Extrinsic GJMS Operators for Submanifolds

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GJMS Operators

GJMS Operators: conformally covariant differential operators on Riemannian manifolds (M^n, g)

$$P_{2\ell} = (-\Delta_g)^\ell + \mathit{lots}$$

If $\widehat{g} = e^{2\omega} g$ with $\omega \in C^\infty(M)$, then

$$\widehat{P}_{2\ell} = e^{(-n/2-\ell)\,\omega} \circ P_{2\ell} \circ e^{(n/2-\ell)\,\omega}$$

Exist for all $\ell \geq 1$ if n odd, only for $1 \leq 2\ell \leq n$ if n even

$P_2 = -\Delta + \frac{n-2}{4(n-1)} R$: Yamabe operator, conformal Laplacian

P_4: Paneitz operator

If n even, P_n is the critical GJMS operator

Fundamental objects in conformal geometry

Branson's Q-curvature

Q-curvature is defined in terms of zeroth order term of $P_{2\ell}$:

$$P_{2\ell}1 = \frac{(n-2\ell)}{2} Q_{2\ell}$$

If n is even, RHS vanishes in critical case $2\ell = n$.

Branson: analytically continue to define critical Q_n.

Important property:

$$e^{n\omega} \widehat{Q_n} = Q_n + P_n \omega$$

The Project

Joint with Jeffrey Case and Tzu-Mo Kuo.

arXiv:2306.11294

Project begun at AIM, August 2022

$\Sigma^k \subset (M^n, g)$ Construct submanifold analogs of GJMS operators

Differential operators on Σ determined by geometry of $\Sigma \subset M$

One possibility: intrinsic operators

Let $h =$ induced metric on Σ

Take usual GJMS operators for (Σ, h).

Not what we want

Main Existence Result

Theorem: Let $n \geq 3$, $1 \leq k \leq n-1$. For the following values of ℓ, there exists a (minimal submanifold) extrinsic GJMS operator $P_{2\ell} : C^\infty(\Sigma) \to C^\infty(\Sigma)$, with leading term $(-\Delta_h)^\ell$.

- $1 \leq \ell < \infty$ if n and k are both odd,
- $1 \leq \ell < n/2$ if n is even and k is odd,
- $1 \leq \ell \leq k/2 + 1$ if k is even. (If $\ell = k/2 + 1$ and n is even, also assume $n > k + 2$.)

$P_{2\ell}$ is natural, self-adjoint, and

$$\widehat{P}_{2\ell} = e^{(-k/2-\ell)\,\omega|_\Sigma} \circ P_{2\ell} \circ e^{(k/2-\ell)\,\omega|_\Sigma}\,, \qquad \widehat{g} = e^{2\omega} g.$$

P_k is the critical operator if k is even.

Have $P_{2\ell} 1 = \frac{k-2\ell}{2} Q_{2\ell}$. If k is even, analytically continue to obtain Q_k: (minimal submanifold) extrinsic critical Q-curvature.

$$e^{k\,\omega|_\Sigma} \widehat{Q}_k = Q_k + P_k(\omega|_\Sigma).$$

P_2, Q_2

Notation: Use local coordinates $z^i = (x^\alpha, u^{\alpha'})$, where $1 \le i \le n$, $1 \le \alpha \le k$, $k+1 \le \alpha' \le n$. Always assume $\Sigma = \{u = 0\}$ and $\partial_\alpha \perp \partial_{\alpha'}$ on Σ. $\quad TM|_\Sigma = T\Sigma \oplus N\Sigma$.

Schouten tensor: $\mathrm{P}_{ij} := \frac{1}{n-2}\left(R_{ij} - \frac{1}{2(n-1)} R g_{ij}\right)$

Decomposes into $\mathrm{P}_{\alpha\beta}$, $\mathrm{P}_{\alpha\alpha'}$, $\mathrm{P}_{\alpha'\beta'}$

Second fundamental form: $L(X,Y) := ({}^g\nabla_x Y)^\perp$. Written $L_{\alpha\beta}^{\alpha'}$

Mean curvature vector: $H^{\alpha'} := \frac{1}{k} g^{\alpha\beta} L_{\alpha\beta}^{\alpha'}$. Then

$$P_2 = -\Delta_h + \frac{k-2}{2} Q_2, \qquad Q_2 = \mathrm{P}_\alpha{}^\alpha + \frac{k}{2}|H|^2.$$

Can rewrite. Set $\overline{P}_2 = -\Delta_h + \frac{k-2}{4(k-1)} R_h =$ intrinsic P_2. Then

$$P_2 = \overline{P}_2 + \frac{k-2}{2}\mathrm{G}, \qquad \mathrm{G} := \frac{1}{2(k-1)}\left(|\mathring{L}|^2 - W_{\alpha\beta}{}^{\alpha\beta}\right).$$

Formula in terms of $\overline{P}_2$ only holds for $k > 1$.

Poincaré Metric

Original construction of usual GJMS operators used ambient metric. Later reformulated in terms of Poincaré metric g_+.

Special case: For $(M, g) = (\mathbb{R}^n, |dx|^2)$, have
$g_+ = r^{-2}(dr^2 + |dx|^2) =$ hyperbolic metric.

Given (M, g), construct a Poincaré metric for g: metric g_+ on $X = M \times [0, \epsilon)_r$ of form

$$g_+ = r^{-2}\left(dr^2 + g_r\right), \qquad g_0 = g$$

with g_r smooth and even in r, and

$$\left| \operatorname{Ric}(g_+) + n g_+ \right|_{g_+} = \begin{cases} O(r^\infty) & n \text{ odd} \\ O(r^n) & n \text{ even} \end{cases}$$

g_r is unique modulo $O(r^\infty)$ for n odd, and modulo $O(r^n)$ for n even.

Construction of Usual GJMS Operators

The GJMS operators arise as obstructions to smooth extension as eigenfunctions of Δ_{g_+}.

Given $\ell \geq 1$ and $f \in C^\infty(M)$, search for $F \in C^\infty(X)$ so that $F|_M = f$ and $u := r^{n/2-\ell}F$ satisfies

$$\left[\Delta_{h+} + \left((n/2)^2 - \ell^2\right)\right] u = O(r^\infty)$$

There is an obstruction at order 2ℓ: can find F unique modulo $O(r^{2\ell})$ so that u gives $O(r^{n/2+\ell})$. The leading term in the error:

$$\left.\left(r^{-n/2-\ell}\left[\Delta_{h+} + \left((n/2)^2 - \ell^2\right)\right] u\right)\right|_M$$

is independent of the $O(r^{2\ell})$ ambiguity in F, and equals $a_\ell P_{2\ell} f$, where $a_\ell^{-1} = (-1)^\ell 2^{2(\ell-1)}(\ell-1)!^2$.

Can carry out the same construction for any AH metric. Get differential operators $P_{2\ell}$ on M. But they depend on the AH metric, not just on (M, g).

Singular Yamabe Extrinsic Operators

Gover-Waldron have constructed extrinsic operators and Q-curvatures when $\Sigma \subset (M, [g])$ is a hypersurface.

Use a distinguished AH representative in $(M, [g])$: asymptotic solution of the singular Yamabe problem.

This is an AH metric in $[g]$ defined near Σ with asymptotically constant scalar curvature $-n(n+1)$, canonically determined to finite order by $\Sigma \subset (M, [g])$.

Method produces operators on vector bundles (tractor bundles) too.

Minimal Submanifold Extension

Our construction associates an AH metric to $\Sigma^k \subset (M^n, g)$.

First extend the background space to (X, g_+) just as before:
$X = M \times [0, \epsilon)_r$, $g_+ = r^{-2}(dr^2 + g_r)$ Poincaré metric.

Then search for a submanifold $Y^{k+1} \subset X^{n+1}$ satisfying

- $Y \cap M = \Sigma$
- Y is asymptotically minimal with respect to g_+
- Y is smooth and even

If n, k both odd, there exists unique Y to infinite order.
If k even, obstructed at order $k+2$

Let $h_+ =$ metric induced on Y by g_+.

Then (Y, h_+) is AH with conformal infinity $(\Sigma, [h])$

Apply GJMS construction on (Y, h_+) to get minimal submanifold extrinsic operators

Factorization for GJMS Operators

Return to usual GJMS operators on (M, g)

Factorization in case g is Einstein (Branson, Fefferman-G, Gover)

Suppose $\mathrm{Ric}(g) = \lambda(n-1)g$. Then

$$P_{2\ell} = \prod_{j=1}^{\ell}(-\Delta_g + \lambda c_j), \qquad c_j = (\tfrac{n}{2} + j - 1)(\tfrac{n}{2} - j).$$

If n is even, then

$$Q_n = \lambda^{n/2}(n-1)!$$

Same factorization holds for minimal submanifold extrinsic operators if (M, g) is Einstein and $\Sigma \subset (M, g)$ is minimal!

Factorization for Minimal Submanifold Extrinsic Operators

Theorem: Suppose $\mathrm{Ric}(g) = \lambda(n-1)g$ and $\Sigma \subset (M, g)$ is minimal. Then

$$P_{2\ell} = \prod_{j=1}^{\ell}(-\Delta_h + \lambda c_j), \qquad c_j = (\tfrac{k}{2} + j - 1)(\tfrac{k}{2} - j).$$

If k is even, then

$$Q_k = \lambda^{k/2}(k-1)!$$

Result fails for the intrinsic operators and the singular Yamabe operators, even for Q_2. Recall $Q_2 = \mathsf{P}_\alpha{}^\alpha + \frac{k}{2}|H|^2$. Q_2 is not constant for the intrinsic and singular Yamabe operators for minimal submanifolds of Einstein manifolds. Involve R_h, $|\mathring{L}|^2$.

Jeffrey will discuss an application to a Gauss–Bonnet Theorem involving renormalized area for minimal submanifolds of Poincaré-Einstein spaces. The application was the motivation for this project: we needed to find extrinsic operators satisfying this factorization.

Analogous to Chang-Qing-Yang result for renormalized volume.

Fractional Order Scattering Operators

There are fractional order scattering operators associated to any AH space (Y^{k+1}, h_+). For $\operatorname{Re} s > k/2$, solve Poisson equation

$$\big(\Delta_{h_+} + s(k-s)\big)u = 0, \qquad u = Fr^{k-s} + Gr^s, \qquad F|_\Sigma = f$$

with $F,\ G \in C^\infty(Y)$. Scattering operator is $S(s)f = G|_\Sigma$.

Chang–Gonzalez: for $\mathbb{H}^{k+1}$, equivalent to Caffarelli-Silvestre extension.

Scattering operators have rich theory for h_+ a Poincaré metric.

Might be interesting to explore possible uses of scattering operators for $h_+ =$ induced metric on minimal submanifold of Poincaré-Einstein space, reflecting extrinsic geometry of a submanifold $\Sigma \subset (M, g)$.

Thanks for listening!