Conformally covariant polydifferential operators associated with CVIs and its applications

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§1 Guideline

- Introduction and Motivation
- Conformally variational Riemannian invariants (CVIs)
- Conformally covariant polydifferential operators associated with CVIs
- Ovsienko-Redou operators and applications

Joint works with Jeffrey Case (Penn State) and Wei Yuan (Sun Yat-Sen University, China)

§1 Motivation from Conformal Geometry

Yamabe Problem:

- Yamabe functional:

$$F(\widetilde{g}) = \frac{\int_M R_{\widetilde{g}} dV_{\widetilde{g}}}{Vol_M(\widetilde{g})^{\frac{n-2}{n}}}.$$

- Conformal Laplacian:

$$L_g(u) := -\Delta_g u + \frac{n-2}{4(n-1)} R_g u;$$

- Sharp Sobolev inequality on $\mathbb{R}^n$:

$$\int_{\mathbb{R}^n} |\nabla \varphi|^2 dx \geq C_n ||\varphi||^2_{L^{\frac{2n}{n-2}}(\mathbb{R}^n)}.$$

- Q: How to generalize to higher order curvature quantities?

§1 Q_4-curvature, a 4th order analogue of scalar curvature

- $n \geq 3$, Branson's Q_4-curvature and the Paneitz operator P_g are

$$P_g u := (-\Delta_g)^2 u - \mathrm{div}(a_n R_g g - b_n Ric_g) du + \frac{n-4}{2} Q_4^g u$$

$$Q_4^g := -\frac{1}{2(n-1)} \Delta_g R_g + c_n R_g^2 - d_n |Ric_g|^2,$$

- **Gauss-Bonnet-Chern Formula in 4D**: Given (M^4, g) a closed Riemannian manifold,

$$\int_{M^4} \left(Q_4^g + \frac{1}{4} |W_g|^2 \right) dV_g = 8\pi^2 \chi(M) \quad W_g : \text{Weyl tensor.}$$

- $k_P := \int_{M^4} Q_4^g dV_g$ is a conformal invariant.

$$(P_g u + Q_4^g) = Q_4^{\tilde{g}} e^{4u}, \qquad \text{for } n = 4 \text{ and } \tilde{g} = e^{2u} g$$

$$P_g(u) = \frac{n-4}{2} Q_4^{\tilde{g}} u^{\frac{n+4}{n-4}}, \qquad \text{for } n \neq 4 \text{ and } \tilde{g} = u^{\frac{4}{n-4}} g$$

§1 Q-curvature in Riemannian geometry

Motivation: **Fischer-Marsden('75)**, **Chang-Gursky-Yang('96)**.

- Through metric deformations of Q_4-curvature, we can prove local stability, local and global rigidity and volume comparison for Q_4-curvature (**L.-Yuan'16, L-Yuan'22**).
- Recall $\gamma_g^* f = (DR_g)^* f = \nabla^2 f - g\Delta_g f - fRic(g)$. Then

$$Ric(g) = -\gamma_g^*(1).$$

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$$Ric(g) = -\gamma_g^*(1).$$
- **L.-Yuan('17)**: On (M^n, g) $(n \geq 3)$, define the symmetric $(0,2)$-tensor associated to Q_4 to be
$$J_g := -\frac{1}{2}\Gamma_g^* 1.$$

J-curvature is a generalization of Ricci curvature:

$$\mathrm{tr}_g J_g = Q_4^g \qquad \text{and} \qquad div_g J_g = \frac{1}{4} dQ_4^g.$$

§2 Conformally Variational Riemannian invariants

A CVI (conformally variational Riemannian invariant), L_g is a

(1) natural Riemannian scalar invariant:

$$L_g = \mathrm{contr}(\nabla^{r_1} Rm \otimes \cdots \otimes \nabla^{r_j} Rm);$$

(2) homogeneous:

$$L_{c^2 g} = c^{\omega} L_g, \quad c > 0,$$

where $\omega = -2k$ is the *weight*.

(3) conformally variational:

$\exists$ a Riemannian functional $\mathcal{F}$ such that

$$\frac{d}{dt}\Big|_{t=0} \mathcal{F}(e^{2t\gamma} g) = \int_M \gamma L_g \ dvol_g,$$

for all metrics g and smooth functions γ.

§2 Properties and examples of CVIs

- If L_g is a CVI of weight $-2k$ on (M^n, g), then

$$DF(\gamma) = (n-2k)\int_M \gamma L_g \, dvol_g,$$

where $F(g) := \int L_g \, dvol_g$. (i.e. a standard conformal primitive)

- (Branson-Gover'08) L is a CVI of weight $-2k$ if and only if $\exists$ a formally self-adjoint operator A_g such that $A_g(1) = 0$ and

$$DL_g(\gamma) := \frac{d}{dt}\Big|_{t=0} L_{e^{2t\gamma}g} = -2k\gamma L_g + A_g(\gamma) = -2k\gamma L_g - \delta(T(d\gamma)).$$

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- **Examples of CVIs**:

weight -2 : scalar curvature - $R_{c^2 g} = c^{-2} R_g$,
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weight -4 : basis $\{Q_4, \sigma_2, |W|^2\}$ - $Q_4^{c^2 g} = c^{-4} Q_4^g$,

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$DQ_4(\gamma) = -4\gamma Q_4^g + (P_g)_0(\gamma)$, where $(P_g)_0(\gamma) := P_g(\gamma) - \gamma P_g(1)$.

weight -6: renormalized volume coefficient v_3, Q_6...etc

§2 Results for CVIs

- L : a CVI of weight $-2k$

  $\Gamma[h] := DL_g(h) = \frac{d}{dt}\Big|_{t=0} L(g(t)), \quad g(t) = g + th$
  — the metric linearization of L at g.

  $\Gamma^*(f) : C^\infty(M) \to S_2(M)$ — the formal L^2-adjoint of Γ.

- Define the associated symmetric $(0,2)$-tensor $S := -\Gamma^*(1)$. We have

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- **Case - L.- Yuan ('19)** Variational properties and many results of scalar and Q_4-curvatures extend to CVIs:

  e.g. Schur's lemma, almost Schur's lemma, locally prescribing CVI, and local rigidity on closed flat manifolds...etc

§3 Motivation

- GJMS operators L_{2k} is a linear, formally self-adjoint operator with leading terms $(-\Delta)^k$ on n-manifolds. Let $\widetilde{g} = e^{2\gamma} g$

  L_2 = conformal Laplacian: $L_2^{\widetilde{g}}(u) = e^{-\frac{n+2}{2}\gamma} L_2^g(e^{\frac{n-2}{2}\gamma} u)$ and $L_2(1) = R_g$.

  L_4 = Paneitz operator: $L_4^{\widetilde{g}}(u) = e^{-\frac{n+4}{2}\gamma} L_2^g(e^{\frac{n-4}{2}\gamma} u)$ and $L_4(1) = \frac{n-4}{2} Q_4^g$.

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- The formally self-adjoint, conformally covariant, tri-differential operator associated to the σ_2-curvature is the polarization of

$$L_{\sigma_2}(u,u,u) := \frac{1}{2}\delta\left(|\nabla u|^2 du\right) - \frac{n-4}{16}\left(u\Delta|\nabla u|^2 - \delta\left((\Delta u^2) du\right)\right)$$
$$-\frac{1}{2}\left(\frac{n-4}{4}\right)^2 u\delta\left(T_1(\nabla u^2)\right) + \left(\frac{n-4}{4}\right)^3 \sigma_2 u^3;$$

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$$L_{\sigma_2}(1,1,1) = \left(\frac{n-4}{4}\right)^3 \sigma_2.$$

Case-Wang ('18): solved a Dirichlet problem on manifolds with boundary under a positivity assumption.

Case('21): used the multi-linearity of L_{σ_2} to give a new proof of a sharp fully nonlinear Sobolev inequality.

§3 Definition

A natural ℓ-differential operator $D : (C^\infty(M))^\ell \to C^\infty(M)$ is associated to a CVI L (of weight $-2k$) if it is
(1) conformally covariant of bidegree (a, b):

$$D^{e^{2\gamma}g}(u_1, ..., u_\ell) = e^{-b\gamma} D^g(e^{a\gamma}u_1, ..., e^{a\gamma}u_\ell), \quad \gamma, u_1, ..., u_\ell \in C^\infty(M);$$

(2) formally self-adjoint:

$$(u_0, \ldots, u_\ell) \mapsto \int_M u_0 \, D(u_1, \ldots, u_\ell) \, dvol_g$$

is symmetric $\forall u_0, ..., u_\ell \in C^\infty(M)$.

(3) recovers L:

for $n > 2k$, $D(1, \ldots, 1) = \left(\frac{n-2k}{\ell+1}\right)^\ell L$;
for $n = 2k$,

$$\frac{1}{\ell!} \frac{\partial^\ell}{\partial t^\ell}\bigg|_{t=0} e^{nt\gamma} L^{e^{2t\gamma}g} = D(\gamma, \ldots, \gamma) \quad \forall \gamma \in C^\infty(M).$$

§3 Existence of CVI associated operators

Theorem (Case-L.- Yuan'22)

Let L be a CVI of weight $-2k$*. There is an integer* $1 \leq j \leq 2k$ *such that*
(1) there is a $(j-1)$*-differential operator associated to L; and*
(2) for any $1 \leq \ell < j-1$, $\nexists$ *an* ℓ*-differential operator associated to L.*
j *is called the* ***rank*** *of L.*

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- A natural ℓ-differential operator D is conformally covariant of bidegree (a, b) if and only if $\frac{\partial}{\partial t}\Big|_{t=0} \left(e^{bt\Upsilon} \circ D^{e^{2t\Upsilon} g} \circ e^{-at\Upsilon} \right) = 0 \ \ \forall \gamma \in C^\infty(M)$.

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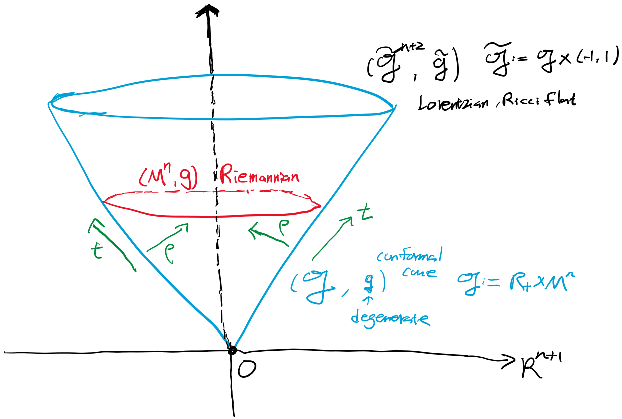
- **Remark:** If a CVI L is of weight $-2k$, then rank $r \leq 2k$.

  Examples: Q_{2k} is rank 2 and σ_k or v_k is rank $2k$.

§3 What is an ambient space?

The canonical sphere $M^n = \mathbb{S}^n \hookrightarrow \mathbb{R}^{n+1,1} = \widetilde{\mathcal{G}}$:



§3 Ambient Space

- **Theorem [Fefferman-Graham'12]**: Let $(M^n, [g])$ be a conformal manifold. Given $g \in [g]$, there is a one-parameter family of metrics g_ρ on M such that $g_0 = g$ and $\widetilde{g} = 2\rho\, dt^2 + 2t\, dt\, d\rho + t^2 g_\rho$.

  $\widetilde{g}$ is an ambient metric on ambient space $\widetilde{\mathcal{G}} := \mathbb{R}_+ \times M \times (-\varepsilon, \varepsilon)$ for some $\varepsilon > 0$.

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- Let $X = t\partial_t$ be the infinitesimal generator of the dilation $\delta_\lambda : \widetilde{\mathcal{G}} \to \widetilde{\mathcal{G}}$. Given $\omega \in \mathbb{R}$, let $\widetilde{\mathcal{E}}[\omega] := \{\widetilde{u} \in C^\infty(\widetilde{\mathcal{G}}) : X\widetilde{u} = \omega\widetilde{u}\}$.

  The space of conformal densities of weight ω is $\mathcal{E}[\omega] := \{\widetilde{u}|_{\mathcal{G}} : \widetilde{u} \in \widetilde{\mathcal{E}}[\omega]\}$.

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  $\iff \quad \widetilde{L}(Qz) \equiv 0 \mod Q \quad \forall z \in \widetilde{\mathcal{E}}[\omega - 2]$, where $Q := |X|^2$ is a defining function for $\mathcal{G} \cong \mathbb{R}_+ \times M$.

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- If $\widetilde{L}$ is tangential, it induces a conformally covariant operator $L^g : C^\infty(M) \to C^\infty(M)$ using identifications $\mathcal{E}[\omega] \cong_g C^\infty(M)$.

§3 Classification of conformally covariant operators

- **Branson ('95)**: The space of conformally covariant differential operators on S^n is the span of restrictions $\widetilde{\Delta}^k \widetilde{u}|_{\mathcal{G}}$, where $\widetilde{\Delta}$ is ambient Laplacian on Minkowski space $(\mathbb{R}^{n+1,1}, -d\tau^2 + dx^2)$ and $\widetilde{u} \in \widetilde{\mathcal{E}}[-\frac{n-2k}{2}]$.

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- On the ambient space of a pseudo-Riemannian n-manifold ($n \geq 2k$):

  $\widetilde{\Delta}^k \longrightarrow$ induce GJMS operators $L_{2k} = (-\Delta)^k +$ l.o.t
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  critical weights:
  $\omega_1, \omega_2 \in \mathcal{I}_k := \{-\frac{n-2k}{2} - \ell\}_{\ell=0}^{k-1}, \quad \omega_1 + \omega_2 \in \mathcal{O}_k := \{-\frac{n-2k}{2} + \ell\}_{\ell=0}^{k-1}$
- **Ovsienko-Redou('03)**: for non-critical weights ω_1, ω_2, the space of conformally covariant bi-differential operators of total order $2k$ on S^n, $D_{2k;\omega_1,\omega_2} : \mathcal{E}[\omega_1] \otimes \mathcal{E}[\omega_2] \to \mathcal{E}[\omega_1 + \omega_2 - 2k]$ is one-dimensional.
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- **Clerc ('16, '17)**: classified the space for remaining weights on S^n - can be one, two or three dimensional.
- **Case-L.-Yuan('23)** - complete classification of tangential Ovsienko-Redou operators of order at most $2k \leq n$ on the ambient space of an n-manifold.

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Theorem (Case-L.-Yuan'23, Case 1)

Let $(M^n, [g])$ be a conformal manifold and $k \leq n/2$ be a positive integer. Let $\omega_1, \omega_2 \in \mathbb{R}$. Suppose that either
(1) at most one of $\omega_1 \in \mathcal{I}_k$ or $\omega_2 \in \mathcal{I}_k$ or $\omega_1 + \omega_2 \in \mathcal{O}_k$ holds;
(2) $\omega_1, \omega_2 \in \mathcal{I}_k$, with $\omega_1 + \omega_2 + n \leq k$, but $\omega_1 + \omega_2 \notin \mathcal{O}_k$;
(3) $\omega_1 \in \mathcal{I}_k$ and $\omega_1 + \omega_2 \in \mathcal{O}_k$ with $\omega_2 \geq k$, but $\omega_2 \notin \mathcal{I}_k$; or
(4) $\omega_2 \in \mathcal{I}_k$ and $\omega_1 + \omega_2 \in \mathcal{O}_k$ with $\omega_1 \geq k$, but $\omega_1 \notin \mathcal{I}_k$. Then the space of Ovsienko-Redou operators is one-dimensional and spanned by

$$\widetilde{D}_{2k;\omega_1,\omega_2}(\widetilde{u}, \widetilde{v}) := \sum_{s=0}^{k} \sum_{t=0}^{k-s} a_{k-s-t,s,t} \widetilde{\Delta}^{k-s-t} \left((\widetilde{\Delta}^s \widetilde{u}) (\widetilde{\Delta}^t \widetilde{v}) \right),$$

where $a_{s,t} = a_{s,t}(s, t, n, k, \text{gamma function})$. Then $\widetilde{D}_{2k;\omega_1,\omega_2}$ is tangential, and induces a natural conformally covariant bi-differential operator $D_{2k;\omega_1,\omega_2} : \mathcal{E}[\omega_1] \otimes \mathcal{E}[\omega_2] \to \mathcal{E}[\omega_1 + \omega_2 - 2k]$.

Remark: For remaining weights ω_1, ω_2 in case 2 and 3, the space of Ovsienko-Redou operators is two and three-dimensional, respectively.

§3 What do we know about Ovsienko-Redou operators?

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Theorem (Case-L.- Yuan'23)

Let (M^n, g) be pseudo-Riemannian manifold and $k \in \{1, 2, 3\}$. Then $D_{2k} := D_{2k,-\frac{n-2k}{3},-\frac{n-2k}{3}}$ is formally self-adjoint.

Conjecture: D_{2k} is formally self-adjoint for all k.

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Theorem (Case-L.- Yuan'23 (commutator formula))

Let $(S^n, d\theta^2)$ be round sphere, $k \in \mathbb{N}$ and $n > 2k$. Let $\{x^i\}_{i=0}^n$ be standard Cartesian coordinates in $\mathbb{R}^{n+1}$. Then

$$\sum_{i=0}^{n} x^i [D_{2k}, x^i] = -C_{n,k} D_{2k-2, -\frac{n-2k+3}{3}, -\frac{n-2k}{3}},$$

where $[D, f](u \otimes v) := D((uf) \otimes v) - fD(u \otimes v)$ for all $u, v, f \in C^\infty(S^n)$.

§3 Applications of Ovsienko-Redou operators

1. Construction of a large family of conformally covariant differential operators:

Theorem (Case-L.- Yuan'23)

Let $(M^n, [g])$ be a conformal manifold and $k \leq n/2$ be a positive integer, and let $\widetilde{I} \in \widetilde{\mathcal{E}}[-2\ell], \ell \leq k$ be a natural scalar Riemannian invariant on $(\widetilde{\mathcal{G}}, \widetilde{g})$. Then the operator $\widetilde{D} : \widetilde{\mathcal{E}}[-\frac{n-2k}{2}] \to \widetilde{\mathcal{E}}[-\frac{n+2k}{2}]$,

$$\widetilde{D}(\widetilde{u}) := \widetilde{D}_{2k-2\ell;-2\ell,-\frac{n-2k}{2}}(\widetilde{I}, \widetilde{u})$$

is tangential, and induces a natural conformally covariant differential operator $D : \mathcal{E}[-\frac{n-2k}{2}] \to \mathcal{E}[-\frac{n+2k}{2}]$. Moreover, if $k \leq \ell + 3$, then D is formally self-adjoint.

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Examples

$\ell = 0$: recover GJMS operators;

$\ell = 2, k = 3$: Inserting $\widetilde{I} = |\widetilde{Rm}|^2$, we have

$$Du = \frac{n-10}{2}\Delta(|W|^2 u) + \frac{n-10}{2}|W|^2\Delta u) + (2\Delta|W|^2 - \frac{(n-6)^2}{2}J|W|^2)u.$$

§3 Applications of Ovsienko-Redou operators

2. Sharp fully nonlinear Sobolev inequalities:

Theorem (Case-L.- Yuan, work in progress)

Let $n \geq 5$ and $\varepsilon \geq 0$. Then

$$\int_{S^n} \left[\left(\frac{16(n-1)}{3(n+2)} + \varepsilon \right) Q_4^g + \frac{16(n-4)^2}{9(n+2)} \sigma_2^g \right] dvol_g \geq C_n Vol_g(S^n)^{\frac{n-4}{n}},$$

for all conformally flat metrics g on S^n. Moreover, equality holds if and only if g has constant sectional curvature.

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Theorem (Case-L.- Yuan, work in progress)

Let $n \geq 5$ and $\varepsilon \geq 0$. Then

$$\int_{\mathbb{R}^n} \varepsilon(\Delta u^3)^2 + 4u^2|\nabla^2 u^2|^2 + \frac{4(2n-5)}{n+2} u^2(\Delta u^2)^2 \, dx \geq C_n \left(\int_{\mathbb{R}^n} |u|^{\frac{6n}{n-4}} \, dx \right)^{\frac{n-4}{n}}$$

for all $u \in \mathcal{S}_\varepsilon$. Moreover, equality holds if and only if
$u(x) = u_{a,\lambda,x_0}(x) := a(\lambda + |x - x_0|^2)^{-\frac{n-4}{6}}$.

Thank you for your attention!