On the conformal Dirac-Einstein equations

Martino Vittorio, Bologna

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Setting of the problem

(M^n, g) closed Riemannian, $n \geq 3$.

$(M^n, g, \Sigma_g M)$ closed Spin, $n \geq 3$.

Einstein field equation

$$Ric_g - \frac{R_g}{2} g =$$

Dirac wave equation, $\psi \in \Sigma_g M$

$$D_g \psi = m\psi, \quad \text{(mass } m\text{)}$$

Critical points equation of

$$\mathcal{E} : \mathcal{G}(M) \longrightarrow \mathbb{R}$$

$$\mathcal{E}(g) = \int_M R_g \, dv_g$$

Critical points equation of

$$\mathcal{F}_g : \Sigma_g M \longrightarrow \mathbb{R}$$

$$\mathcal{F}_g(\psi) = \int_M \langle D_g \psi, \psi \rangle - m|\psi|^2 \, dv_g$$

$$\boxed{S_g = \mathcal{F}_g}$$

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$$\boxed{S_g = \mathcal{F}_g}$$

Let M^n be closed, $n \geq 3$, and $g \in \mathcal{G}(M), \psi \in \Sigma_g M, \lambda > 0$

The Dirac-Einstein functional is

$$\mathcal{E}(g,\psi) = \int_M R_g + \langle D_g \psi, \psi \rangle - \lambda |\psi|^2 \, dv_g$$

Critical points of $\mathcal{E}$ solve the coupled Dirac-Einstein equations

$$\begin{cases} Ric_g - \frac{R_g}{2} g = T_{g,\psi} \\ D_g \psi = \lambda \psi \end{cases}$$

with $\quad T_{g,\psi}(X,Y) = -\frac{1}{4} \langle X \cdot \nabla_Y \psi + Y \cdot \nabla_X \psi, \psi \rangle, \quad X, Y \in TM$

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Conformal restriction

$$g_0 \in \mathcal{G}(M), \quad [g_0] = \left\{ g \in \mathcal{G}(M),\ g = u^{\frac{4}{n-2}} g_0,\ u \in C^\infty(M),\ u > 0 \right\}$$

For $\varphi \in \Sigma_{g_0} M$, we set $\psi = u^{\frac{1-n}{n-2}} \varphi \in \Sigma_g M \ \Rightarrow \ D_g \psi = u^{-\frac{n+1}{n-2}} D_{g_0} \varphi$

$$\mathcal{E}\left(g, \psi\right) = \int_M u L_{g_0} u + \langle D_{g_0} \varphi, \varphi \rangle - \lambda u^{\frac{2}{n-2}} |\varphi|^2 \, dv_{g_0} =: E(u, \varphi)$$

where $\qquad L_{g_0} u = -a_n \Delta_{g_0} u + R_{g_0} u, \qquad a_n = \frac{4(n-1)}{n-2}$

Critical points of E solve the conformal Dirac-Einstein equations

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Classification result

Let $\left(M^3, g_0, \Sigma_{g_0} M\right)$ be a closed spin manifold of dimension three.

Let
$$E : H^1(M) \times H^{\frac{1}{2}}(\Sigma_{g_0} M) \to \mathbb{R}$$

$$E(u, \varphi) = \int_M u L_{g_0} u + \langle D_{g_0}\varphi, \varphi\rangle - u^2|\varphi|^2 \, dv_{g_0}$$

Critical points of E are (weak) solutions of
$$\begin{cases} L_{g_0} u = |\varphi|^2 u \\ D_{g_0}\varphi = u^2\varphi \end{cases} \qquad (CDE)$$

Given a function $F \in C^1(X, \mathbb{R})$ on a Hilbert space X, a Palais-Smale sequence for F at level $c \in \mathbb{R}$ is a sequence $\{x_k\} \subseteq X$ such that

$$\begin{cases} F(x_k) \xrightarrow[k \to \infty]{\mathbb{R}} c \\ \nabla F(x_k) \xrightarrow[k \to \infty]{X} 0 \end{cases}$$

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Theorem (A. Maalaoui, V. M.)

Suppose that (M^3, g_0) *has a positive conformal Yamabe invariant* $Y^M_{[g_0]}$. *Let* $\{(u_n, \varphi_n)\}$ *be a Palais-Smale sequence for* E *at level* $c \in \mathbb{R}$. *Then there exist* $u_\infty \in C^\infty(M)$, $\varphi_\infty \in C^\infty(\Sigma M)$ *with* $(u_\infty, \varphi_\infty)$ *solution of* (CDE), m *sequences of points* $x_n^1, \cdots, x_n^m \in M$ *such that* $\lim_{n\to\infty} x_n^k = x^k \in M$, *for* $k = 1, \ldots, m$ *and* m *sequences of real numbers* $R_n^1, \cdots, R_n^m$ *converging to zero, such that:*

$$\text{(i)}\quad u_n = u_\infty + \sum_{k=1}^{m} u_n^{*k} + o(1)_{H^1}, \quad u_n^{*k} = \exp^*_{R_n^k, x_n}(U_\infty^k),$$

$$\text{(ii)}\quad \varphi_n = \varphi_\infty + \sum_{k=1}^{m} \varphi_n^{*k} + o(1)_{H^{\frac{1}{2}}}, \quad \varphi_n^{*k} = \exp^*_{R_n^k, x_n}(\Phi_\infty^k),$$

$$\text{(iii)}\quad E(u_n, \varphi_n) = E(u_\infty, \varphi_\infty) + \sum_{k=1}^{m} E_{\mathbb{R}^3}(U_\infty^k, \Phi_\infty^k) + o(1)_{\mathbb{R}}.$$

$(U_\infty^k, \Phi_\infty^k)$ *are critical points of* $E_{\mathbb{R}^3}$ *with the standard Euclidian metric.*

- H.C. Wente, Large solutions to the volume constrained Plateau problem, Arch. Rational Mech. Anal. 75, no. 1, 59-77, (1980/81).
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Remark.
The assumption on M of having a positive conformal Yamabe invariant implies that there are no harmonic spinors, that is the Dirac operator D_{g_0} has no kernel.

This follows from the conformal invariance of $ker(D_g)$

$$D_{g_0}\varphi = u^4 D_g \psi$$

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Existence results - I

As in the Yamabe problem, one defines a conformal invariant $I^M_{[g_0]}$.

We have the following Aubin type result:

Theorem (A. Maalaoui, V. M.)

Let $\left(M^3, g_0, \Sigma_{g_0}M\right)$ be a closed spin manifold of dimension three. It holds:

$$I^M_{[g_0]} \leq I^{S^3}_{[g_S]}.$$

Moreover, if

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then problem (CDE) has a non-trivial ground state solution.

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Existence results - II

Let us consider the following $(CDES)$ $\begin{cases} L_{g_s} u = K|\varphi|^2 u \\ \\ D_{g_s}\varphi = K u^2 \varphi \end{cases}$ on $\mathbb{S}^3$

Theorem (C. Guidi, A. Maalaoui, V. M.)

Let $k \in C^2(\mathbb{S}^3)$ be Morse. Let us set $h = k \circ \pi_{ste}^{-1}$ and assume the following Bahri-Coron type conditions:

$$(i)\ \Delta h(\xi) \neq 0,\ \forall\, \xi \in \mathrm{crit}[h], \qquad (ii) \sum_{\substack{\xi \in \mathrm{crit}[h] \\ \Delta h(\xi) < 0}} (-1)^{m(h,\xi)} \neq -1,$$

Then, $\exists\, \varepsilon_0 > 0$ such that for $K = 1 + \varepsilon k$ and $|\varepsilon| < \varepsilon_0$, the system $(CDES)$ has a solution.

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Classification and existence results, $\partial M \neq \emptyset$

$(M^n, \partial M)$ compact with boundary, $n \geq 3$, $g \in \mathcal{G}(M), \psi \in \Sigma_g M$

We consider the Dirac-Einstein functional

$$\mathcal{E}_{\mathcal{B}}(g, \psi) = \int_M R_g + \langle D_g \psi, \psi \rangle - \lambda |\psi|^2 \, dv_g + \frac{1}{2} \int_{\partial M} h_g \, d\sigma_g$$

where h_g is the mean curvature of ∂M induced by g.

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Thanks for the attention

Dirac operator on $\mathbb{R}^n$

Let $(M, g) = (\mathbb{R}^n, g_E)$ equipped with the standard Euclidean metric. The spinor bundle is given by

$$\Sigma_{g_E}\mathbb{R}^n = \Sigma\mathbb{R}^n = \mathbb{R}^n \times \mathbb{C}^N, \quad N = 2^{[\frac{n}{2}]}$$

A spinor is a function $\psi : \mathbb{R}^n \to \mathbb{C}^N$

There exist n complex constant matrices σ_k, $N \times N$, satisfying

$$\sigma_k\sigma_j + \sigma_j\sigma_k = -2\delta_{kj}I$$

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Some examples.

$n = 1$. For $x \in \mathbb{R}$, a spinor is $\psi(x) = u(x) + iv(x) \in \mathbb{C}$, $\sigma_1 = i$, so

$$D\psi = i\partial_x\psi = -v' + iu', \quad D^2\psi = -\Delta\psi = -u'' - iv''$$

$n = 2$. For $(x, y) \in \mathbb{R}^2$, a spinor is $\psi(x, y) = \begin{pmatrix} \psi_1(x, y) \\ \psi_2(x, y) \end{pmatrix} \in \mathbb{C}^2$

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$n = 2$. For $(x, y) \in \mathbb{R}^2$, a spinor is $\psi(x, y) = \begin{pmatrix} \psi_1(x, y) \\ \psi_2(x, y) \end{pmatrix} \in \mathbb{C}^2$

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \quad \sigma_2 = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}$$

$$D\psi = \sigma_1 \begin{pmatrix} \partial_x \psi_1 \\ \partial_x \psi_2 \end{pmatrix} + \sigma_2 \begin{pmatrix} \partial_y \psi_1 \\ \partial_y \psi_2 \end{pmatrix} = \begin{pmatrix} \partial_x \psi_2 + i\partial_y \psi_2 \\ -\partial_x \psi_1 + i\partial_y \psi_1 \end{pmatrix}$$

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$n = 3$. For $(x, y, z) \in \mathbb{R}^3$, a spinor is $\psi(x,y,z) = \begin{pmatrix} \psi_1(x,y,z) \\ \psi_2(x,y,z) \end{pmatrix} \in \mathbb{C}^2$

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which are the classic Pauli matrices

$$D\psi = \sigma_1 \partial_x \psi + \sigma_2 \partial_y \psi + \sigma_3 \partial_z \psi = D_{xy}\psi + \begin{pmatrix} -D_z\psi_1 \\ D_z\psi_2 \end{pmatrix}$$

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Dirac-Einstein equations on $\mathbb{R}^3$

Consider $\mathbb{R}^3$ equipped with the standard Euclidean metric.

The conformal Dirac-Einstein functional is

$$E_{\mathbb{R}^3} : H^1(\mathbb{R}^3) \times H^{\frac{1}{2}}(\Sigma\mathbb{R}^3) \to \mathbb{R}$$

$$E_{\mathbb{R}^3}(U,\Psi) = \int_{\mathbb{R}^3} a_3|\nabla U|^2 + \langle D\Psi,\Psi\rangle - U^2|\Psi|^2 \, dx, \quad x \in \mathbb{R}^3, \; a_3 = 8$$

Critical points of $E_{\mathbb{R}^3}$ are solutions of

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Bubbles on $\mathbb{R}^3$

Let (U, Ψ) be a ground state solution of $(CDE_{\mathbb{R}^3})$, with $U \geq 0$.

Then there exist $\lambda > 0$, $x_0 \in \mathbb{R}^3$ and $\Psi_0 \in \mathbb{C}^2$, $|\Psi_0| = \frac{1}{\sqrt{2}}$ such that:

$$U(x) = U_{\lambda,x_0}(x) = \left(\frac{2\lambda}{\lambda^2 + |x - x_0|^2}\right)^{1/2}$$

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