Conformal metrics with prescribed curvatures

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The problem

The geometric problem

Let (M, g) be an $n-$dimensional compact Riemannian manifold.
Given smooth functions K and H, do there exist any conformal metric $\tilde{g}$ whose scalar curvature is K and mean curvature is H?

The PDEs formulation

- The conformal class of g is $[g] := \{\phi g : \phi \in C^\infty(M), \phi > 0\}$
- If $\tilde{g} = u^{\frac{4}{m-2}} g$, $u > 0$, (u is the *conformal factor*) then

$$\mathcal{S}_{\tilde{g}} = \left(-\frac{4(n-1)}{n-2} \Delta_g u + \mathcal{S}_g u \right) u^{-\frac{n+2}{n-2}}$$

  is the scalar curvature associated to $\tilde{g}$ and

$$h_{\tilde{g}} = \left(\frac{2}{n-2} \frac{\partial u}{\partial \nu} + h_g u \right) u^{-\frac{n}{n-2}}$$

  is the boundary mean curvature $h_{\tilde{g}}$ associated to the metric g
- Δ_g is the Laplace-Beltrami operator
- $\mathcal{S}_g$ is the scalar curvature associated to the metric g
- h_g is the boundary mean curvature associated to the metric g
- ν is the outward unit normal vector to ∂M.

An equivalent doubly critical problem

There exist a conformal metric $\tilde{g}$ whose scalar curvature is K and mean curvature is H

$$\Updownarrow$$

There exists a positive solution of

$$\begin{cases} -\Delta_g u + \dfrac{n-2}{4(n-1)}\mathcal{S}_g u = K u^{\frac{n+2}{n-2}} \text{ in } M \\ \dfrac{\partial u}{\partial \nu} + \dfrac{n-2}{2} h_g u = H u^{\frac{n}{n-2}} \text{ on } \partial M. \end{cases}$$

The elliptic problem is *doubly* critical

- $2^* = \frac{2n}{n-2}$ is the critical Sobolev exponent for the embedding $H^1_g(M) \hookrightarrow L^{\frac{2n}{n-2}}(M)$
- $2^\sharp = \frac{2(n-1)}{n-2}$ is the critical exponent for the trace embedding $H^1_g(M) \hookrightarrow L^{\frac{2(n-1)}{n-2}}(\partial M)$

Prescribing constant curvatures

Minimal boundary: $H = 0$ (Escobar, JDG 1992)

$$\begin{cases} -\Delta_g u + \frac{n-2}{4(n-1)} \mathcal{S}_g u = K u^{\frac{n+2}{n-2}} \text{ in } M \\ \frac{\partial u}{\partial \nu} + \frac{n-2}{2} h_g u = 0 \text{ on } \partial M \end{cases}$$

u is a critical point of the functional

$$Q(u) := \frac{\int_M \left(|\nabla_g u|_g^2 + \frac{n-2}{4(n-1)} \mathcal{S}_g u^2 \right) dx + \frac{n-2}{2} \int_{\partial M} h_g u^2 d\sigma}{\left(\int_M |u|^{\frac{2n}{n-2}} \right)^{\frac{n-2}{n}}}, \quad u \in H_g^1(M), \ u \neq 0$$

The infimum $I(M) := \inf_{\substack{u \in H_g^1(M) \\ u \neq 0}} Q(u)$ depends only on the conformal class of g

$I(M) < I(\mathbb{S}^n_+)^{(*)} \ \Rightarrow$ the infimum is achieved! (Aubin's argument)

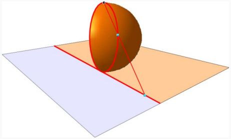
Find a test function $u \in H_g^1(M)$ such that

$$Q(u) < I(\mathbb{S}^n_+)$$

(*) $\mathbb{S}^n_+$ is the upper standard hemisphere.

The minimizers on the half-sphere

The upper standard half-sphere $\mathbb{S}^n_+$ is conformally equivalent to the half euclidean space $\mathbb{R}^n_+$ via the stereographic projection



The minimizers of $I(\mathbb{S}^n_+)$ are the *bubbles*

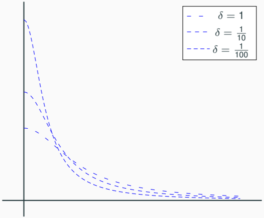
$$U_{\delta,z}(x) := \frac{1}{\delta^{\frac{n-2}{2}}} U\left(\frac{x-z}{\delta}\right), \ \delta > 0, \ z = (z', 0) \in \partial\mathbb{R}^n_+$$

with

$$U(x) := \frac{1}{(1+|x|^2)^{\frac{n-2}{2}}},$$

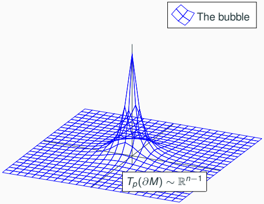
which are the solutions of

$$\begin{cases} -\Delta U = n(n-2)U^{\frac{n+2}{n-2}} \text{ in } \mathbb{R}^n_+ \\ \frac{\partial U}{\partial \nu} = 0 \text{ on } \partial\mathbb{R}^n_+. \end{cases}$$



The test functions on the manifold

We transplant the approximate extremal functions on the manifold via normal coordinates



The infimum is achieved when

- $n = 3, 4, 5$
- ∂M has a non-umbilic point $^{(*)}$
- ∂M is umbilic and M is locally conformally flat
- ∂M is umbilic and $n \geq 6$ and the Weyl tensor does not vanish identically on ∂M

$^{(*)}$ the boundary point is non-umbilic is the tensor $\pi - h_g g$ does not vanish where π is the second fundamental form

Scalar flat metric: $K = 0$ (Escobar, Annals Math.1992)

$$\begin{cases} -\Delta_g u + \frac{n-2}{4(n-1)} \mathcal{S}_g u = 0 \text{ in } M \\ \frac{\partial u}{\partial \nu} + \frac{n-2}{2} h_g u = H u^{\frac{n}{n-2}} \text{ on } \partial M. \end{cases}$$

u is a critical point of the functional

$$Q(u) := \frac{\int_M \left(|\nabla_g u|_g^2 + \frac{n-2}{4(n-1)} \mathcal{S}_g u^2 \right) dx + \frac{n-2}{2} \int_{\partial M} h_g u^2 d\sigma}{\left(\int_{\partial M} |u|^{\frac{2(n-1)}{n-2}} \right)^{\frac{n-2}{n-1}}}, \quad u \in H_g^1(M), \ u \neq 0$$

The infimum $I(M) := \inf_{\substack{u \in H_g^1(M) \\ u \neq 0}} Q(u)$ depends only on the conformal class of g

$I(M) < I(\mathbb{B}^n)^{(*)}$ $\Rightarrow$ the infimum is achieved! (Aubin's argument)

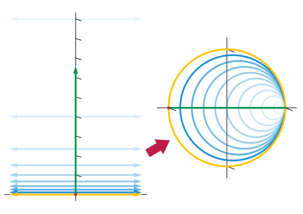
Find a test function $u \in H_g^1(M)$ such that

$$Q(u) < I(\mathbb{B}^n)$$

(*) $\mathbb{B}^n$ is the unit ball in $\mathbb{R}^n$ endowed with the euclidean metric.

The minimizers on the ball

The Euclidean ball $\mathbb{B}^n$ is conformally equivalent to the euclidean space $\mathbb{R}^n_+$ via the inversion map



The minimizers of $I(\mathbb{B}^n)$ are the *bubbles*

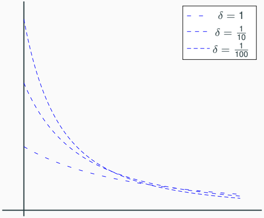
$$U_{\delta,z}(x) := \frac{1}{\delta^{\frac{n-2}{2}}} U\left(\frac{x-z}{\delta}\right), \ \delta > 0, \ z = (z', 0) \in \partial\mathbb{R}^n_+$$

with

$$U(x', x_n) := \frac{1}{\left((1+x_n)^2 + |x'|^2\right)^{\frac{n-2}{2}}}$$

which are the solutions of

$$\begin{cases} -\Delta U = 0 \text{ in } \mathbb{R}^n_+ \\ \frac{\partial U}{\partial \nu} = (n-2) U^{\frac{n}{n-2}} \text{ on } \partial\mathbb{R}^n_+. \end{cases}$$



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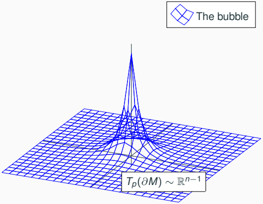
We transplant the approximate extremal functions on the manifold via normal coordinates



The infimum is achieved when

- $n = 3$
- $n = 4, 5$ and ∂M is umbilic
- $n \geq 6$, ∂M is umbilic and M is locally conformally flat
- $n \geq 7$ and ∂M has a non-umbilic point

Constant curvatures (Han & Li Duke Math J 1999)

$$\begin{cases} -\Delta_g u + \dfrac{n-2}{4(n-1)}\mathcal{S}_g u = n(n-2)Ku^{\frac{n+2}{n-2}} \text{ in } M \\ \dfrac{\partial u}{\partial \nu} + \dfrac{n-2}{2}h_g u = (n-2)Hu^{\frac{n}{n-2}} \text{ on } \partial M. \end{cases}$$

They approximate problem with sub-critical problems

$$\begin{cases} -\Delta_g u_\epsilon + \dfrac{n-2}{4(n-1)}\mathcal{S}_g u_\epsilon = n(n-2)Ku_\epsilon^{\frac{n+2}{n-2}-\epsilon} \text{ in } M \\ \dfrac{\partial u_\epsilon}{\partial \nu} + \dfrac{n-2}{2}h_g u_\epsilon = (n-2)Hu_\epsilon^{\frac{n}{n-2}-\epsilon} \text{ on } \partial M. \end{cases}$$

The set of positive solutions is compact and far away from zero (via a fine analysis of possible blow-up behavior of solutions) when

- $K = +1$
- M is of positive type (i.e. $\lambda_1(M) > 0$) $^{(*)}$
- M is locally conformally flat and ∂M is umbilic (and not conformally equivalent to the standard hemisphere)

- $K = -1$ and $H < 1$
- M is of negative type (i.e. $\lambda_1(M) < 0$) $^{(*)}$

$^{(*)}$ $\lambda_1(M) = \min\limits_{u \in H^1_g(M)\setminus\{0\}} \left(\int\limits_M \left(|\nabla_g u|_g^2 + \frac{n-2}{4(n-1)}\mathcal{S}_g u^2 \right) dx + \frac{n-2}{2} \int\limits_{\partial M} h_g u^2 d\sigma \right) \Big/ \int\limits_M |u|^2$

Constant curvatures (Han & Li Comm. An. Geo. 2000)

$$\begin{cases} -\Delta_g u + \dfrac{n-2}{4(n-1)}\mathcal{S}_g u = n(n-2)u^{\frac{n+2}{n-2}} & \text{in } M \\ \dfrac{\partial u}{\partial \nu} + \dfrac{n-2}{2} h_g u = (n-2)Hu^{\frac{n}{n-2}} & \text{on } \partial M. \end{cases}$$

u is a critical point of the functional

$$J(u) := \frac{1}{2}\left(\int_M \left(|\nabla_g u|_g^2 + \frac{n-2}{4(n-1)}\mathcal{S}_g u^2\right) dx + \frac{n-2}{2}\int_{\partial M} h_g u^2 d\sigma\right) - \frac{(n-2)^2}{2}\int_M |u|^{\frac{2n}{n-2}} - \frac{n-2}{2(n-1)}H\int_{\partial M} |u|^{\frac{2(n-1)}{n-2}}$$

There exists a nontrivial critical point via the Mountain Pass Lemma of Ambrosetti and Rabinowitz:

- The geometry holds true: $\lim_{t\to+\infty} J(tu) = -\infty$ if $u \not\equiv 0$ and if M is of positive type then $J(u) > 0$ if $\|u\| = r_0$ is small
- The Palais-Smale condition holds true below a threshold value S_H

The threshold value S_H

$$S_H := (n-2)\left(\int_{\mathbb{R}^n_+} U^{\frac{2n}{n-2}} + \frac{1}{2(n-1)} H \int_{\partial\mathbb{R}^n_+} U^{\frac{2(n-1)}{n-2}}\right)$$

The threshold is given in terms of the solution to the extremal problem

$$\begin{cases} -\Delta U = n(n-2)U^{\frac{n}{n-2}} & \text{in } \mathbb{R}^n_+ \\ \dfrac{\partial U}{\partial \nu} = (n-2)HU^{\frac{n}{n-2}} & \text{on } \partial\mathbb{R}^n_+. \end{cases}$$

whose solutions are the bubbles

$$U_{\delta,z}(x) := \frac{1}{\delta^{\frac{n-2}{2}}} U\left(\frac{x-z}{\delta}\right), \ \delta > 0, \ z = (z', 0) \in \partial\mathbb{R}^n_+$$

where

$$U(x', x_n) = \frac{1}{\left(1 + |x'|^2 + |x_n + H|^2\right)^{\frac{n-2}{2}}}$$

They transplant this extremal functions on the manifold via normal coordinates to build the test function u such that iif $n \geq 5$ and ∂M has a non-umbilic point satisfies

$$\max_{0<t<\infty} J(tu) < S_H$$

and this ensures the existence of the Mountain Pass critical point!

Variable curvatures

Some partial results

- Minimal boundary (i.e. $H = 0$) on the half-sphere
  - M. Ben Ayed, K. El Mehdi & M. O. Ahmedou (2002, 2005, 2021)
  - Y.Y. Li (1995)
- Scalar flate metric (i.e. $K = 0$) on the euclidean ball
  - W. Abdelhedi, H. Chtioui & M. O. Ahmedou (2008)
  - Z. Djadli, A. Malchiodi & M. O. Ahmedou (2004)
  - X. Xu & H. Zhang (2016)
  - S.-Y. A. Chang, X. Xu & P. C. Yang (1998)
- If both K and H do not vanish
  - A. Ambrosetti, Y. Li & A. Malchiodi (2002) in a perturbative setting on the euclidean ball
  - Z. Djadli, A. Malchiodi & M. O. Ahmedou (2003) on the three-dimensional half-sphere.
  - X. Chen, P. T. Ho & L. Sun (2018) when H and K are negative functions and the boundary has negative Yamabe invariant.

Cruz-Blázquez, Malchiodi & Ruiz, CRELLE 2022

$$\begin{cases} -\Delta_g u + \frac{n-2}{4(n-1)} \mathcal{S}_g u = n(n-2) K u^{\frac{n+2}{n-2}} \text{ in } M \\ \frac{\partial u}{\partial \nu} = (n-2) H u^{\frac{n}{n-2}} \text{ in } \partial M. \end{cases} \quad (*)$$

u is a critical point of

$$F(u) := \frac{1}{2} \int_M \left(|\nabla_g u|_g^2 + \frac{n-2}{4(n-1)} \mathcal{S}_g u^2 \right) - \frac{(n-2)^2}{2} \int_M K |u|^{\frac{2n}{n-2}} - \frac{(n-2)^2}{2(n-1)} \int_{\partial M} H |u|^{\frac{2(n-1)}{n-2}}$$

F is coercive and has a global minimum point if

$$\mathcal{S}_g < 0, \quad \max_{p \in M} K(p) < 0 \quad \text{and} \quad \max_{p \in \partial M} \mathfrak{D}(p) < 1, \quad \mathfrak{D}(p) = \frac{H(p)}{\sqrt{|K(p)|}}$$

(*) Via the conformal change of metric due to Escobar, one can assume that the mean curvature $h_g = 0$ and $\mathcal{S}_g$ has constant sign.

What happens when $\mathfrak{D}(p) > 1$ somewhere on the boundary?

F has a mountain pass critical point if $n = 3$ and

$$\mathcal{S}_g = 0, \quad \max_{p\in M} K(p) < 0, \int_{\partial M} H(p) d\sigma < 0 \quad \text{and} \quad \max_{p\in\partial M} \mathfrak{D}(p) > 1, \quad \mathfrak{D}(p) = \frac{H(p)}{\sqrt{|K(p)|}}$$

- If there exists a point $p \in \partial M$ with $\mathfrak{D}(p) > 1$, the functional F is not bounded from below anymore and the geometry of the mountain-pass is satisfied!
- To show that Palais-Smale sequences of approximate solutions converge, they have to prove that the solutions of suitable sub-critical approximating problems

$$\begin{cases} -\Delta_g u_\epsilon + \dfrac{n-2}{4(n-1)} \mathcal{S}_g u_\epsilon = K_\epsilon u^{\frac{n+2}{n-2}-\epsilon} \text{ in } M \\ \dfrac{\partial u_\epsilon}{\partial \nu} = H_\epsilon u^{\frac{n}{n-2}-\epsilon} \text{ on } \partial M. \end{cases}$$

where

$$K_\epsilon \to K \text{ in } C^2(M) \quad \text{and} \quad H_\epsilon \to H \text{ in } C^2(\partial M).$$

are uniformly bounded.
As a consequence, they will converge to a solution of the original problem.

The blow-up analysis

$$\begin{cases} -\Delta_g u_\epsilon + \dfrac{n-2}{4(n-1)}\mathcal{S}_g u_\epsilon = K_\epsilon u^{\frac{n+2}{n-2}-\epsilon} \text{ in } M \\ \dfrac{\partial u_\epsilon}{\partial \nu} = H_\epsilon u^{\frac{n}{n-2}-\epsilon} \text{ on } \partial M. \end{cases}$$

Let

$$\mathscr{B} := \{p \in M \;:\; \text{there exists } p_\epsilon \to p \text{ and } u_\epsilon(p_\epsilon) \to +\infty\}$$

For any $n \geq 3$

$$\mathscr{B} \subset \left\{ p \in \partial M \;:\; \mathfrak{D}(p) := \frac{H(p)}{\sqrt{|K(p)|}} \geq 1 \right\}$$

$n = 3 \quad \Rightarrow \quad \mathscr{B} = \emptyset \quad \Rightarrow \quad$ The Palais-Smale condition holds true!

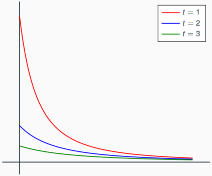
The blow-up profiles

There are two types of blow-up profiles, which are solutions of the following problem in the half-space: (∗)

$$\begin{cases} -\Delta u = n(n-2)K(p)u^{\frac{n+2}{n-2}} \text{ in } \mathbb{R}^n_+ & \text{(with } K(p) < 0) \\ \dfrac{\partial u}{\partial \nu} = (n-2)H(p)u^{\frac{n}{n-2}} \text{ on } \mathbb{R}^n_+ & \end{cases}$$

- if $\mathfrak{D}(p) = \frac{H(p)}{\sqrt{|K(p)|}} < 1$ there are no solutions
- if $\mathfrak{D}(p) = \frac{H(p)}{\sqrt{|K(p)|}} = 1$ the only solutions are one-dimensional and given by

$$U_t(x_n) = \frac{1}{(2H(p)x_n + t)^{\frac{n-2}{2}}}, \ t > 0$$



(∗) The solutions have been classified by Y. Li & M. Zhu (1995) and M. Chipot, I. Shafrir & M. Fila (1996)

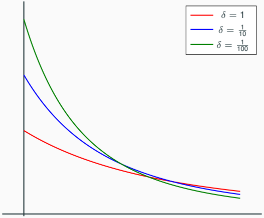
$$\begin{cases} -\Delta u = n(n-2)K(p)u^{\frac{n+2}{n-2}} \text{ in } \mathbb{R}^n_+ & \text{(with } K(p) < 0\text{)} \\ \dfrac{\partial u}{\partial \nu} = (n-2)H(p)u^{\frac{n}{n-2}} \text{ on } \mathbb{R}^n_+ \end{cases}$$

- if $\mathfrak{D}(p) = \frac{H(p)}{\sqrt{|K(p)|}} > 1$ the solutions are the bubbles

$$U_{\delta,z}(x) = \frac{1}{\delta^{\frac{n-2}{2}}} U\left(\frac{x-z}{\delta}\right), \ \delta > 0, \ z \in \partial\mathbb{R}^n_+$$

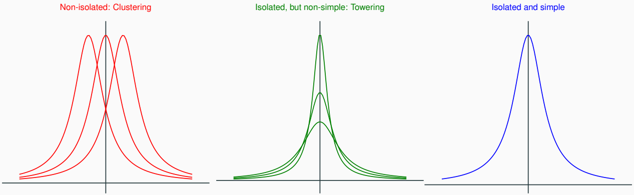
where

$$U(x', x_n) = \frac{1}{|K(p)|^{\frac{n-2}{4}}} \frac{1}{(|x'|^2 + |x_n + \mathfrak{D}(p)|^2 - 1)^{\frac{n-2}{2}}}$$



The key point in the blow-up analysis and a natural question

If $n = 3$ all the blow-up points are isolated and simple!



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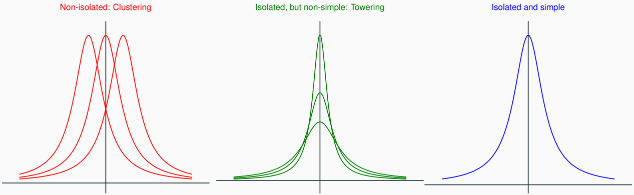

In higher dimensions $n \geq 4$, are the blow-up points still isolated and simple?

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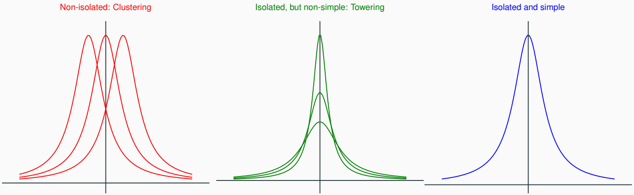
If $n = 3$ all the blow-up points are isolated and simple!



In higher dimensions $n \geq 4$, are the blow-up points still isolated and simple?

A partial answer

If $4 \leq n \leq 7$ a clustering blow-up point exists!

Our main result

Theorem (Cruz-Blázquez, Pistoia & Vaira 2022)

We consider the perturbed problem$^{(*)}$

$$\begin{cases} -\Delta_g u + \dfrac{n-2}{4(n-1)}\mathcal{S}_g u = n(n-2)K u^{\frac{n+2}{n-2}} & \text{in } M \\ \dfrac{\partial u}{\partial \nu} + \epsilon u = (n-2)H u^{\frac{n}{n-2}} & \text{on } \partial M \end{cases}$$

where ϵ is a small and positive parameter.
Assume

(I) $4 \leq n \leq 7$ and $\mathcal{S}_g > 0$,

(II) $H > 0$ and $K < 0$ are constant functions such that $\mathfrak{D} := \frac{H}{\sqrt{|K|}} > 1$,

(III) $p \in \partial M$ is a non-degenerate minimum point of $p \to \|\pi_g(p)\|^2$ with $\pi_g(p) \neq 0$ (i.e. p is non-umbilic) . $^{(**)}$

Then p is a clustering blow-up point, i.e.

for any $k \in \mathbb{N}$, there exist k points $p^1_\epsilon, \dots, p^k_\epsilon \in \partial M$ for and $\epsilon_k > 0$ such that for all $\epsilon \in (0, \epsilon_k)$ the problem has a solution u_ϵ with k positive peaks at p^j_ϵ and $p^j_\epsilon \to p$ as $\epsilon \to 0$.

$^{(*)}$ We choose Escobar's metric so that $h_g = 0$

$^{(**)}$ π_g is the second fundamental form of the boundary.

Some remarks

- The function $p \to \|\pi_g(p)\|^2$ is a Morse function for generic metrics g with $h_g = 0$ (Cruz & Pistoia 2022)
- The clustering blow-up point exists if $4 \leq n \leq 7$. If $n \geq 8$ our construction does not work. Could it be refined?
- We build a family of solutions with a clustering blow-up point. It is also possible to find solutions with a towering blow-up point. This is a work in progress by Cruz-Blázquez.

The proof

- The proof of our results relies on the finite-dimensional Ljapunov-Schmidt reduction method.
    - The first step is finding a good ansatz
    - The second step is studying the linear theory
    - The third step concerns the study of the reduced problem

The ansatz

- The building blocks of our construction are the bubbles

$$U_{\delta,z}(x) := \frac{1}{\delta^{\frac{n-2}{2}}} U\left(\frac{x-z}{\delta}\right), \ \delta < 0, \ z \in \partial\mathbb{R}^{n-1}$$

where

$$U(x) := \frac{1}{|K|^{\frac{n-2}{4}}} \frac{1}{(|\tilde{x}|^2 + (x_n + \mathfrak{D})^2 - 1)^{\frac{n-2}{2}}} \text{ with } \mathfrak{D} = \frac{H}{\sqrt{|K|}} > 1$$

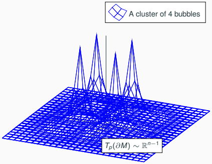
which solve the limit problem

$$\begin{cases} -\Delta u = n(n-2)Ku^{\frac{n+2}{n-2}} \text{ in } \mathbb{R}^n_+ \\ \dfrac{\partial u}{\partial \nu} = (n-2)Hu^{\frac{n}{n-2}} \text{ on } \partial\mathbb{R}^n_+ \end{cases}$$

The ansatz is the sum of k positive bubbles which concentrate at the same boundary point p with the same speeds, i.e. in local coordinates around p

$$W(x) = \sum_{j=1}^{k} \frac{1}{\delta_j^{\frac{n-2}{2}}} U\left(\frac{x-z_j}{\delta_j}\right)$$

- all concentration parameters δ_j have the same speed with respect to ϵ
- all the concentration points z_j collapse to 0 as $\epsilon \to 0$



A refinement of the ansatz

- Unfortunately this first approximation is not as good as one can expect and we need to refine as
$$W(x) = \sum_{j=1}^{k} \frac{1}{\delta_j^{\frac{n-2}{2}}} U\left(\frac{x-z_j}{\delta_j}\right) + \sum_{j=1}^{k} \frac{1}{\delta_j^{\frac{n-4}{2}}} V\left(\frac{x-z_j}{\delta_j}\right)$$
where the new function $V : \mathbb{R}^n_+ \to \mathbb{R}$ solves the linear problem $^{(*)}$
$$\begin{cases} -\Delta V - (n+2)KU^{\frac{4}{n-2}} V = c_n \sum_{i,j=1}^{n-1} h^{ij}(p) x_n \frac{\partial^2 U(x)}{\partial x_i \partial x_j} & \text{in } \mathbb{R}^n_+, \\ \frac{\partial V}{\partial \nu} - nHU^{\frac{2}{n-2}} V = 0 & \text{on } \partial\mathbb{R}^n_+, \end{cases}$$

How to solve the linear problem?

$^{(*)}$ $h^{ij}(p)$ are the coefficients of the second fundamental form of M at the point $p \in \partial M$

The linear theory

The bubbles are non-degenerate$^{(*)}$, i.e. all the solution of the linearized problem

$$\begin{cases} -\Delta\varphi - (n+2)KU^{\frac{4}{n-2}}\varphi = 0 \text{ in } \mathbb{R}^n_+, \\ \dfrac{\partial\varphi}{\partial\nu} - nHU^{\frac{2}{n-2}}\varphi = 0 \text{ on } \partial\mathbb{R}^n_+, \end{cases}$$

are a linear combination of the functions

$$\varphi_1(x) := \frac{\partial U}{\partial x_1}(x), \ \ldots, \ \varphi_{n-1}(x) := \frac{\partial U}{\partial x_{n-1}}(x)$$

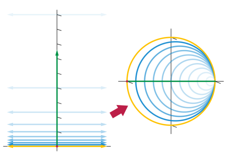
and

$$\varphi_n(x) := \frac{n-n}{2}U(x) + \nabla U(x)\cdot x$$

$^{(*)}$ The case $K=0$ by Almaraz 2011. The case $K>0$ by Han-Li 2000. Our case $K<0$ was unknown.

An equivalent linear problem on the hyperbolic ball

The half-space $\mathbb{R}^n_+$ is conformally equivalent to the hyperbolic ball $(\mathbb{B}^n_\rho, g_{\mathbb{H}})$ with radius $\rho := \mathfrak{D} - \sqrt{\mathfrak{D}^2 - 1} < 1$ and metric $g_{\mathbb{H}} := \frac{4}{1-|x|^2}|dx|^2$



Therefore, the linear problem on the euclidean halfspace

$$\begin{cases} -\Delta\varphi - (n+2)KU^{\frac{4}{n-2}}\varphi = 0 \text{ in } \mathbb{R}^n_+, \\ \dfrac{\partial\varphi}{\partial\nu} - nHU^{\frac{2}{n-2}}\varphi = 0 \text{ on } \partial\mathbb{R}^n_+, \end{cases}$$

is equivalent to the linear problem on the hyperbolic ball

$$\begin{cases} \Delta_{\mathbb{H}}\hat{\varphi} - n\hat{\varphi} = 0 \text{ in } \mathbb{B}^n_\rho \\ \dfrac{\partial\hat{\varphi}}{\partial\nu_{\mathbb{H}}} - \mathfrak{D}\hat{\varphi} = 0 \text{ on } \partial\mathbb{B}^n_\rho \end{cases}$$

The Neumann eigenvalue problem on the hyperbolic ball

The Neumann boundary problem

$$\begin{cases} \Delta_{\mathbb{H}}\phi - n\phi = 0 \text{ in } \mathbb{B}^n_\rho \\ \dfrac{\partial \phi}{\partial \nu_{\mathbb{H}}} = \Lambda\phi \text{ on } \mathbb{B}^n_\rho \end{cases}$$

has

- a first simple eigenvalue $\Lambda_0 = \frac{2\rho}{1+\rho^2} = \frac{1}{\mathfrak{D}}$ with eigenfunction $\phi_0(x) = \frac{1+|x|^2}{1-|x|^2}$
- a second eigenvalue $\Lambda_1 = \frac{1+\rho^2}{2\rho} = \mathfrak{D}$ with an $n-$dimensional eigenspace

$$\left\{\phi_1^i(x) = \frac{|x|x_i}{1-|x|^2},\ i = 1, \dots, n\right\}.$$

Proof.

- We introduce the new variable $t = \ln \frac{1+|x|}{1-|x|}$ with $0 < t < T = \ln \frac{1+\rho}{1-\rho}$ and we decompose the solution in spherical harmonics
- We have to study the boundary value problem

$$\begin{cases} \gamma_i'' + (n-1)(\coth t)\,\gamma_i' - \left(\dfrac{i(i+n-2)}{\sinh^2 t} + n\right)\gamma_i = 0 \text{ if } 0 < t < T, \\ \gamma_i'(T) - \Lambda\gamma_i(T) = 0 \end{cases}$$

  - If $i = 0$, then $\Lambda_0 := \tanh T$ and the bounded solutions are multiple of $\gamma_0(t) = \cosh t$
  - If $i = 1$, then $\Lambda_1 := (\tanh T)^{-1}$ and the bounded solutions are multiple of $\gamma_1(t) = \sinh t$
  - If $i \geq 2$, then $\Lambda \geq \Lambda_1$ and the linear ODE does not have any bounded solutions.

The reduced problem

The function

$$u(x) \sim W_{\delta,\mathbf{z}}(x) = \sum_{j=1}^{k} \left(\frac{1}{\delta_j^{\frac{n-2}{2}}} U\left(\frac{x - z_j}{\delta_j}\right) + \frac{1}{\delta_j^{\frac{n-4}{2}}} V\left(\frac{x - z_j}{\delta_j}\right) \right)$$

solves the problem

$$\Updownarrow$$

The concentration parameters $\boldsymbol{\delta} := (\delta_1, \ldots, \delta_k)$ and the concentration points $\mathbf{z} := (z_1, \ldots, z_k)$ are a critical point of the reduced energy

$$\widetilde{F}(\boldsymbol{\delta}, \mathbf{z}) := F\left(W_{\boldsymbol{\delta},\mathbf{z}}\right)$$

where

$$F(u) := \frac{1}{2} \int_M \left(|\nabla_g u|_g^2 + \frac{n-2}{4(n-1)} \mathcal{S}_g u^2 \right) - \frac{(n-2)^2}{2} K \int_M |u|^{\frac{2n}{n-2}}$$
$$- \frac{(n-2)^2}{2(n-1)} H \int_{\partial M} |u|^{\frac{2(n-1)}{n-2}} - \frac{\epsilon}{2} \int_{\partial M} u^2$$

(Note that critical points of F are solutions of the problem!)

The reduced energy looks like

$$\widetilde{F}(\delta,\mathbf{z}) = \underbrace{\sum_{j=1}^{k}\left(\delta_j^2-\epsilon\delta_j\right)\|\pi(p)\|^2}_{\text{the contribution of each peak } z_j} + \underbrace{\sum_{j=1}^{k}\delta_j^2\mathcal{Q}(p)(z_i,z_j)+\sum_{\substack{i,j=1\\ i\neq j}}^{k}\frac{(\delta_i\delta_j)^{\frac{n-2}{2}}}{|z_i-z_j|^{n-2}}}_{\text{the interaction between different peaks } z_i \text{ and } z_j} + h.o.t.$$

where

- $\mathcal{Q}(p)$ is the quadratic form associated with the second derivative of $p \to \|\pi(p)\|^2$ at the point p which is positively definite because p is non-degenerate minimum point $p \to \|\pi(p)\|^2$
- the "h.o.t." term is an higher order term in low dimensions $4 \leq n \leq 7$

The reduced energy has a minimum point!

- If $n = 5, 6, 7$ we choose

$$\delta_j \sim \epsilon \quad \text{and} \quad |z_j| \sim \zeta \quad \text{with} \quad \epsilon^2\zeta^2 \sim \frac{\epsilon^{n-2}}{\zeta^{n-2}}$$

  and we can minimize the reduced energy.
- If $n = 4$ the expansion of the reduced energy is more involved, but a minimum point still exists!

A related result

Prescribing nearly constant curvatures on balls

Let us consider the problems

$$(P^2_\epsilon) \quad \begin{cases} -\Delta u = 2(-1+\epsilon K)e^u \text{ in } \mathbb{B}^2 \\ \partial_\nu v + 4 = 2\mathfrak{D}(1+\epsilon H)e^{\frac{v}{2}} \text{ on } \partial\mathbb{B}^2. \end{cases}$$

and if $n \geq 3$

$$(P^n_\epsilon) \quad \begin{cases} -\Delta u = n(n-2)(-1+\epsilon K)u^{\frac{n+2}{n-2}} \text{ in } \mathbb{B}^n, \\ \frac{\partial u}{\partial \nu} + (n-2)u = (n-2)\mathfrak{D}(1+\epsilon H)u^{\frac{n}{n-2}} \text{ in } \partial\mathbb{B}^n \end{cases}$$

where

- $\mathfrak{D} > 1$
- $K \in C^1(\mathbb{B}^n)$, $H \in C^1(\partial\mathbb{B}^n)$
- $\epsilon \in \mathbb{R}$ is small

Problems share many similarities, not only for their geometric importance, but also from an analytic point of view: they both have critical terms in the interior and in the boundary nonlinearities

- in (P^2_ϵ) the exponential nonlinearities are critical in view of the Moser-Trudinger inequalities,
- in (P^n_ϵ) we have the critical Sobolev exponent and the critical trace exponent.

The case $n = 2$ (Battaglia, Cruz-Blázquez & Pistoia 2023)

Let

$$\psi(\xi) := K(\xi) - \mathfrak{c}H(\xi), \ \mathfrak{c} = \frac{2\mathfrak{D}}{\mathfrak{D} - \sqrt{\mathfrak{D}^2 - 1}} > 0$$

If $\mathfrak{D} \neq \frac{2}{\sqrt{3}}$ and one of the following holds true:

1. $\partial_\nu K(\xi) - \mathfrak{c}(-\Delta)^{\frac{1}{2}} H(\xi) < 0$ at any global maximum ξ of ψ;
2. $\partial_\nu K(\xi) - \mathfrak{c}(-\Delta)^{\frac{1}{2}} H(\xi) > 0$ at any global minimum ξ of ψ;
3. $\partial_\nu K(\xi) - \mathfrak{c}(-\Delta)^{\frac{1}{2}} H(\xi) \neq 0$ at any critical point ξ of ψ, ψ is Morse and

$$\sum_{\{\xi : \nabla\psi(\xi)=0,\, \partial_\nu K(\xi) - \mathfrak{c}(-\Delta)^{\frac{1}{2}} H(\xi) < 0\}} (-1)^{\mathrm{ind}_\xi \nabla\psi} \neq 1;$$

then, the problem

$$(P_\epsilon^2) \quad \begin{cases} -\Delta u = -2(1 + \epsilon K)e^u \text{ in } \mathbb{B}^2 \\ \partial_\nu v + 4 = 2\mathfrak{D}(1 + \epsilon H)e^{\frac{v}{2}} \text{ on } \partial\mathbb{B}^2. \end{cases}$$

has a solution for ϵ small enough.

The case $n \geq 3$ (Battaglia, Cruz-Blázquez & Pistoia 2023)

Let

$$\psi(\xi) := K(\xi) - \mathfrak{c}H(\xi), \ \mathfrak{c} = \mathfrak{c}(n, \mathfrak{D}) > 0$$

If one of the following holds true:

1. $\partial_\nu K(\xi) < 0$ at any global maximum ξ of ψ;
2. $\partial_\nu K(\xi) > 0$ at any global minimum ξ of ψ;
3. $\partial_\nu K(\xi) \neq 0$ at any critical point ξ of ψ, ψ is Morse and

$$\sum_{\{\xi : \nabla\psi(\xi)=0, \, \partial_\nu K(\xi)<0\}} (-1)^{\mathtt{ind}_\xi \nabla\psi} \neq 1;$$

then, the problem

$$(P_\epsilon^n) \quad \begin{cases} -\Delta u = n(n-2)(-1+\epsilon K)u^{\frac{n+2}{n-2}} \text{ in } \mathbb{B}^n, \\ \dfrac{\partial u}{\partial \nu} + (n-2)u = (n-2)\mathfrak{D}(1+\epsilon H)u^{\frac{n}{n-2}} \text{ in } \partial\mathbb{B}^n \end{cases}$$

has a solution for ϵ small enough.

Comments

- Battaglia, Cozzi, Fernandez & Pistoia 2022 studied the case of zero curvature in the interior of the disk, namely
$$\begin{cases} -\Delta u = 0 \text{ in } \mathbb{B}^2 \\ \partial_\nu v + 4 = 2(1+\epsilon H)e^{\frac{v}{2}} \text{ on } \partial\mathbb{B}^2. \end{cases}$$
- A. Chang, X. Xu & P.Yang 1998 studied the case of zero curvature in the interior of the unit ball ($n \geq 3$)
$$\begin{cases} -\Delta u = 0 \text{ in } \mathbb{B}^n, \\ \partial\nu u + (n-2)u = (n-2)(1+\epsilon H)u^{\frac{n}{n-2}} \text{ in } \partial\mathbb{B}^n \end{cases}$$
- Ambrosetti, Li & Malchiodi 2002 perturbed the positive constant curvature on the unit ball, namely
$$\begin{cases} -\Delta u = n(n-2)(1+\epsilon K)u^{\frac{n+2}{n-2}} \text{ in } \mathbb{B}^n, \\ \partial\nu u + (n-2)u = (n-2)(c+\epsilon H)u^{\frac{n}{n-2}} \text{ in } \partial\mathbb{B}^n \end{cases}$$

An idea of the proof

- The proof relies on a perturbation argument introduced by Ambrosetti, Garcia Azorero & Peral 1999 who found a metric whose scalar curvature is a perturbation of the constant on the n-dimensional sphere, namely studied the perturbed problem

$$-\Delta u = (1 + \epsilon\kappa(x))u^{\frac{n+2}{n-2}} \text{ in } \mathbb{R}^n,\ n \geq 3$$

- The same idea was used by Grossi & Prashant 2005, who studied the problem on the $2-$dimensional sphere, namely

$$-\Delta u = (1 + \epsilon\kappa(x))e^u \text{ in } \mathbb{R}^2$$

- The main ingredient is non-degeneracy of the bubbles.

The non-degeneracy of the bubbles

If $\mathfrak{D} > 1$ the unperturbed problem

$$\begin{cases} \Delta u = 2e^{u} \text{ in } \mathbb{B}^2 \\ \partial_\nu v + 2 = 2\mathfrak{D}\, e^{\frac{v}{2}} \text{ on } \partial\mathbb{B}^2. \end{cases}$$

is conformally equivalent

$$\begin{cases} \Delta u = 2e^{u} \text{ in } \mathbb{R}^2_+, \\ \partial_\nu u = 2\mathfrak{D}\, e^{\frac{u}{2}} \text{ on } \partial\mathbb{R}^2_+, \end{cases}$$

whose solutions are the bubbles

$$U_{\delta,\xi}(x,t) := 2\ln \frac{2\delta}{(x-\xi)^2 + (t+\delta\mathfrak{D})^2 - \delta^2}, \quad \delta > 0,\ \xi \in \mathbb{R}$$

which are non-degenerate (*), i.e. the solutions of the linear problem

$$\begin{cases} \Delta v = 2e^{U_{\delta,\xi}}\, v \text{ in } \mathbb{R}^2_+, \\ \partial_\nu v = \mathfrak{D}\, e^{\frac{U_{\delta,\xi}}{2}}\, v \text{ on } \partial\mathbb{R}^2_+, \end{cases}$$

are linear combination of

$$z^0_{\delta,\xi} = \partial_\delta U_{\delta,\xi} \text{ and } z^1_{\delta,\xi} = \partial_{\xi_1} U_{\delta,\xi}$$

(*) Jevnikar, Lopez-Soriano, Medina & Ruiz 2022

If $\mathfrak{D} > 1$ the unperturbed problem

$$\begin{cases} \Delta u = n(n-2)u^{\frac{n+2}{n-2}} \text{ in } \mathbb{B}^n, \\ \partial_\nu u + (n-2)u = (n-2)\mathfrak{D}\, u^{\frac{n}{n-2}} \text{ in } \partial\mathbb{B}^n \end{cases}$$

is conformally equivalent

$$\begin{cases} \Delta u = n(n-2)u^{\frac{n+2}{n-2}} \text{ in } \mathbb{R}^n_+, \\ \partial_\nu u = (n-2)\mathfrak{D}\, u^{\frac{n}{n-2}} \text{ in } \partial\mathbb{R}^n_+ \end{cases}$$

whose solutions are the bubbles

$$U_{\delta,\xi}(x,t) := \frac{\delta^{\frac{n-2}{2}}}{\left(|x-\xi|^2 + (t+\delta\mathfrak{D})^2 - \delta^2\right)^{\frac{n-2}{2}}}, \quad \delta > 0,\ \xi \in \mathbb{R}^{n-1}$$

which are non-degenerate (*), i.e. the solutions of the linear problem

$$\begin{cases} \Delta v = n(n+2)U_{\delta,\xi}^{\frac{4}{n-2}}\, v \text{ in } \mathbb{R}^n_+, \\ \partial_\nu v = n\mathfrak{D}\, U_{\delta,\xi}^{\frac{2}{n-2}}\, v \text{ on } \partial\mathbb{R}^n_+, \end{cases}$$

are linear combination of

$$z^0_{\delta,\xi} = \partial_\delta U_{\delta,\xi} \text{ and } z^i_{\delta,\xi} = \partial_{\xi_i} U_{\delta,\xi},\quad i = 1,\dots,n-1$$

(*) Cruz-Blázquez, Pistoia & Vaira 2022

THANK YOU FOR YOUR ATTENTION.