Yamabe flow of asymptotically flat metrics

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Outline

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Yamabe flow

Let (M^n, g_0) be a Riemannian manifold. The Yamabe flow $g(t) \in [g_0]$ is the evolution that satisfies

$$\begin{cases} \partial_t g = -R_g g, \\ g(0) = g_0. \end{cases} \tag{1}$$

We may write $g(t) = u(x,t)^{\frac{4}{n-2}} g_0$. The relation between R_g and R_{g_0} is:

$$-a(n)\Delta_{g_0} u + R_{g_0} u = R_g u^{\frac{n+2}{n-2}}, \tag{2}$$

where $a(n) = 4\frac{n-1}{n-2}$.

The Yamabe flow

The Yamabe flow can be rewritten as an evolution equation of the conformal factor $u(x,t)$:

$$\partial_t u^N = \frac{n+2}{4}\left[a(n)\Delta_{g_0} u - R_{g_0} u\right]. \tag{3}$$

We denote $N = \frac{n+2}{n-2}$.

The Yamabe flow

The conformal Laplacian

$$L_g := -\Delta_g + \frac{1}{a(n)} R_g.$$

Under conformal change of metric $\hat{g} = u^{\frac{4}{n-2}} g$, $u > 0$

$$a(n) \cdot L_g u = \hat{R} u^{\frac{n+2}{n-2}}. \tag{4}$$

- The Yamabe problem: Let (M^n, g_0) $n \geq 3$ be a compact Riemannian manifold. Then there exists a constant scalar curvature metric $g \in [g_0]$. This problem is equivalent to solving the equation (4) for a constant $\hat{R}$. [Trudinger '68], [Aubin '76], [Schoen '84].

The Yamabe flow

In the 80s, Hamilton proposed the Yamabe flow as a heat flow approach to study the Yamabe problem.

- The uniformization theorem in dimension 2. Hamilton '88. Chow '91, etc.
- The Yamabe problem in dimension $n \geq 3$.

- Hamilton '88, Chow '92, Ye '94, Schwetlick-Struwe '03, Brendle '05, '07
- Ye '94, Yamabe flow on locally conformally flat manifolds
- Schwetlick-Struwe '03, Yamabe flow in dimension $3 \leq n \leq 5$ with small initial energy
- Brendle '05 Convergence of the Yamabe flow in dimension $3 \leq n \leq 5$ with arbitrary initial energy
- Brendle '07 Convergence of the Yamabe flow in dimension 6 and higher

AF manifolds

Definition of AF manifolds: We call a smooth Riemannian manifold (M^n, g) asymptotically flat of order $\tau > 0$ if for some compact set $K \subset M^n$, there exists an $R_0 > 0$ and a diffeomorphism $\Phi : M^n \backslash K \to \mathbb{R}^n \setminus B_{R_0}(0)$ such that for some $\tau > 0$,

$$g_{ij}(x) = \delta_{ij} + O(|x|^{-\tau}) \quad \text{and} \quad \partial^\alpha g_{ij}(x) = O(|x|^{-\tau-|\alpha|}), \tag{5}$$

for partial derivatives ∂^α of any order, as $|x| \to \infty$ in $\mathbb{R}^n$. Φ is called the asymptotically flat coordinate system, and we write $M_\infty = M \setminus K$.

Remark: 1. For simplicity, we assume M^n has only 1 end. One can consider manifolds with multiple ends as well.

2. When $\tau > n - 2/2$, the ADM mass m is well defined.

AF manifolds

Definitions of weighted Sobolev spaces and Hölder spaces (Bartnik '86):
The weighted Lebesgue spaces $L^q_\beta(M)$, for $q \geq 1$ and $\beta \in \mathbb{R}$, consists of functions such that the following norms are finite.

$$\|v\|_{L^q_\beta(M)} = \begin{cases} \left(\int_M |v|^q r^{-\beta q - n} dx\right)^{\frac{1}{q}}, & q < \infty, \\ \operatorname{ess\,sup}_M \left(r^{-\beta}|v|\right), & q = \infty. \end{cases} \tag{6}$$

The weighted Sobolev spaces $W^{k,q}_\beta(M)$ are then defined in the usual way with the norms

$$\|v\|_{W^{k,q}_\beta(M)} = \sum_{j=0}^{k} \|D^j_x v\|_{L^q_{\beta-j}(M)} \,. \tag{7}$$

AF manifolds

The weighted C^k spaces $C^k_\beta(M)$ consist of the C^k functions for which the following respective norms are finite (Bartnik '86):

$$\|v\|_{C^k_\beta(M)} = \sum_{j=0}^{k} \sup_M r^{-\beta+j} \left|D^j_x v\right|. \tag{8}$$

The weighted Hölder spaces $C^{k+\alpha}_\beta(M)$, $\alpha \in (0,1)$, consist of those $v \in C^k_\beta(M)$ for which the following respective norms are finite:

$$\|v\|_{C^{k+\alpha}_\beta(M)} \tag{9}$$

$$= \|v\|_{C^k_\beta(M)} + \sup_{x\neq y\in M} \min(r(x), r(y))^{-\beta+k+\alpha} \frac{\left|D^k_x v(x) - D^k_x v(y)\right|}{d(x,y)^\alpha}. \tag{10}$$

Short-time existence

- Definition of fine solutions: We say that $g(t)$ is a fine solution of the Yamabe flow on a complete manifold (M^n, g_0) on a maximal time interval $[0, T_0)$ if $g(t) = u(t)^{\frac{4}{n-2}} g_0$ with $u(0) \equiv 1$ and for any $T \in (0, T_0)$ there exists $\delta = \delta(T)$ and $C = C(T)$ such that on $[0, T]$, $0 < \delta \leq |u(t,x)| \leq C$, $\sup_{[0,T]\times M^n} |\nabla_{g_0} u(x,t)| \leq C$, and $\sup_{[0,T]\times M^n} |Rm(g)|(t,x) \leq C$, and moreover either $T_0 < \infty$ and $\lim_{t\to T_0} |Rm|(t,\cdot) = \infty$, or $T_0 = \infty$.
Remark: the blowup alternative in the definition above also holds true if we write either $T_0 < \infty$ and $\lim_{t\to T_0} |R|(t,\cdot) = \infty$, or $T_0 = \infty$ by Ma.

Short-time existence

- Short-time existence and the preservation of asymptotic flatness by Cheng-Zhu '15

Theorem (Cheng-Zhu '15) If (M^n, g_0) is a $C^{2+\alpha}_{-\tau}$ AF manifold, then there exists a fine solution of the Yamabe flow starting from (M^n, g_0) on a maximal time interval $[0, T_0)$ with $T_0 > 0$. Moreover, $g_{ij}(t) - \delta_{ij} \in C^{2+\alpha}_{-\tau}$ for $t \in [0, T_0)$.

Long-time existence

The main results of Chen-Wang are long-time exisence and convergence.

- Theorem ('21, Eric Chen and W.) The Yamabe flow starting from a $C^{k+\alpha}_{-\tau}$ AF manifold $\tau > 0$ exists for all positive times $t > 0$.

Long-time existence

Under the extra condition $Y(M^n, [g_0]) > 0$, we derive the long-time existence together with uniformly decay of the scalar curvature along the flow. The quantitative decay estimate of R_g:

- Theorem ('21, Eric Chen and W.) Suppose the Yamabe flow starts from a $C^{k+\alpha}_{-\tau}$ AF manifold with $Y(M^n, [g_0]) > 0$. Then for any $\delta < \frac{\tau}{2}$, there exists $C > 0$ such that $\|R_{g(t)}\|_{L^\infty} \leq Ct^{-1-\delta}$.

Convergence of $g(t)$

Convergence of $g(t)$

- Theorem ('21, Eric Chen and W.) Suppose the Yamabe flow starts from a $C^{k+\alpha}_{-\tau}$ AF manifold with $Y(M^n, [g_0]) > 0$. Then $g(t) \to g_\infty$ as $t \to \infty$ in $C^{k+\alpha}_{-\tau'}$, for all $0 < \tau' < \min\{\tau, n-2\}$. Here g_∞ is a scalar flat metric on M^n.

Divergence of $g(t)$

Divergence of $g(t)$

- Theorem ('22, Gilles Carron, Eric Chen and W.)Let (M^n, g_0) be a $C^{k+\alpha}_{-\tau}$ AF manifold with $k \geq 3$. If $Y(M^n, [g_0]) \leq 0$, then the Yamabe flow $(M^n, g(t))$ diverges.
- Remark: we have analyzed the blowup rate.

Uniform Sobolev inequality

L^2-Sobolev inequality on $(M^n, g(t))$ is one of the key analytic tool to study the flow. We hope to derive a uniform bound for the Sobolev constant.

We consider two cases:

1) g_0 is a general case of an AF metric;

2) g_0 is an AF metric with positive Yamabe constant.

Sobolev inequality

Proposition

If $u(x,t)$ is the solution of (3) corresponding to a fine solution of the Yamabe flow starting from a $C^{k+\alpha}_{-\tau}$ AF manifold (M^n, g_0), then for any T for which the Yamabe flow exists on $[0,T]$, there exists a $C(g_0,T) > 0$ depending only on an upper bound for T and g_0 such that for every $\phi \in W^{1,2}_0(M, g(t))$, the following Sobolev inequality holds:

$$\left(\int |\phi|^{\frac{2n}{n-2}}\, dV_{g(t)}\right)^{\frac{n-2}{n}} \le C(g_0,T)\int |\nabla\phi|^2\, dV_{g(t)}. \tag{11}$$

Uniform Sobolev inequality

Proposition

If $u(x,t)$ is the solution of (3) corresponding to a fine solution of the Yamabe flow starting from a $C^{k+\alpha}_{-\tau}$ AF manifold (M^n, g_0) satisfying $Y(M,[g_0]) > 0$, then there exists a constant $D = D(g_0)$ such that for any T for which the Yamabe flow exists on $[0,T]$ the following Sobolev inequality holds for every $\phi \in W^{1,2}_0(M, g(t))$, :

$$\left(\int |\phi|^{\frac{2n}{n-2}}\, dV_{g(t)}\right)^{\frac{n-2}{n}} \leq D \int |\nabla\phi|^2\, dV_{g(t)}. \qquad (12)$$

Remark: In the case of the Ricci flow, one needs to consider the μ-functional introduced by Perelman to derive a weighted L^2 Sobolev inequality as a substitute. See the work of [Ye-Chen].

Conformal deformations of asymptotically flat metrics

- Cantor-Brill '81, Let $p \in \left(1, \frac{2n}{n-2}\right)$, $\tau \in \left(2 - n\left(1 - \frac{1}{p}\right), \frac{n}{p}\right)$, $k > \frac{n}{p} + 2$, and $g - \delta_{ij} \in W^{k,p}_{-\tau}$. Then the following are equivalent:

Conformal deformations of asymptotically flat metrics

1. $$Y(M,[g]) := \inf_{\substack{u\in C_0^\infty(M),\\ u\neq 0}} \int_M \frac{c_n|\nabla u|^2 + R_g u^2 \; dV_g}{\left(\int |u|^{\frac{2n}{n-2}} \; dV_g\right)^{\frac{n-2}{n}}} > 0 \qquad (13)$$

   Here $c_n = 4\frac{n-1}{n-2}$.

2. There is a $\tilde{g} - \delta_{ij} \in W^{k,p}_{-\tau}$ such that $\tilde{g}$ is conformally equivalent to $\tilde{g}$ and $R_{\tilde{g}} \equiv 0$.

Moser iteration

The evolution equation of R_g

$$\partial_t R = (n-1)\Delta R + R^2, \quad \partial_t dV_t = -\frac{n}{2} R \, dV_t \tag{14}$$

Monotonicity of integral norms of $\|R\|_{L^{n/2}}$.

Lemma

$\int |R|^{\frac{n}{2}} \, dV_t$ is non-increasing along the flow.

In fact this is also true for all p in a neighborhood of $n/2$.

Moser iteration

Proposition

Let $(M^n, g(t))$ be a Yamabe flow starting from a $C^{k+\alpha}_{-\tau}$ AF manifold with $Y(M^n,[g_0]) > 0$, $k \geq 3$ and suppose that

$$\|R\|_{L^q} \leq \alpha_1 t^{-\gamma_1}, \quad \text{for some } \alpha_1 \geq 0,\ \gamma_1 \geq 0, \text{ and } q > \frac{n}{2}, \tag{15}$$

$$\|R\|_{L^{p_0}} \leq \alpha_2 t^{-\gamma_2}, \quad \text{for some } \alpha_2 \geq 0,\ \gamma_2 \geq 0, \text{ and } p_0 > \frac{n}{2+\tau}. \tag{16}$$

If the flow exists on $[0, T]$ then we have the following estimate for $|R|$:

$$\sup_{x\in\mathbb{R}^n} |R(T,x)| \leq C \cdot \max\left(T^{\frac{1}{p_0}-\gamma_1\frac{q(n+2)}{p_0(2q-n)}-\gamma_2}, T^{-\frac{n}{2p_0}-\gamma_2} \right). \tag{17}$$

Quantitative decay of R_g

When $Y(M^n, [g_0]) > 0$, we obtain the decay estimate of curvature along the Yamabe flow.

Proposition

Along the Yamabe flow $(M^n, g(t))$ starting from a $C^{k+\alpha}_{-\tau}$ AF manifold with $Y(M^n, [g_0]) > 0$, we have that

$$\sup_{x \in \mathbb{R}^n} |R(T,x)| \xrightarrow{T \to \infty} 0. \tag{18}$$

Quantitative decay of R_g

We can perform the Moser iteration again, and derive a quantitative decay estimate of the L^∞-norm of the scalar curvature.

Theorem

Suppose the Yamabe flow starts from a $C^{k+\alpha}_{-\tau}$ AF manifold with $Y(M^n, [g_0]) > 0$. Then for any $\delta < \frac{\tau}{2}$, there exists $C > 0$ such that $\|R\|_{L^\infty} \leq Ct^{-1-\delta}$.

This quantitative decay estimate, together with standard parabolic equation estimate allows us to obtain the convergence of metric.

Convergence of $g(t)$

Two steps to prove convergence of $g(t)$:

- Step 1. Unweighted convergence
- Step 2. Weighted convergence

Convergence of $g(t)$

- Step 1. Unweighted convergence

Theorem

Let (M^n, g_0) be a $C^{k+\alpha}_{-\tau}$ AF manifold with $k \geq 3$. If $Y(M^n, [g_0]) > 0$, then the Yamabe flow $(M^n, g(t))$ starting from (M^n, g_0) converges uniformly in $C^{k+\alpha}_0$ to the unique $C^{k+\alpha}_{-\tau}$ AF metric $g_\infty \in [g_0]$ as $t \to \infty$.

Convergence of $g(t)$

- Existence of limit:

Proposition

There exists a continuous function $u_\infty(x) > 0$ on M such that for all $\delta < \frac{\tau}{2}$,

$$\|u(x,t) - u_\infty(x)\|_{L^\infty(M)} \le \frac{C}{t^\delta},$$

where u_∞ satisfies $u_\infty(x) - 1 \to 0, \quad$ as $r \to \infty$.

Convergence of $g(t)$

- Local convergence:

Proposition

For all $0 < \alpha' < \alpha$, $0 < \delta < \tau/2$

$$\|u(x,t) - u_\infty(x)\|_{C^{k+\alpha'}_{loc}} \le \frac{C}{t^\delta}, \tag{19}$$

Moreover, $u_\infty \in C^{k+\alpha}_{loc}(M)$ *and* $g_\infty = u_\infty(x)^{\frac{4}{n-2}} g_0$ *is a scalar flat metric, i.e.*

$$\Delta_{g_0} u_\infty(x) - \frac{1}{a(n)} R_{g_0}(x) u_\infty(x) = 0.$$

Convergence of $g(t)$

- $u_\infty - 1 \in C^{k+\alpha}_{-\tau}$

Proposition

$u_\infty - 1 \in C^{k+\alpha}_{-\tau}$ and satisfies the estimates

$$|u_\infty(x) - 1| \leq \frac{C}{r^\tau}, \quad \text{as } r \to \infty, \tag{20}$$

and

$$|\partial_j u_\infty(x)| \leq \frac{C}{r^{1+\tau}}, \quad \text{as } r \to \infty. \tag{21}$$

Convergence of $g(t)$

- Step 2. Weighted convergence

Theorem

Let (M^n, g_0) be a $C^{k+\alpha}_{-\tau}$ AF manifold with $Y(M, [g_0]) > 0$, $k \geq 3$, and $\tau > 1$. Then there exists a Yamabe flow $(M^n, g(t))$ starting from (M^n, g_0) defined for all positive times and a metric g_∞ on M^n which is $C^{k+\alpha}_{-\tau'}$ AF for all $\tau' < \min\{\tau, n-2\}$ so that for any such τ' we have

$$\|g(t) - g_\infty\|_{C^{k+\alpha}_{-\tau'}} = O(t^{-\delta_0}), \quad \text{as } t \to \infty, \tag{22}$$

for some $\delta_0 > 0$. In particular, this Yamabe flow converges in $C^{k+\alpha}_{-\tau'}$ to the asymptotically flat, scalar flat metric g_∞.

Convergence of $g(t)$

- Decay estimate for the scalar curvature in both space and time:

Proposition

For any $\tau' < \tau$ there exists some $\delta_0 > 0$ and $C > 0$ depending only on g_0 such that

$$|R(x,t)| \leq \frac{C}{r^{\tau'}(1+t)^{1+\delta_0}} \tag{23}$$

for all $(x,t) \in M \times [0,\infty)$ along the flow.

Main difficulty: there is no weighted version of Shi's estimate for *Rm* and its derivatives as in Ricci flow case at the moment.

Convergence of $g(t)$

To overcome this difficulty, we need a lemma to control $h(x) := r(x)^a$ by a mixed decay estimate. (Here $a > 0$ is a suitable power.)

Lemma

For $r(x) >> 1, t >> 1$ and any $\beta < \frac{1}{2}$,

$$|\nabla_{g(t)} h|_{g(t)} \leq C |\nabla_{g_0} h|_{g(0)} \leq \frac{Ch}{r}, \tag{24}$$

$$|\Delta_{g(t)} h| \leq \frac{Ch}{r} \cdot \left(\frac{1}{r} + \frac{1}{t^\beta} \right). \tag{25}$$

Convergence of $g(t)$

Proposition

For any $\tau' < \tau$ we have

$$u(x,t) - u_\infty(x) \to 0 \quad \text{in } C^0_{-\tau'}, \quad \text{as } t \to \infty,$$

where $u_\infty \in C^{k+\alpha}_{-\tau}$. Moreover, there is a $\delta_0 > 0$ such that

$$\|u(x,t) - u_\infty(x)\|_{C^0_{-\tau'}} \leq \frac{C}{t^{\delta_0}}. \tag{26}$$

Convergence of $g(t)$

We also need uniform control of $u(x,t)$ in parabolic weighted Sobolev and Hölder spaces.

Lemma

(1) *For any $p>1$, $\tau'<\tau$, there exists a constant $C>0$ such that for any $0<\tau_1<\tau'$ and all $t_0\geq 0$.*

$$\|u-1\|_{W^{k,k/2,p}_{-\tau_1}(M\times[t_0,t_0+1])}\leq C.$$

(2) *For any $\tau'<\tau$, there exists a constant $C>0$, such that for all $t_0\geq 0$,*

$$\|u-1\|_{C^{k+\alpha,\frac{k+\alpha}{2}}_{-\tau'}(M\times[t_0,t_0+1])}\leq C.$$

Here C is independent of time t_0.

Convergence of $g(t)$

Using parabolic bootstrap argument, we complete the main theorem.

Theorem

For all $l \leq k$ and $\tau' < \tau$

$$\|u(x,t) - u_\infty(x)\|_{C^{l+\alpha,\frac{l+\alpha}{2}}_{-\tau'}(M\times[t_0,t_0+1])} \leq \frac{C}{t_0^{\delta_0}}. \tag{27}$$

Convergence of $g(t)$

When $Y(M^n, [g_0]) \leq 0$, we have proved the flow must diverge.

Theorem

If $Y(M^n, [g_0]) \leq 0$, then the Yamabe flow $(M^n, g(t))$ starting from (M^n, g_0) does not converge. In particular, $g(t) = u(x,t)^{\frac{4}{n-2}} g_0$ will fail to remain uniformly equivalent to g_0 as $t \to \infty$, and both $\|u(x,t)\|_{L^\infty}$ and the L^2 Euclidean-type Sobolev constant of $g(t)$ will tend to positive infinity.

Thus it gives a relatively clean picture of the Yamabe flow on asymptotically flat manifolds.

Divergence of the flow and the rescaled flow

The paper with Gilles Carron and Eric Chen discusses two cases—one when $Y(M^n, [g_0]) < 0$, and the other when $Y(M^n, [g_0]) = 0$. In both cases, the Yamabe flow $(M^n, g(t))$ blows up at the rate of $u(x, t) = O(t^{\frac{n-2}{4}})$. But the limiting profiles of $\tilde{u}(x, t) := t^{-\frac{n-2}{4}} u(x, t)$ behave differently.

Divergence of the flow and the rescaled flow

Theorem

('22, Gilles Carron, Eric Chen and W.)Let (M^n, g_0) be a $C^{k+\alpha}_{-\tau}$ AF manifold with $k \geq 3$. If $Y(M^n, [g_0]) < 0$, then the Yamabe flow $(M^n, g(t))$ blows up at the rate of $u(x,t) = O(t^{\frac{n-2}{4}})$. Moreover, $\tilde{u}(x,t) := t^{-\frac{n-2}{4}} u(x,t)$ converges to a limiting function $\tilde{u}_\infty > 0$, where $\tilde{u}_\infty$ is the unique solution to the prescribing -1 constant scalar curvature equation:

$$\begin{cases} -a_n \Delta_{g_0} \tilde{u}_\infty + R_{g_0} \tilde{u}_\infty = -\tilde{u}_\infty^N, \\ \tilde{u}_\infty \to 0 \quad \text{as} \quad |x| \to \infty. \end{cases} \tag{28}$$

Divergence of the flow and the rescaled flow

In other words, $\tilde{u}_{\infty}^{\frac{4}{n-2}} g_0$ extends to the unique -1 constant scalar curvature metric on (the compactified space) $\overline{M}$.

Divergence of the flow and the rescaled flow

Theorem

('22, Gilles Carron, Eric Chen and W.)Let (M^n, g_0) be a $C^{k+\alpha}_{-\tau}$ AF manifold with $k \geq 3$. If $Y(M^n, [g_0]) = 0$, then the Yamabe flow $(M^n, g(t))$ blows up at the rate of $u(x,t) = o(t^{\frac{n-2}{4}})$. In other words, $\tilde{u}(x,t) := t^{-\frac{n-2}{4}} u(x,t)$ converges to 0.

Divergence of the flow and the rescaled flow

When $Y(M^n, [g_0]) = 0$, the Yamabe flow $(M^n, g(t))$ blows up at the rate of $u(x,t) = O(t^{\frac{n-2}{4}})$ (including the case $o(t^{\frac{n-2}{4}})$) as well. But the limiting function cannot be positive everywhere on M, and thus cannot be used as a conformal factor. By Harnack inequality one can show that $u(x,t) = o(t^{\frac{n-2}{4}})$ for all x.

Divergence of the flow and the rescaled flow

Thus in order to describe the blow-up profile of the limit, one needs more delicate estimate.

Theorem

For any compact set $K \subset M$, there exists a sequence $\{t_i\} \to \infty$ and points $x(t_i) \in K$ such that the renormalized sequence $\frac{u(x,t_i)}{u(x(t_i),t_i)}$ has a limit $\phi(x) > 0$ as $t_i \to \infty$. ϕ is the unique positive solution up to a multiplicative constant on (M, g_0) that satisfies

$$-a_n \Delta_{g_0} \phi + R_{g_0} \phi = 0. \tag{29}$$

Moreover, $\phi(x)$ has the decay rate $\phi(x) = o(|x|^{2-n})$.

The end

Thank you!