}^{\beta}\mathrm{F}_{\alpha\beta} + (k-3)(\overline{\mathrm{J}} + \mathrm{G})\mathrm{G} \\ &+ \frac{(k-1)(k-5)}{k}\langle \mathrm{F} - \mathrm{G}i^*g, \overline{\mathrm{J}} + \mathrm{F}\rangle + \frac{2(k-2)(k-5)}{k}\left(\frac{1}{2(n-3)}B_{\alpha}{}^{\alpha} \right. \\ &\left. - H^{\alpha'}C_{\alpha\alpha'}{}^{\alpha} + \frac{1}{2}H^{\alpha'}H^{\beta'}W_{\alpha'\alpha\beta'}{}^{\alpha} - \frac{1}{2}|\mathrm{D}|^2 \right). \end{aligned}$$

Sketch of Proof — Key Ideas

1. Fix a representative $h_{(0)}$ of the conformal infinity of (Y, i^*g_+).
2. Let $h = \rho^2 h_+$ be the geo. compactification det'd by $h_{(0)}$. Then

$$\frac{1}{k!}\int_Y Q_{k+1} = \frac{(2\pi)^{(k+1)/2}}{k!!}\chi(Y) + \frac{1}{k!}\int_Y \mathcal{W}_{k+1}.$$

3. Let $\widehat{h}$ be the scattering compactification det'd by $h_{(0)}$; i.e.

$$-\overline{\Delta}_{h_+} v = k, \quad v = \log\rho + A + B\rho^k, \quad \widehat{h} = v^2 h_+.$$

   Then $Q^{\widehat{h}}_{k+1} = 0$ and $\mathcal{A} = \oint_{\partial Y} B|_{\partial Y}$.

4. Compute:

$$\int_Y Q^{\widehat{h}}_{k+1} = \int_Y Q^h_{k+1} + P^h_{k+1}(A + B\rho^k) \equiv (-1)^{\frac{k-1}{2}} k! \oint_{\partial Y} B|_{\partial Y}.$$

Sketch of Proof — Identities For Extrinsic Q-curvature

Theorem (CGKTW (WIP))

Let $i\colon Y^{k+1} \to (X^{n+1}, [g])$ *be a conformal immersion. Then*

$$Q_2 = \overline{Q}_2 + \mathsf{G}, \qquad \text{if } k = 1,$$

$$Q_4 = \overline{Q}_4 + Q_{4,\mathsf{G}} + k\mathcal{I} + 2|\mathsf{F}|^2 - \frac{k+1}{2}\mathsf{G}^2, \qquad \text{if } k = 3,$$

where

$$P_{4,\mathsf{G}} := \overline{\nabla}^{\alpha} \circ (4F_{\alpha\beta} - (k-1)\mathsf{G}g_{\alpha\beta}) \circ \overline{\nabla}^{\beta} + \frac{k-3}{2} Q_{4,\mathsf{G}} u,$$

$$Q_{4,\mathsf{G}} := 4\overline{\Delta}\mathsf{G} - (k-5)\overline{\nabla}^{\alpha}\overline{\nabla}^{\beta}\mathsf{F}_{\alpha\beta} + (k-3)(\cdots)$$

are such that

$$P_{4,\mathsf{G}}^{e^{2u}g} \circ e^{-\frac{k-3}{2}u} = e^{-\frac{k+3}{2}u} \circ P_{4,\mathsf{G}}^{g}.$$

The Submanifold Alexakis Conjecture — Statement

Conjecture (CGKTW (WIP))

Fix $k \in \mathbb{N}$ odd. Let I^g be a natural scalar Riemannian submanifold invariant such that $\int_Y I^g$ is conformally invariant for every Riemannian immersion $i\colon Y^{k+1} \to (X^{n+1}, g)$. Then there is a constant c, a scalar conformal submanifold invariant $\mathcal{W}$ of weight $-k-1$, and a natural extrinsic Riemannian vector field V^α such that

$$I = c\overline{\mathrm{Pf}} + \mathcal{W} + \overline{\nabla}_\alpha V^\alpha.$$

Remark

Alexakis (2012) proved the corresponding result for natural scalar Riemannian invariants.

The Submanifold Alexakis Conjecture — Partial Results

1. Mondino–Nguyen (2018) proved the case $k = 1$ assuming the immersion has codimension one or two.
2. Mondino–Nguyen (2018) also proved the case of general $k \in \mathbb{N}$ assuming the immersion has codimension one and the invariant can be written as a sum of an intrinsic invariant and a polynomial in $\mathring{L}_{\alpha\beta\alpha'}$.
3. Astaneh–Solodukhin (2021) and Chalabi–Herzog–O'Bannon–Robinson–Sisti (2022) classified the conformally invariant integrals in dimension four.
4. Juhl (2023) proved that the decomposition is true for the fourth-order singular Yamabe Q-curvature (codimension one).
5. The previous slide gives the decomposition for Q_2 and Q_4.

The Submanifold Alexakis Conjecture — Partial Results

Theorem (CGKTW (WIP))

The Submanifold Alexakis Conjecture is true in dimensions two and four.

Proof when dim $Y = 2$

The natural scalar Riemannian submanifold invariants of weight -2 are spanned by

$$\overline{Pf}, |\mathring{L}|^2, \mathsf{G}, |H|^2, P_{\alpha'}{}^{\alpha'}.$$

Now compute conformal linearizations.

The Submanifold Alexakis Conjecture — Partial Results

Proof when $\dim Y = 4$

Modulo scalar conformal submanifold invariants and tangential divergences, the natural scalar Riemannian submanifold invariants of weight -4 are spanned by

$$\langle \mathrm{F}, \overline{\mathrm{P}} \rangle, W_{\alpha\beta\alpha'}{}^{\beta}\mathrm{D}^{\alpha\alpha'}, |\mathring{L}|^2\overline{\mathrm{J}}, \langle \mathring{L}^2, \overline{\mathrm{P}} \rangle, \overline{\mathrm{J}}^2, |\overline{\mathrm{P}}|^2, \mathrm{G}\overline{\mathrm{J}}, |\mathrm{D}|^2,$$

$$H^{\alpha'}\overline{\Delta}H_{\alpha'}, H^{\alpha'}\overline{\nabla}^{\alpha}\mathrm{D}_{\alpha\alpha'}, H^{\alpha'}\overline{\nabla}^{\alpha}W_{\alpha\beta\alpha'}{}^{\beta}, |H|^4, |H|^2|\mathring{L}|^2, |H^{\alpha'}\mathring{L}_{\alpha\beta\alpha'}|^2,$$

$$|H|^2\overline{\mathrm{J}}, \mathrm{G}|H|^2, H^{\alpha'}\mathring{L}_{\alpha}{}^{\beta}{}_{\alpha'}\mathring{L}_{\beta}{}^{\gamma\beta'}\mathring{L}_{\gamma}{}^{\alpha}{}_{\beta'}, H^{\alpha'}\mathring{L}_{\alpha\beta\alpha'}\mathrm{F}^{\alpha\beta}, H^{\alpha'}\mathring{L}_{\alpha\beta\alpha'}\overline{\mathrm{P}}^{\alpha\beta},$$

$$H^{\alpha'}H^{\beta'}W_{\alpha'\alpha\beta'}{}^{\alpha}, H^{\alpha'}\mathring{L}^{\alpha\beta\beta'}W_{\alpha\alpha'\beta\beta'}, H^{\alpha'}C_{\alpha\alpha'}{}^{\alpha},$$

$$\mathrm{G}\mathrm{P}_{\alpha'}{}^{\alpha'}, \mathrm{P}^{\alpha'\beta'}W_{\alpha'\alpha\beta'}{}^{\alpha}, |H|^2\mathrm{P}_{\alpha'}{}^{\alpha'}, |\mathring{L}|^2\mathrm{P}_{\alpha'}{}^{\alpha'}, H^{\alpha'}H^{\beta'}\mathrm{P}_{\alpha'\beta'},$$

$$\mathring{L}_{\alpha\beta}{}^{\alpha'}\mathring{L}^{\alpha\beta\beta'}\mathrm{P}_{\alpha'\beta'}, \overline{\mathrm{J}}\mathrm{P}_{\alpha'}{}^{\alpha'}, \mathrm{P}_{\alpha'}{}^{\alpha'}\mathrm{P}_{\beta'}{}^{\beta'}, \mathrm{P}_{\alpha'\beta'}\mathrm{P}^{\alpha'\beta'},$$

$$H^{\alpha'}\nabla_{\alpha'}\mathrm{J},$$

$$\Delta\mathrm{J},$$

Now compute conformal linearizations.

Thank You!