Collective behaviour of active particles with mean-field interaction

Oscar de Wit, PhD student, University of Cambridge

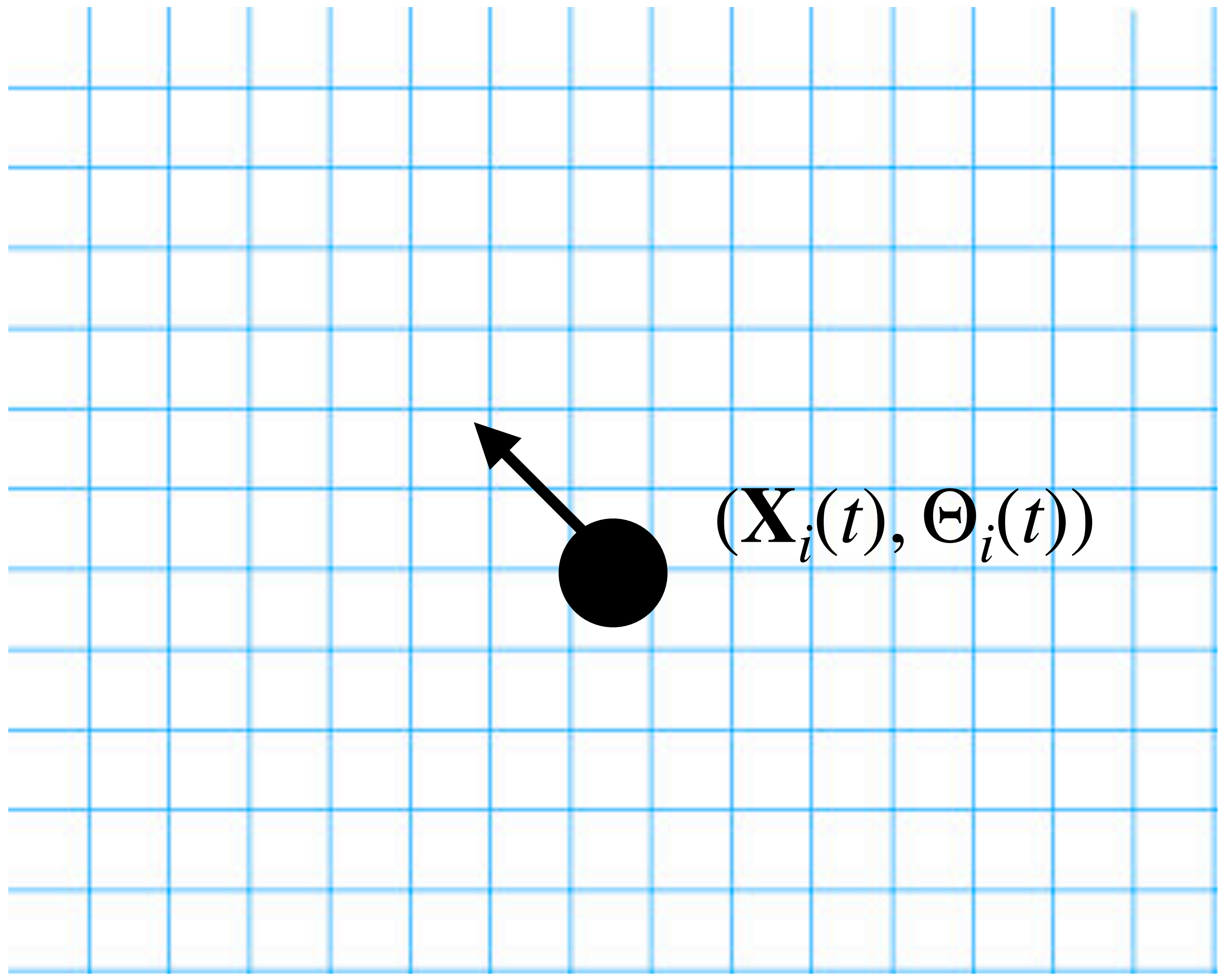
Model

System of SDEs

$$\mathrm{d}\mathbf{X}_t^{i,N} = \mathrm{Pe}\,\mathbf{e}(\Theta_t^{i,N})\mathrm{d}t + \sqrt{2D_T}\mathrm{d}\mathbf{W}_t^{i,N}$$

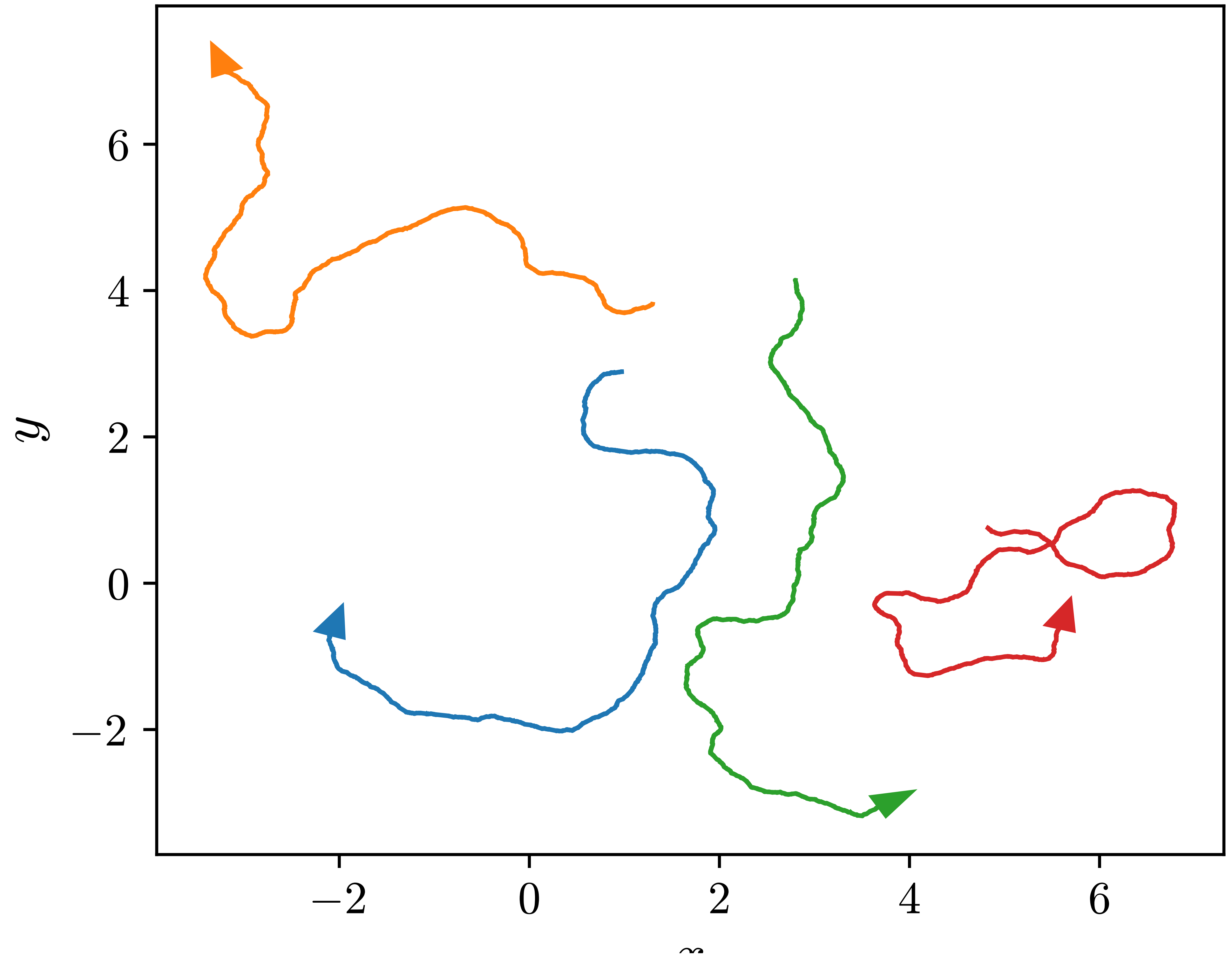
$$\mathrm{d}\Theta_t^{i,N} = \sqrt{2}\mathrm{d}W_t^{i,N}$$

- Active Brownian Particles
- $(\mathbf{X}, \Theta) \in \mathbb{T}_L^2 \times \mathbb{T}_{2\pi}$, $\mathbf{e}(\Theta) = (\cos\Theta, \sin\Theta)$, N particles, Brownian motions $\mathbf{W}$ and W, constants Pe, D_T



$$\mathrm{d}\mathbf{X}_t^{i,N} = \mathrm{Pe}\,\mathbf{e}(\Theta_t^{i,N})\mathrm{d}t + \sqrt{2D_T}\mathrm{d}\mathbf{W}_t^{i,N}$$

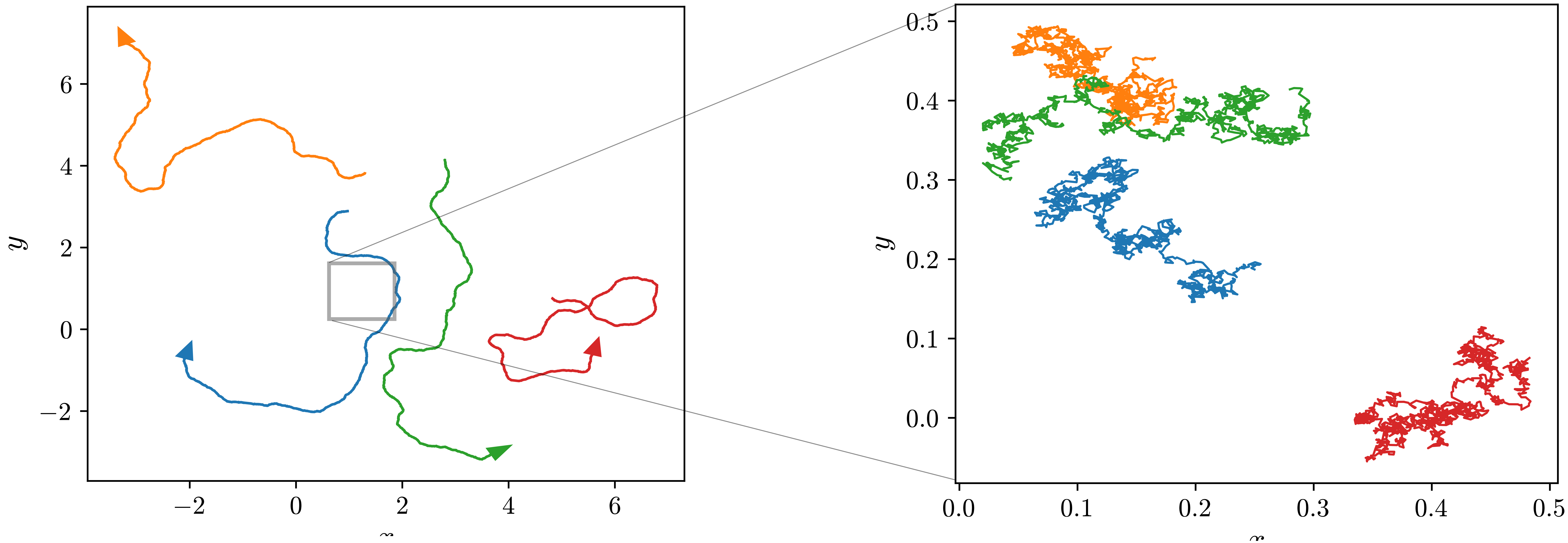
$$\mathrm{d}\Theta_t^{i,N} = \sqrt{2}\mathrm{d}W_t^{i,N}$$



$\mathrm{Pe} = 2,\ D_T = 10^{-3}$

$$\mathrm{d}\mathbf{X}_t^{i,N} = \mathrm{Pe}\,\mathbf{e}(\Theta_t^{i,N})\mathrm{d}t + \sqrt{2D_T}\,\mathrm{d}\mathbf{W}_t^{i,N}$$

$$\mathrm{d}\Theta_t^{i,N} = \sqrt{2}\,\mathrm{d}W_t^{i,N}$$



$\mathrm{Pe} = 2,\ D_T = 10^{-3}$

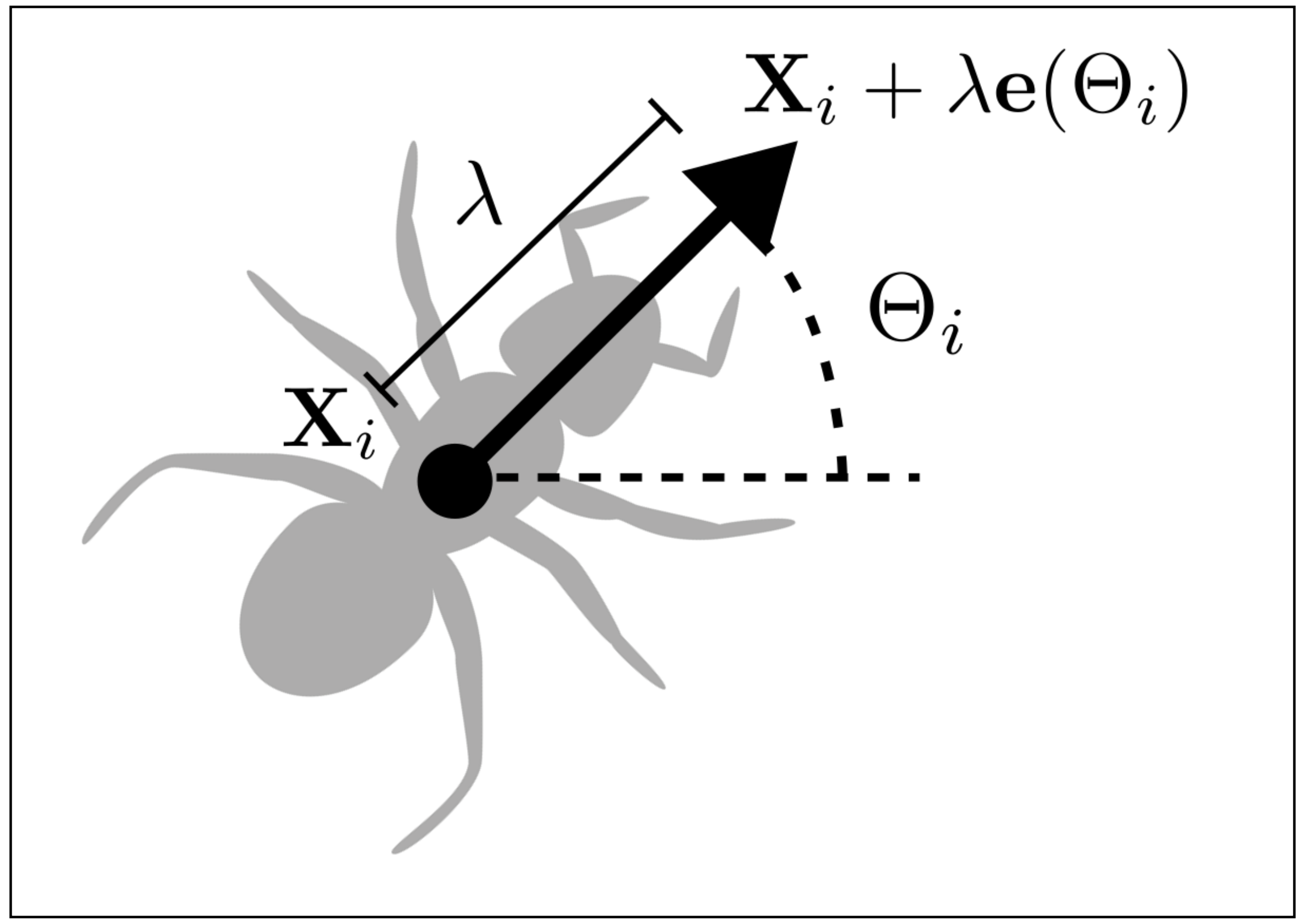
$\mathrm{Pe} = 0,\ D_T = 10^{-3}$

Model

System of SDEs

$$\mathrm{d}\mathbf{X}_t^{i,N} = \mathrm{Pe}\mathbf{e}(\Theta_t^{i,N})\mathrm{d}t + \sqrt{2D_T}\mathrm{d}\mathbf{W}_t^{i,N}$$

$$\mathrm{d}\Theta_t^{i,N} = \gamma\mathbf{n}(\Theta_t^{i,N}) \cdot \frac{1}{N}\sum_{j\neq i} \nabla K(\mathbf{X}_t^{i,N} + \lambda\mathbf{e}(\Theta_t^{i,N}) - \mathbf{X}_t^{j,N})\mathrm{d}t + \sqrt{2}\mathrm{d}W_t^{i,N}$$



Model

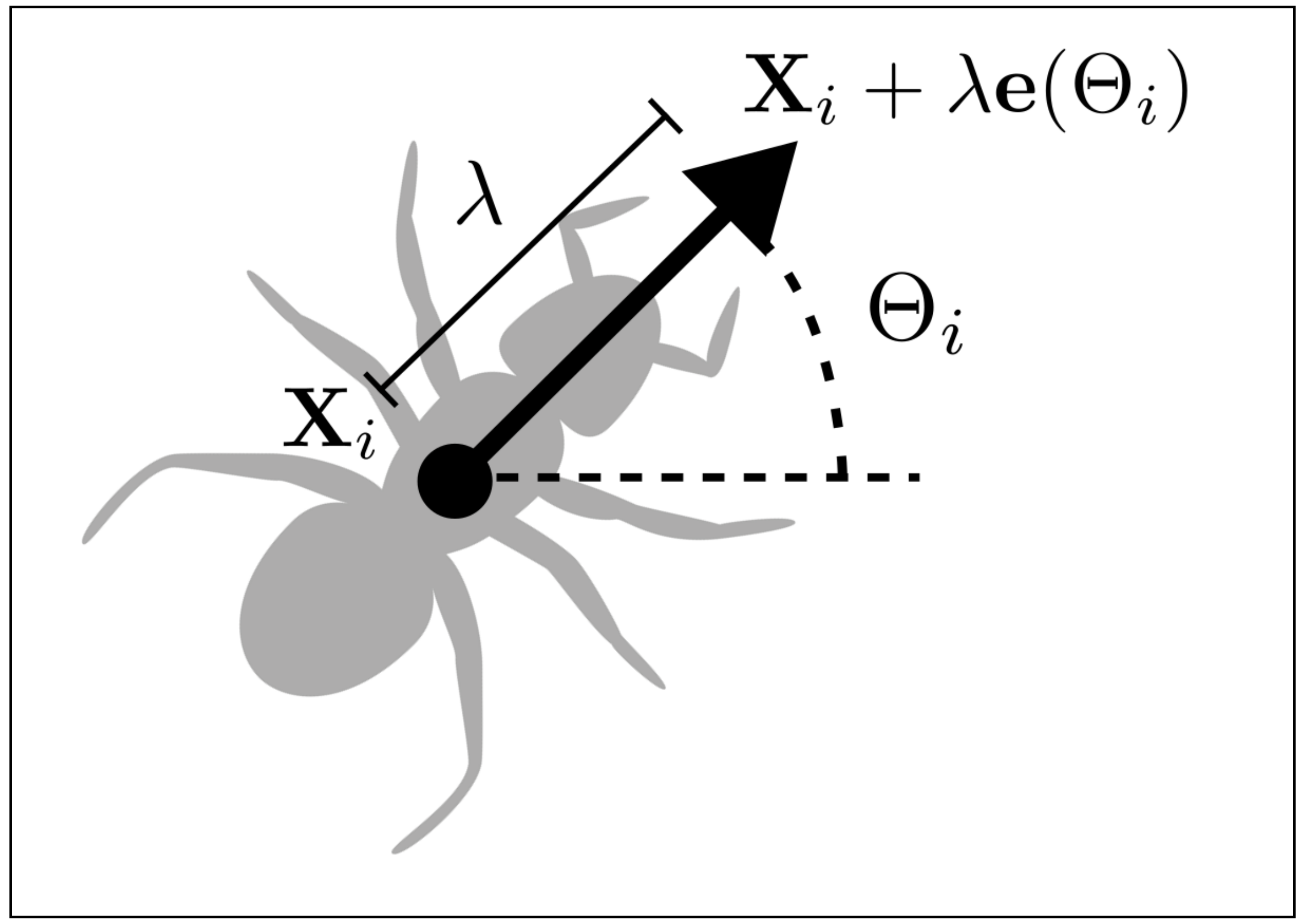
System of SDEs

$$\mathrm{d}\mathbf{X}_t^{i,N} = \mathrm{P}\mathrm{e}\mathbf{e}(\Theta_t^{i,N})\mathrm{d}t + \sqrt{2D_T}\mathrm{d}\mathbf{W}_t^{i,N}$$

$$\mathrm{d}\Theta_t^{i,N} = \gamma\mathbf{n}(\Theta_t^{i,N}) \cdot \frac{1}{N}\sum_{j\neq i} \nabla K(\mathbf{X}_t^{i,N} + \lambda\mathbf{e}(\Theta_t^{i,N}) - \mathbf{X}_t^{j,N})\mathrm{d}t + \sqrt{2}\mathrm{d}W_t^{i,N}$$

Interaction via a torque on the orientation

$\mathbf{n}(\theta) \cdot \nabla K = \mathbf{e}(\theta) \times \nabla K$ so that $\mathbf{e}(\theta) \to \nabla K$



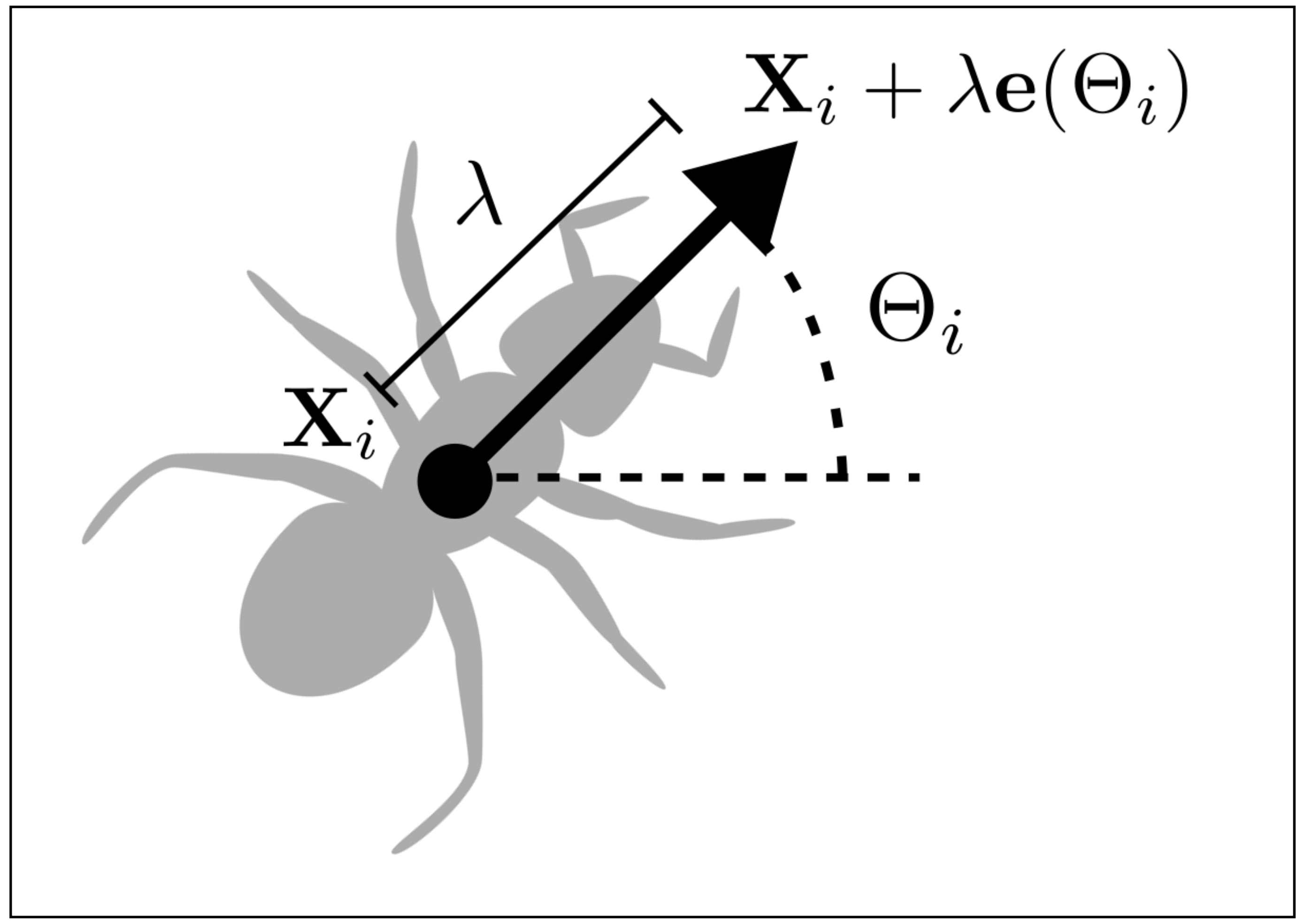
Model

System of SDEs

$$\mathrm{d}\mathbf{X}_t^{i,N} = \mathrm{Pe}\,\mathbf{e}(\Theta_t^{i,N})\mathrm{d}t + \sqrt{2D_T}\mathrm{d}\mathbf{W}_t^{i,N}$$

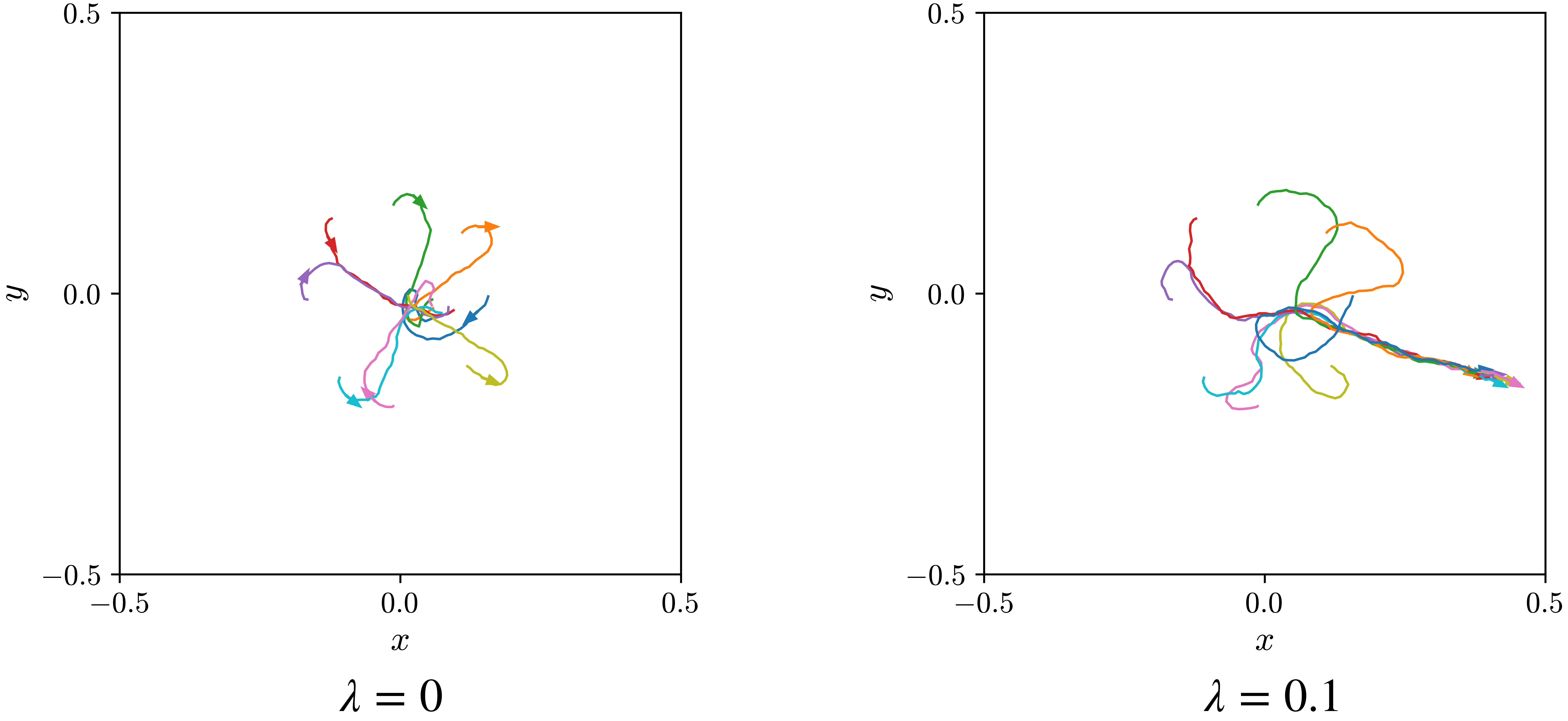
$$\mathrm{d}\Theta_t^{i,N} = \gamma\mathbf{n}(\Theta_t^{i,N}) \cdot \frac{1}{N}\sum_{j\neq i}\nabla K(\mathbf{X}_t^{i,N} + \lambda\mathbf{e}(\Theta_t^{i,N}) - \mathbf{X}_t^{j,N})\mathrm{d}t + \sqrt{2}\mathrm{d}W_t^{i,N}$$

- Constants $\mathrm{Pe}, D_T, \gamma, \lambda$, $\mathbf{n}(\theta) = (-\sin\theta, \cos\theta)^\mathsf{T}$
- Vicsek model, Keller-Segel model, anticipation



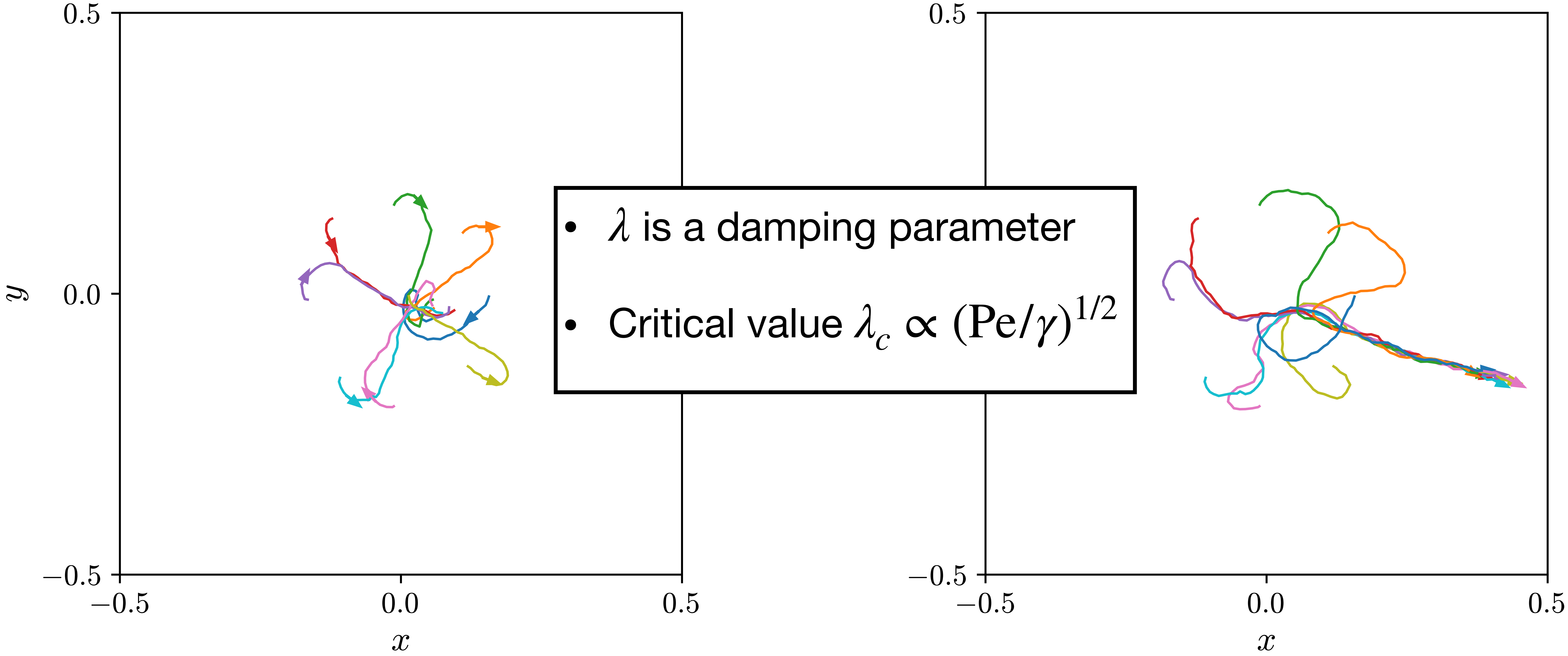
$$\mathrm{d}\mathbf{X}_t^{i,N} = \mathrm{Pe}\,\mathbf{e}(\Theta_t^{i,N})\mathrm{d}t + \sqrt{2D_T}\mathrm{d}\mathbf{W}_t^{i,N}$$

$$\mathrm{d}\Theta_t^{i,N} = \gamma\mathbf{n}(\Theta_t^{i,N}) \cdot \frac{1}{N}\sum_{j\neq i} \nabla K(\mathbf{X}_t^{i,N} + \lambda\mathbf{e}(\Theta_t^{i,N}) - \mathbf{X}_t^{j,N})\mathrm{d}t + \sqrt{2}\mathrm{d}W_t^{i,N}$$



$$\mathrm{d}\mathbf{X}_t^{i,N} = \mathrm{Pe}\,\mathbf{e}(\Theta_t^{i,N})\mathrm{d}t + \sqrt{2D_T}\,\mathrm{d}\mathbf{W}_t^{i,N}$$

$$\mathrm{d}\Theta_t^{i,N} = \gamma\mathbf{n}(\Theta_t^{i,N}) \cdot \frac{1}{N}\sum_{j\neq i} \nabla K(\mathbf{X}_t^{i,N} + \lambda\mathbf{e}(\Theta_t^{i,N}) - \mathbf{X}_t^{j,N})\mathrm{d}t + \sqrt{2}\,\mathrm{d}W_t^{i,N}$$

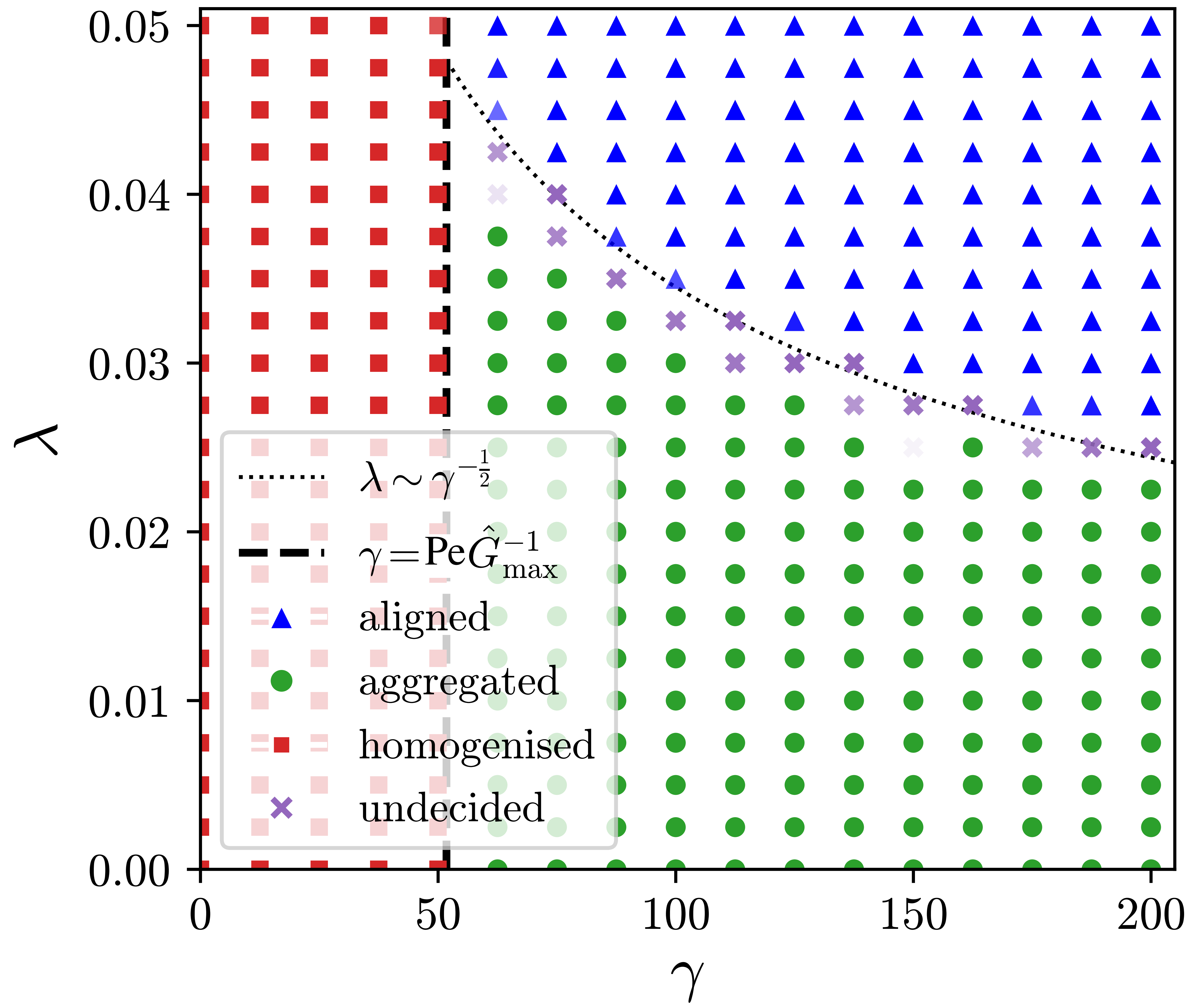


- λ is a damping parameter
- Critical value $\lambda_c \propto (\mathrm{Pe}/\gamma)^{1/2}$

$\lambda = 0$ $\qquad\qquad$ $\lambda = 0.1$

Model

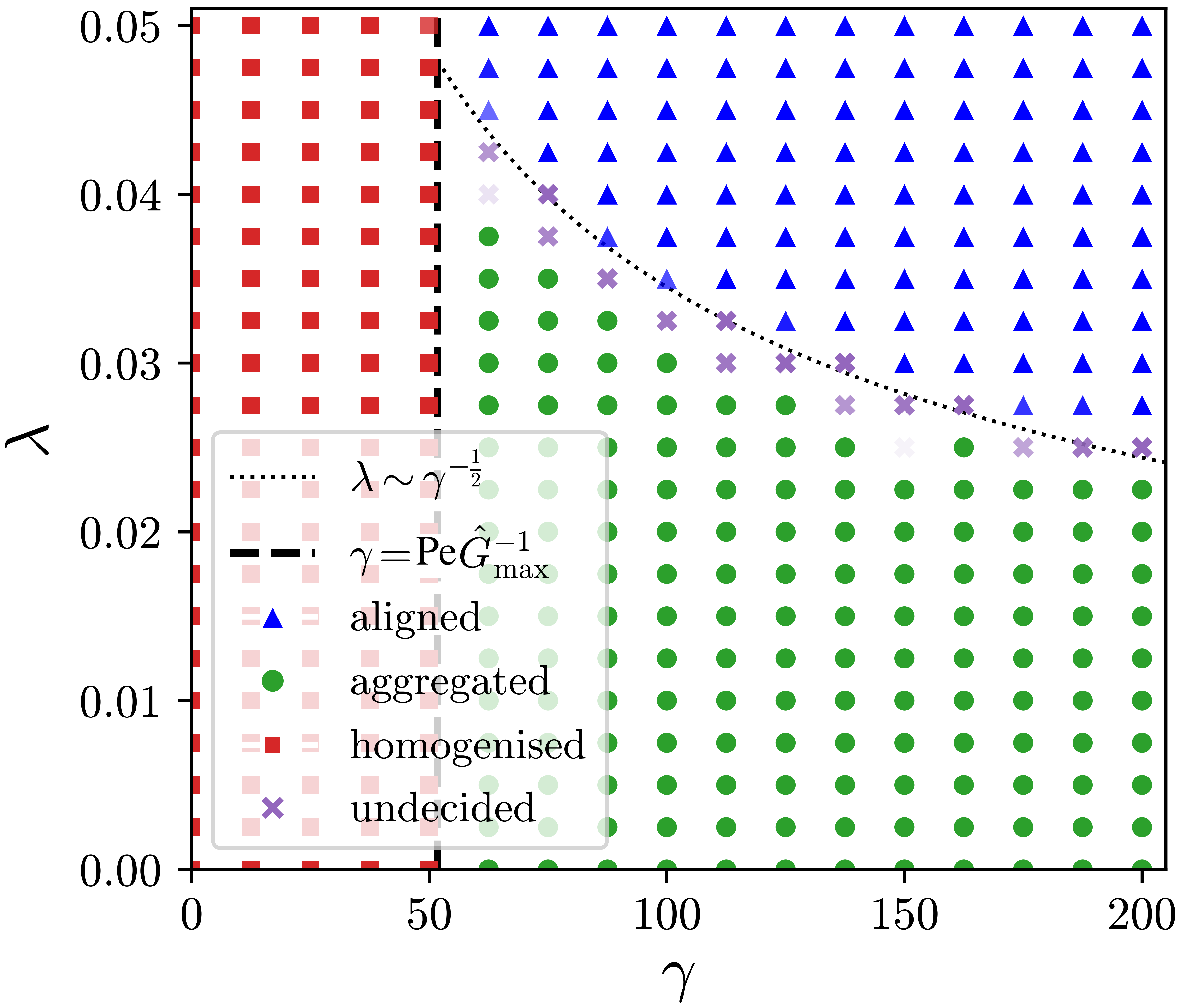
Phase diagram



- Classification via alignment $\langle e^{\mathrm{i}\Theta_t^i} \rangle$ and aggregation $\langle \, | \mathbf{X}_t^i - \mathbf{X}_t^{\mathrm{com}} | \, \rangle$ parameters (com=Center Of Mass)
- Dashed line: mean-field PDE linear (in)stability
- Dotted line: dimensional scaling of critical damping value

Model

Phase diagram



Phase transitions

1. Subcritical to supercritical
2. Supercritical to supercritical

Mean-field limit PDE

Propagation of chaos

- **Globally Lipschitz** K: classical result by McKean, MFL PDE
  $$\left\{ \partial_t f = \nabla_{\mathbf{x}} \cdot (D_T \nabla_{\mathbf{x}} f - \mathrm{Pe}\mathbf{e}_\theta f) + \partial_\theta(\partial_\theta f - \gamma \mathbf{n}_\theta \cdot (\nabla K * \rho)_\lambda f) \right.$$
- Using notation $\rho(t, \mathbf{x}) = \int_0^{2\pi} f(t, \mathbf{x}, \theta)\mathrm{d}\theta$ and $g_\lambda(\mathbf{x}, \theta) = g(\mathbf{x} + \lambda \mathbf{e}_\theta)$
- **Newtonian** $K = -\frac{1}{2\pi}\log|\mathbf{x}|$: compactness mean-field limit (dW, thesis, 2025)

Mean-field limit PDE

Dynamics

$$\left\{\partial_t f = \nabla_{\mathbf{x}} \cdot (D_T \nabla_{\mathbf{x}} f - \mathrm{Pe}\mathbf{e}_\theta f) + \partial_\theta(\partial_\theta f - \gamma \mathbf{n}_\theta \cdot (\nabla K * \rho)_\lambda f)\right. \quad \neq \quad \left\{\partial_t f = \nabla \cdot \left(f \nabla \frac{\delta F}{\delta f}\right)\right.$$

- **Subcritical**: in analogy with continuity gradient flows $f = \nabla \cdot \left(f \nabla \frac{\delta F}{\delta f}\right)$ [e.g. Carrillo, Gvalani, Wu, Delgadino, …], convergence to homogeneous state and uniform-in-time propagation of chaos if and only if the homogeneous state is the unique stationary state?

- **Supercritical**: without anticipation ($\lambda = 0$), Kuramoto-like aggregation? [Bertini, Giacomin, Poquet, …]

- **Supercritical**: for $\lambda > 0$ more than two types of attracting states?

Mean-field limit PDE

Sub- and supercritical $\lambda \geq 0$

$$\begin{cases} \partial_t f = \nabla_{\mathbf{x}} \cdot (D_T \nabla_{\mathbf{x}} f - \mathrm{Pe}\mathbf{e}_\theta f) + \partial_\theta (D_R \partial_\theta f - \gamma \mathbf{n}_\theta \cdot (\nabla c + \lambda \nabla^2 c \mathbf{e}_\theta) f) \\ 0 = \Delta_{\mathbf{x}} c - \alpha c + \rho \end{cases}$$

- **Theorem.** *The PDE system defines a semidynamical system on a Banach space Y. The system possesses a compact global attractor $A \subset \subset Y$. Furthermore two distinct cases arise:*

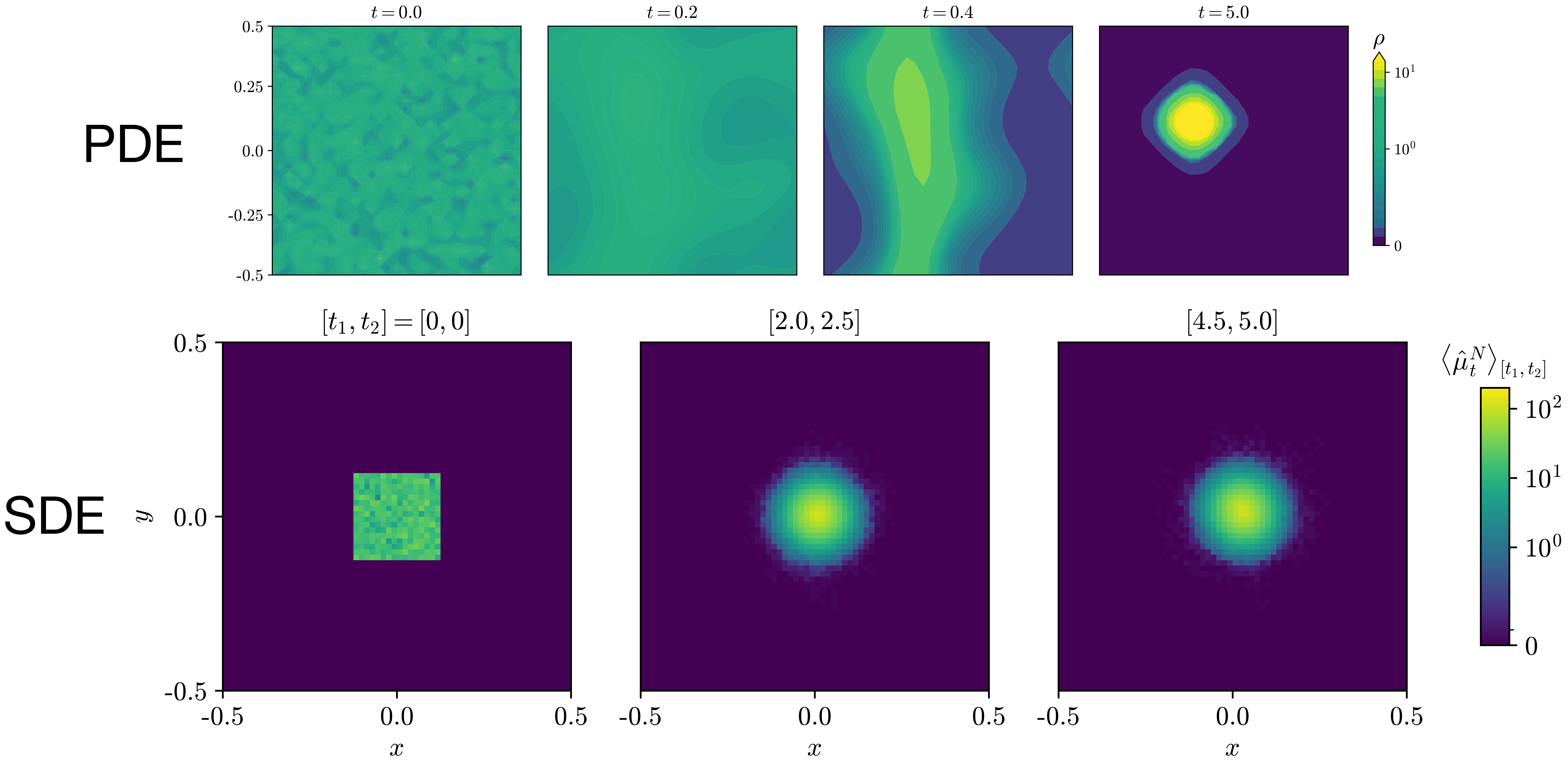
1. *There exists a γ^* such that for any $0 \leq \gamma < \gamma^*$ and $A = \{1/2\pi\}$.*

2. *If $\gamma > \mathrm{Pe}\hat{K}_{\max}^{-1}$ (linearly unstable) then $4 \leq \dim A$.*

- Rakotomalala & dW, arxiv, (2025), submitted

Mean-field limit PDE

Supercritical, $\lambda = 0$, Kuramoto-like aggregation?

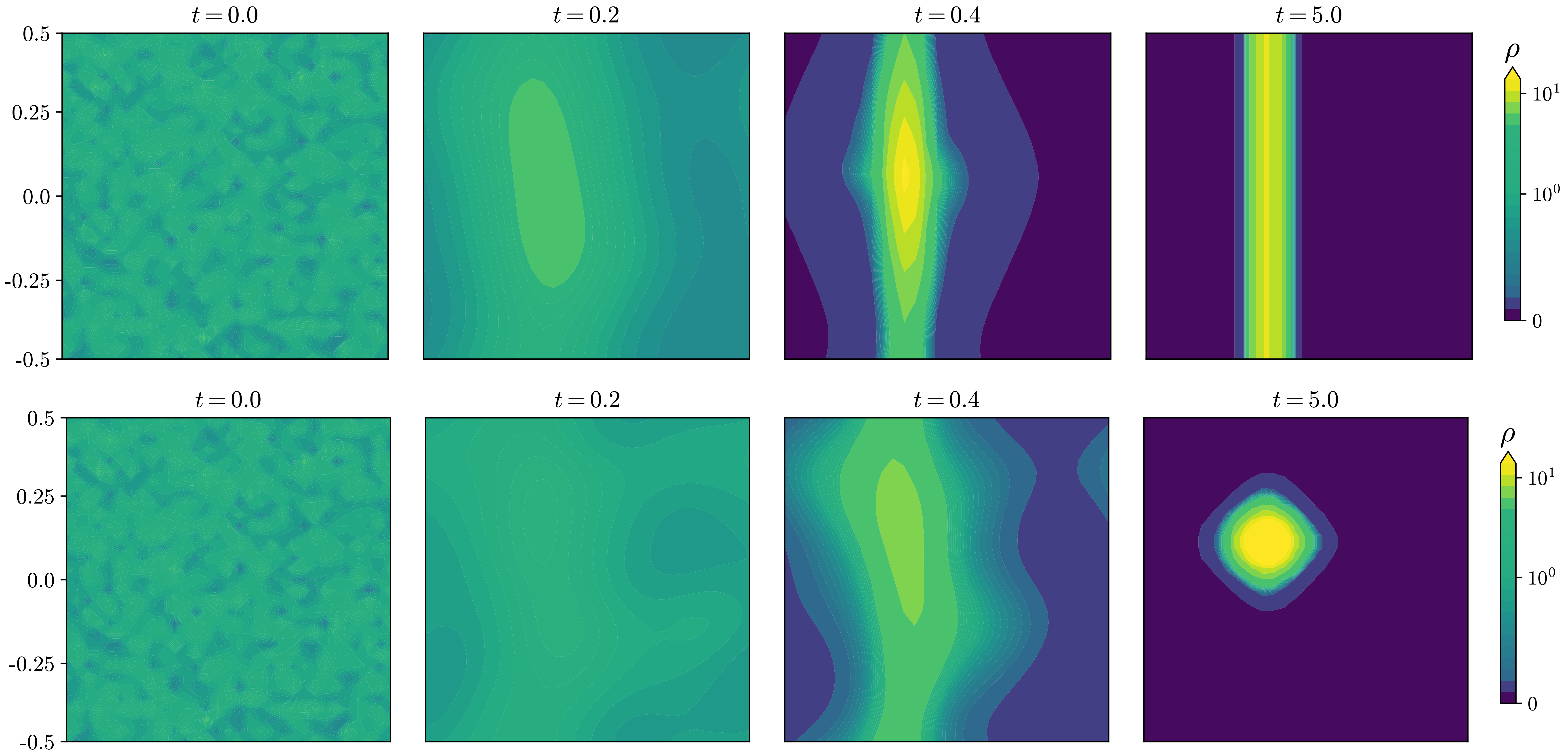


$N = 2000, \lambda = 0, D_T = 0.01, \mathrm{Pe} = 5, \gamma = 200$

Mean-field limit PDE

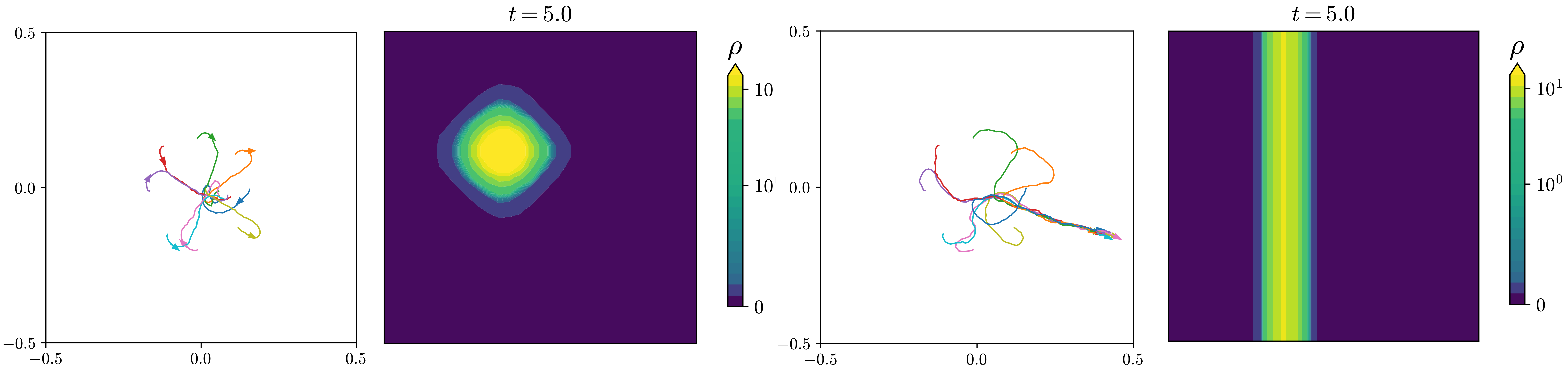
Supercritical, $\lambda \geq 0$

- Convergent finite volume scheme
- Bruna, Schmidtchen & dW (2025), ESAIM: M2AN, to appear



Mean-field limit PDE

Supercritical, $\lambda \geq 0$, large deviations



To be continued…

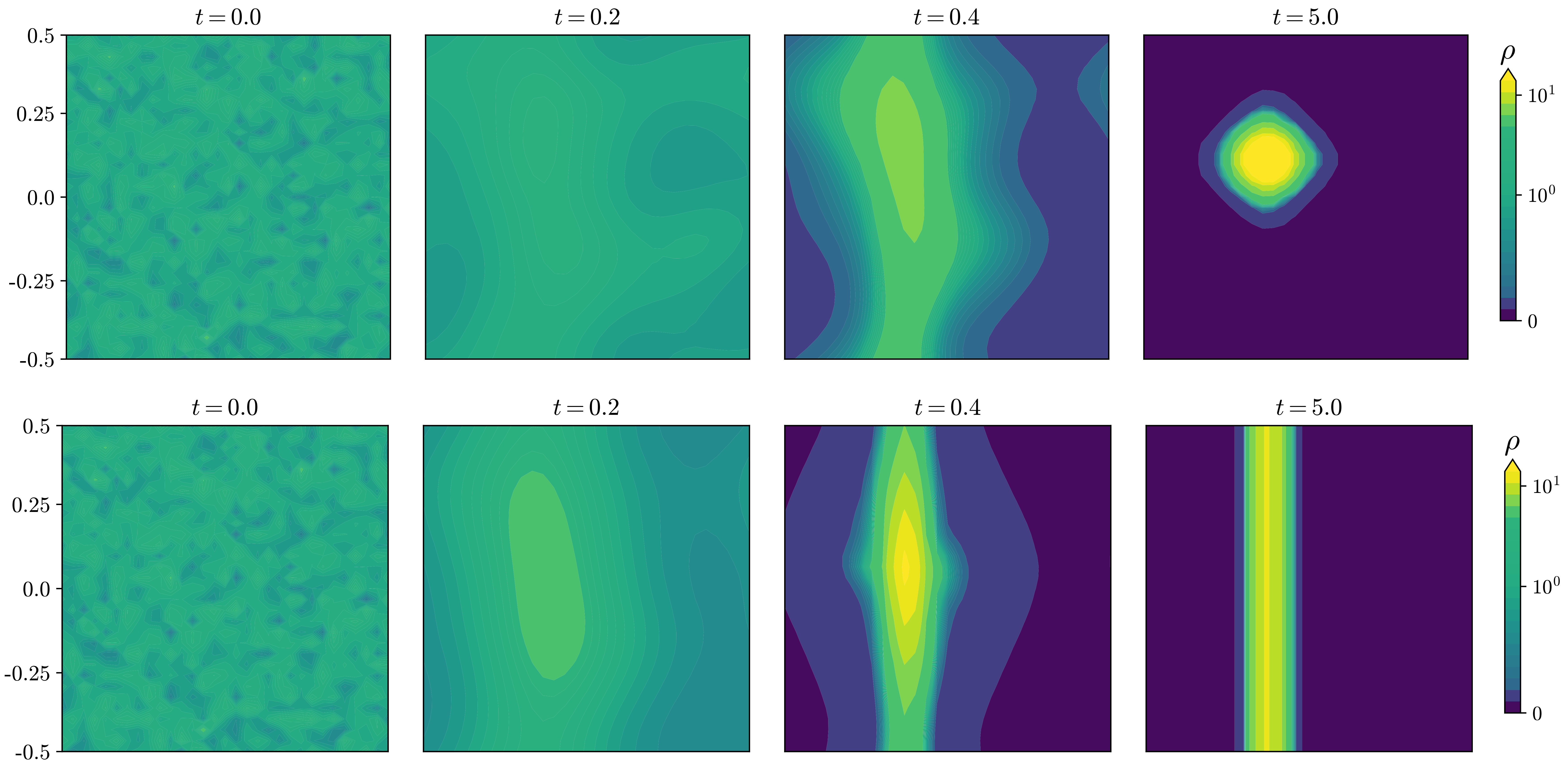
Thank you!

- Thanks to Maria Bruna, Martin Burger, Markus Schmidtchen and Matthias Rakotomalala
- Thank you IMAG and Granada

Mean-field limit PDE

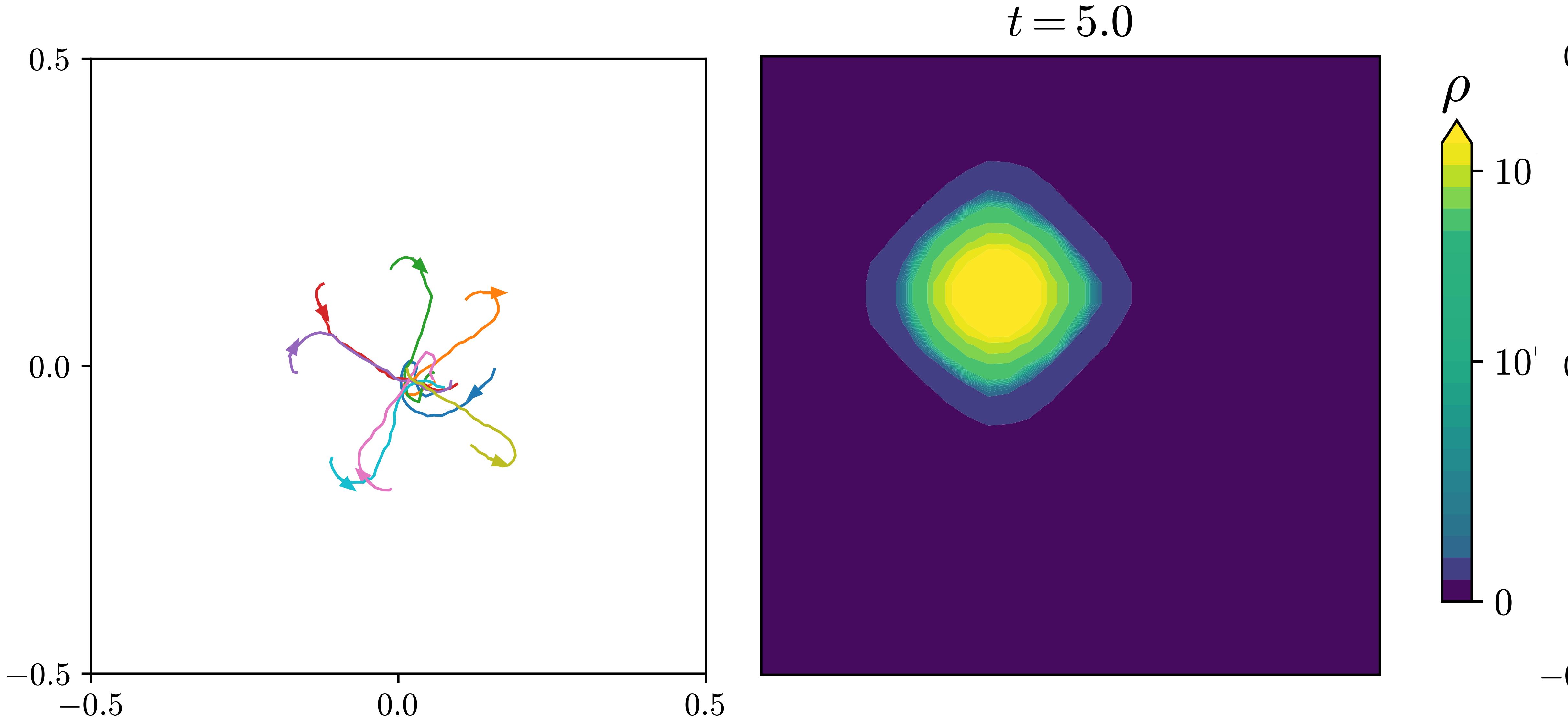
$$\partial_t f = \nabla_{\mathbf{x}} \cdot (D_T \nabla_{\mathbf{x}} f - \mathrm{Pe}\mathbf{e}_\theta f) + \partial_\theta(\partial_\theta f - \gamma \mathbf{n}_\theta \cdot \nabla c_\lambda f), \quad 0 = \Delta c - \alpha c + \rho$$

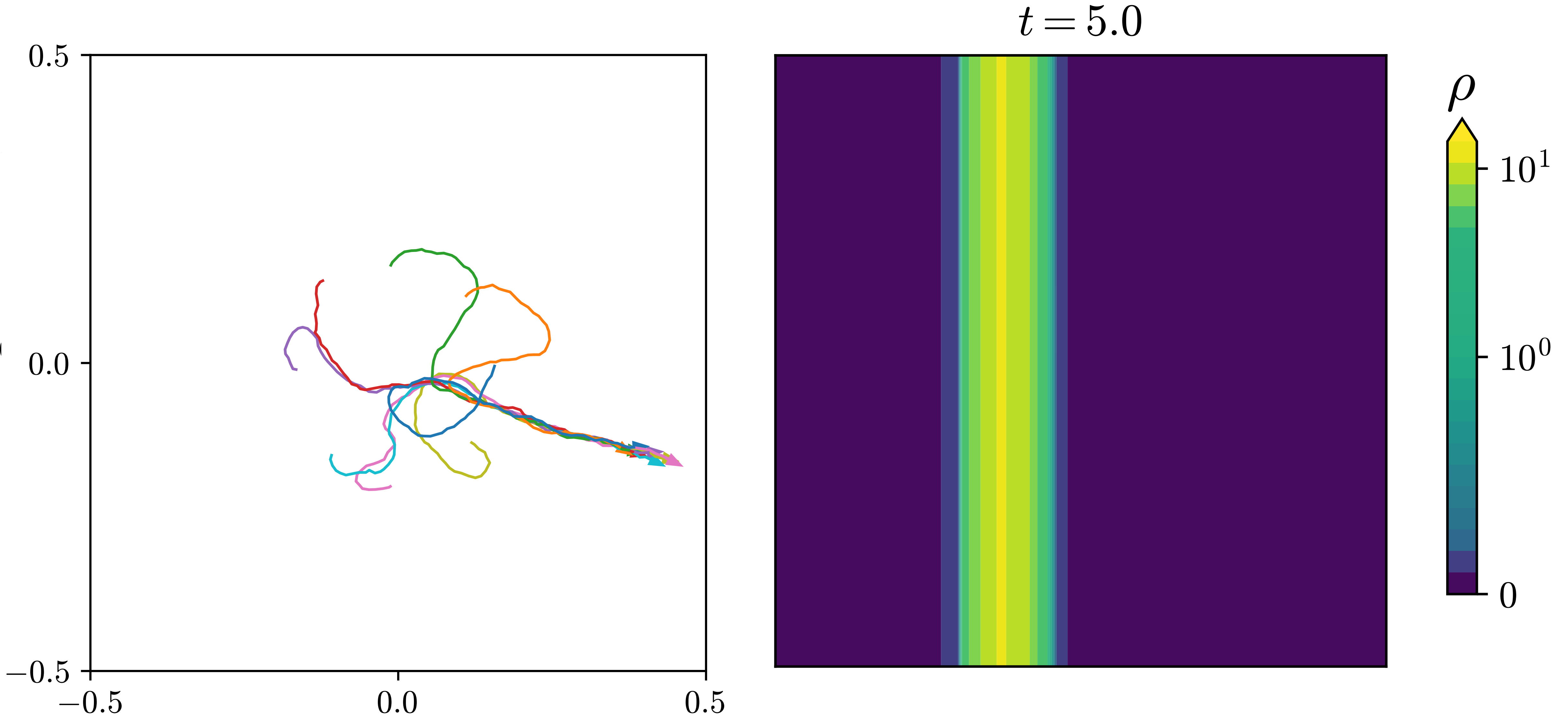
- Convergent finite volume scheme
- Bruna, Schmidtchen & dW (2025), ESAIM: M2AN, to appear



Mean-field limit PDE

$$\partial_t f = \nabla_{\mathbf{x}} \cdot (D_T \nabla_{\mathbf{x}} f - \mathrm{Pe}\,\mathbf{e}_\theta f) + \partial_\theta(\partial_\theta f - \gamma \mathbf{n}_\theta \cdot \nabla c_\lambda f), \quad 0 = \Delta c - \alpha c + \rho$$





Mean-field limit PDE

Globally Lipschitz interaction kernel

Theorem. (McKean.) *Let* $\mathbf{Z}_t^i = (\mathbf{X}_t^i, \Theta_t^i)$ *be the solutions to the interacting particle system with globally Lipschitz kernel* ∇K *and let* $\bar{\mathbf{Z}}_t^i$ *be a solution to the associated McKean-Vlasov process, then*

- For the random empirical measure $\mu_t^N = \frac{1}{N}\sum_i \delta_{\mathbf{Z}_t^i}$ we have $\mathbb{E}\left[W_1(\mu_t^N, f_t)\right] \leq \dfrac{C}{\sqrt{N}}$,

  where $\partial_t f = \nabla_{\mathbf{x}} \cdot (D_T \nabla_{\mathbf{x}} f - \mathrm{Pe}\mathbf{e}_\theta f) + \partial_\theta(\partial_\theta f - \gamma \mathbf{n}_\theta \cdot (\nabla K * \rho)_\lambda f)$

- Using notation $\rho(t, \mathbf{x}) = \displaystyle\int_0^{2\pi} f(t, \mathbf{x}, \theta)\mathrm{d}\theta$ and $g_\lambda(\mathbf{x}, \theta) = g(\mathbf{x} + \lambda \mathbf{e}_\theta)$

Mean-field limit PDE

Singular interaction kernel

Theorem. *If* $0 \leq 2\lambda^2 + \dfrac{\lambda\gamma}{2\pi} < 8D_T$*, the sequence of empirical measures* $(\mu^N(t))_N$ *converges to* $f(t)$ *where* f *is the unique weak solution to*

$$\begin{cases} \partial_t f = \nabla_{\mathbf{x}} \cdot (D_T \nabla_{\mathbf{x}} f - \mathrm{Pe}\mathbf{e}_\theta f) + \partial_\theta(\partial_\theta f - \gamma \mathbf{n}_\theta \cdot \nabla_{\mathbf{x}} c_\lambda f), \\ 0 = \Delta_{\mathbf{x}} c - \alpha c + \rho \, . \end{cases}$$

- $\mu^N(t=0) \sim f(t=0) \in L^\infty \cap L^1(\mathbb{R}^3), \int |z|^2 f(t=0)\mathrm{d}z < \infty$
- Fournier & Jourdain, Ann. Appl. Probab., (2017), Prokhorov's Theorem