Compactness of singular constant Q-curvature metrics on punctured spheres

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joint work with

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Setting

We are interested in singular Yamabe metrics with constant Q-curvature of order 2σ on a finitely punctured sphere. These are metrics of the form $g = U^{\frac{4}{n-2\sigma}} g_0$ on $\mathbf{S}^n \backslash \Lambda$, where $\Lambda = \{p_1, \ldots, p_k\}$, with $Q_g^{2\sigma}$ constant. This curvature condition is equivalent to the equation

$$P_{g_0}^{2\sigma} U = c_{n,\sigma} U^{\frac{n+2\sigma}{n-2\sigma}}, \qquad \liminf_{x \to p_i} U(x) = \infty. \tag{1}$$

One can formulate and study this problem for $n \geq 3$ and $0 < \sigma < n/2$, most famously when $\sigma = 1$. I'll state some results we can prove when (i) $n \geq 5$ and $\sigma = 2$ and (ii) when $n \geq 7$ and $\sigma = 3$.

Moduli spaces

We define the marked moduli spaces

$$\mathcal{M}^4_\Lambda = \left\{ g \in [g_0] : Q^4_g = \frac{n(n^2-4)}{8}, g \text{ complete on } \mathbf{S}^n \backslash \Lambda \right\}$$

and the unmarked moduli spaces

$$\begin{aligned} \mathcal{M}^4_k &= \left\{ g \in [g_0] : Q^4_g = \frac{n(n^2-4)}{8}, \right. \\ & \qquad \left. g \text{ complete on } \mathbf{S}^n \backslash \Lambda \text{ with } \#\Lambda = k \right\}. \end{aligned}$$

We give both moduli spaces the Gromov-Hausdorff topology.
One can similary define the marked moduli space $\mathcal{M}^6_\Lambda$ and the unmarked moduli space $\mathcal{M}^6_k$ in the sixth order setting.

Main theorems, first statement

We give a rough statement of our main theorems.

Theorem (Andrade, do Ò, –)

Let $k \geq 3$ and let δ_1, δ_2 be two positive numbers. Let $\{g_i\} \subset \mathcal{M}_k^4$ be a sequence of singular Yamabe metrics with k punctures. If the distance with respect to the round metric between the punctures is bounded below by δ_1 and a certain geometric quantity to be explained later is bounded below by δ_2 then a subsequence of $\{g_i\}$ converges uniformly on compact subsets.

Theorem (Andrade, do Ò, –, Wei)

The theorem above also holds for the moduli space $\mathcal{M}_k^6$.

Delauany metrics

We describe the Delaunay solutions of the singular Yamabe problem. These are solutions with two punctures, and after stereographic projection we can rephrase (1) as

$$u : \mathbf{R}^n \backslash \{0\} \to (0, \infty), \qquad (-\Delta_0)^\sigma u = c_{n,2} u^{\frac{n+2\sigma}{n-2\sigma}}. \tag{2}$$

The functions u and U are related by

$$u = U u_{\mathrm{sph}}, \qquad u_{\mathrm{sph}}(x) = \left(\frac{1 + |x|^2}{2} \right)^{\frac{2\sigma - n}{2}}.$$

We can more easily describe these solutions after the Emden-Fowler change of coordinates:

$$t = -\log|x|, \quad \theta = \frac{x}{|x|}, \quad v(t, \theta) = e^{\left(\frac{2\sigma - n}{2}\right)t} u(e^{-t}\theta). \tag{3}$$

We state the remaining formulas for the Delaunay metrics only in the case of $\sigma = 2$; one can describe Delaunay metrics for any admissible σ, but their expressions become more complicated.

The change of variables (3) tranforms (2) to

$$v : \mathbf{R} \times \mathbf{S}^{n-1} \to (0, \infty), \qquad P^4_{\mathrm{cyl}}(v) = c_{n,2} v^{\frac{n+4}{n-4}}, \tag{4}$$

where

$$\begin{aligned} P^4_{\mathrm{cyl}} &= \partial_t^4 + \Delta_\theta^2 + 2\Delta_\theta \partial_t^2 - \left(\frac{n(n-4)}{2} \right) \Delta_\theta && (5) \\ &\quad - \left(\frac{n(n-4)+8}{2} \right) \partial_t^2 + \frac{n^2(n^2-4)}{16}. && (6) \end{aligned}$$

A theorem of C. S. Lin states all solutions on $\mathbf{R} \times \mathbf{S}^{n-1}$ are rotationally invariant, reducing this PDE to an ODE (in the variable t).

The spherical solution transforms into

$$v_{\rm sph} = (\cosh t)^{\frac{4-n}{2}}$$

and the cylindrical solution is the only constant solution,

$$v_{\rm cyl} = \varepsilon_n = \left(\frac{n(n-4)}{n^2-4} \right)^{\frac{n-4}{8}} .$$

Theorem (Schoen for $\sigma = 1$, Frank and König for $\sigma = 2$, Andrade and Wei for $\sigma = 3$)

For each $\varepsilon \in (0, \varepsilon_n]$ there exists a unique, positive, periodic solution v_ε of (4) attaining its minimal value of ε at $t = 0$. Moreover, if v is any global solution of (4) then either $v = v_{\rm sph}$ or $v = v_\varepsilon(\cdot + T)$ for some $\varepsilon \in (0, \varepsilon_n]$ and $T \in \mathbf{R}$.

As a consequence, $\mathcal{M}^4_{p,q} = (0, \varepsilon_n]$ for any $p \neq q \in \mathbf{S}^n$.
We can transform back to Euclidean coordinates by inverting the Emden-Fowler change of coordinates (3), obtaining

$$u_\varepsilon(x) = |x|^{\frac{2\sigma-n}{2}} v_\varepsilon(-\log|x|).$$

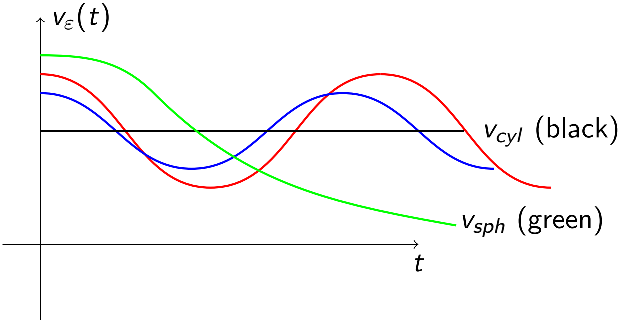

Figure: This figure shows some profile curves of Delaunay solutions.

We summarize what is currently known about the Delaunay metrics of all orders here.

- Delaunay metrics with constant $Q_g^{2\sigma}$-curvature exist for each order $\sigma < n/2$, even for fractional orders (delaTorre, del Pino, González and Wei for $0 < \sigma < 1$ and Jin and Xiong for all orders)
- The Delaunay metrics give all possible singular Yamabe metrics with two punctures and are ordered by a Hamiltonian energy if $\sigma = 1$ (Schoen), $\sigma = 2$ (Frank and König) or $\sigma = 3$ (Andrade and Wei).
- For $0 < \varepsilon < \varepsilon_n$ each v_ϵ achieves its minimum ε once and its maximum once within each period. This maximal value is strictly less than 1.
- The period T_ε of v_ε is a decreasing function of ε and $\lim_{\varepsilon \to 0^+} T_\varepsilon = \infty$.
- When $\varepsilon > 0$ is small the Delaunay solution is close to a string of spheres connected by small necks. The necks are close to (rescaled) fundamental solutions of $(-\Delta)^\sigma$.

Asymptotics

Jin and Xiong used the method of moving planes in its integral from to prove the following.

Theorem (Jin and Xiong)

Let $u : \mathbf{B}_1(0)\backslash\{0\} \to (0,\infty)$ satisfy $(-\Delta)^\sigma u = u^{\frac{n+2\sigma}{n-2\sigma}}$ and $(-\Delta)^\gamma u \geq 0$ for each $\gamma < \sigma$. Then

$$u(x) = |x|^{\frac{2\sigma-n}{2}}(\overline{u}(|x|) + \mathcal{O}(|x|)), \qquad \overline{u}(r) = \frac{1}{n\omega_n r^{n-1}} \int_{\mathbf{B}_r(0)} u(x)d\sigma(x).$$

Later work of Ngô and Ye show that if u is a conformal factor of a singular Yamabe metric with finitely many punctures (*i.e.* $(-\Delta_0)^\sigma u = c_{n,\sigma} u^{\frac{n+2\sigma}{n-2\sigma}}$ on $\mathbf{R}^n\backslash\{p_1,\dots,p_k\}$) then $(-\Delta_0)^\gamma u \geq 0$ for each $\gamma < \sigma$, so this additional hypothesis is redundant in our setting.

We now use the uniqueness of the Delaunay metrics if $\sigma = 1$, $\sigma = 2$ or $\sigma = 3$ to define an asymptotic necksize for each puncture of a metric $g = U^{\frac{4}{n-2\sigma}} g_0 \in \mathcal{M}_k^4$ or $g \in \mathcal{M}_k^6$.

We let p_i be in the singular set of U. By the asymptotics of Jin and Xiong we know u is asymptotically radial near p_i, but by the classification of two-puncture solutions we know that the Delaunay metrics are the only possible radial solutions. Thus, letting $u = U u_{\mathrm{sph}}$, we have

$$\begin{aligned} u(x) &= |x|^{\frac{2\sigma-n}{2}}(u_{\varepsilon_i}(Rx) + \mathcal{O}(|x|)) &(7)\\ &= |x|^{\frac{2\sigma-n}{2}}(v_{\varepsilon_i}(-\log|x| + T) + \mathcal{O}(|x|)). &(8)\end{aligned}$$

Thus we can define the asymptotic necksize of $g = U^{\frac{4}{n-2\sigma}} g_0$ at the puncture p_i as ε_i, the necksize of the Delaunay asymptote.

Main theorems, second statement

Here are precise statements of our theorems.

Theorem (Andrade, do Ò, –)

Let $k \geq 3$ and let δ_1, δ_2 be two positive numbers. Let $\{g_i\} \subset \mathcal{M}_k^4$ be a sequence of singular Yamabe metrics with k punctures. If the distance with respect to the round metric between the punctures is bounded below by δ_1 and the asymptotic necksize at each puncture is bounded below by δ_2 then a subsequence of $\{g_i\}$ converges uniformly on compact subsets.

Theorem (Andrade, do Ò, –, Wei)

The theorem above also holds for the moduli space $\mathcal{M}_k^6$.

Remark

- *Dan Pollack proved the corresponding theorem in the scalar curvature setting, and our proof uses much of his technique.*
- *We can only prove the compactness result in the $\sigma = 2$ and $\sigma = 3$ settings because otherwise we don't know if there are rotationally invariant constant Q-curvature metrics on $\mathbf{R} \times \mathbf{S}^{n-1}$ beside the Delaunay metrics.*
- *This theorem is maybe the zeroth step in addressing the following question: how well does the asymptotic data of a metric $g \in \mathcal{M}_k^\sigma$ determine the metric itself?*
- *Finally, it is worthwhile to consider possible degenerations in moduli space: each punctures must coalesce or necksizes must go to zero. (See the figure on the next slide.) Thus one can imagine compactifying $\mathcal{M}_k^\sigma$ by gluing on copies of $\mathcal{M}_l^\sigma$ with $l < k$.*

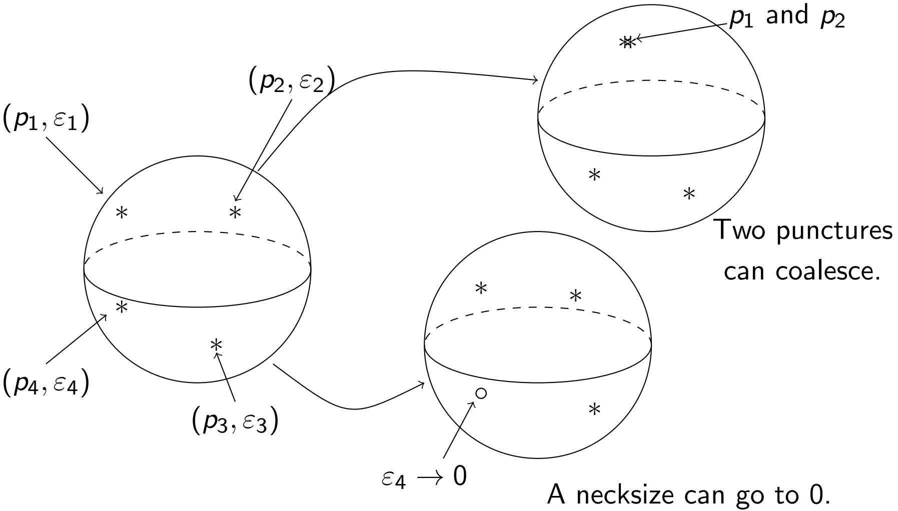

Figure: The two possible degenerations in the moduli space $\mathcal{M}_k$.

Pohozaev invariant

Much of our work depends on a first integral for the Delaunay metrics. For $v : (t_1, t_2) \times \mathbf{S}^{n-1} \to \mathbf{R}$ we define

$$\mathcal{H}(v) = \mathcal{H}_{\mathrm{rad}}(v) + \mathcal{H}_{\mathrm{ang}}(v) + \frac{(n-4)^2(n^2-4)}{32}|v|^{\frac{2n}{n-4}},$$

where

$$\mathcal{H}_{\mathrm{rad}}(v) = -\frac{\partial v}{\partial t}\frac{\partial^3 v}{\partial t^3} + \frac{1}{2}\left(\frac{\partial^2 v}{\partial t^2}\right)^2 + \left(\frac{n(n-4)+8}{4}\right)\left(\frac{\partial v}{\partial t}\right)^2 - \frac{n^2(n-4)^2}{32}v^2$$

and

$$\mathcal{H}_{\mathrm{ang}}(v) = -\frac{1}{2}(\Delta_\theta v)^2 - \left|\nabla_\theta \frac{\partial v}{\partial t}\right|^2 + \frac{n(n-4)}{4}|\nabla_\theta v|^2.$$

This is the Hamiltonian energy for Q_g^4 on a cylinder. There is a similar, but more complicated formula for the Hamiltonian energy in the 6th order case.

Integration by parts shows that if v satisfies (4) then

$$\frac{d}{dt}\int_{\{t\}\times\mathbf{S}^{n-1}} \mathcal{H}(v)d\theta = 0. \tag{9}$$

In our proof we also use a slight variation of the Pohozaev integral: if v satisfies

$$P^4_{\mathrm{cyl}} v = A v^{\frac{n+4}{n-4}}$$

then

$$\frac{d}{dt}\int_{\{t\}\times\mathbf{S}^{n-1}} \mathcal{H}_{\mathrm{rad}} + \mathcal{H}_{\mathrm{ang}} + \frac{n-4}{2n} A v^{\frac{2n}{n-4}} d\theta = 0.$$

We now define the radial Pohozaev invariant as follows: Let $g = u^{\frac{4}{n-4}} g_0$ be a complete, constant Q-curvature metric on $\mathbf{S}^n \backslash \{p_1, \dots, p_k\}$. For each $j = 1, 2, \dots, k$ select $r > 0$ sufficiently small such that u is smooth on $\mathbf{B}_r(p_j) \backslash \{p_j\}$, and perform the cylindrical change of variables (3) in a punctured ball centered at p_j to obtain a function v_j satisfying (4). Finally define

$$\mathcal{P}(g, p_j) = \int_{\{t\} \times \mathbf{S}^{n-1}} \mathcal{H}(v_j) d\theta.$$

By the asymptotic expansion (7), we know that

$$v_j(t,\theta) = v_{\varepsilon_j}(t + T_j) + \mathcal{O}(e^{-t})$$

for some Delaunay parameter ε_j and translation parameter T_j, so

$$\mathcal{P}(g, p_j) = |\mathbf{S}^{n-1}|\mathcal{H}_{\mathrm{rad}}(v_{\varepsilon_j}) < 0.$$

Furthermore, the radial Pohozaev invariant uniquely determines the asymptotic Delaunay necksize ε. In particular, bounding all the asymptotic necksizes of a sequence of metrics $g_i = u_i^{\frac{4}{n-4}} g_0$ away from zero is equivalent to bounding all the radial Pohozaev invariants away from zero.

We can give the Pohozaev invariant $\mathcal{P}(g, p_j)$ a more geometric interpretation. (See work of Gover and Ørsted.) Constant Q-curvature metrics are critical points of a conformally invariant energy functional. Thus one can define the full Pohozaev invariant as an $\mathfrak{so}(n+1,1)^*$-valued cohomology class defined by

$$\mathcal{PO}([\Sigma])(X) = \int_\Sigma T(X, \nu) d\sigma,$$

where Σ is a hypersurface, $X \in \mathfrak{so}(n+1,1)$ is a conformal Killing field, and T is a rank 2 tensor computing the first variation of Q^4_g among all nearby metrics (not necessarily conformal). (Y.-J. Lin computed this tensor explicitly.)

The Pohozaev invariant $\mathcal{P}(g, p_j)$ defined above is $\mathcal{PO}(\partial \mathbf{B}_r(p_j))(r\partial_r)$.

Some related results and open questions

- C. S. Lin ($\sigma = 2$), Wei and Xu (σ a positive integer) and Chen, Li and Ou (fractional σ) proved Obata-type theorems, which in our context state $\mathcal{M}_0^\sigma = SO(n+1,1)$.
- They also show $\mathcal{M}_1^\sigma = \varnothing$.
- By the uniqueness of the Delaunay solutions we know that $\mathcal{M}^\sigma_{\{p,q\}} = (0, \varepsilon_n]$ for $\sigma = 1, 2, 3$ and $p \neq q \in \mathbf{S}^n$.
- For $k \geq 3$ the moduli space $\mathcal{M}_k^2$ is not empty. Baraket and Rebhi constructed examples for k a positive, even integer, and forthcoming work of Andrade, Caju, do Ò, – and Silva Santos constructs solutions for all positive integers greater than 2.
- A related construction attaches k Delaunay ends to a nondegenerate compact Riemannian manifold for any k.

- Can one show uniqueness of the Delaunay solutions for each admissible order σ?
- What is the link between super-harmonicity and positive Q_g^γ curvature? For instance, in our setting we know u satisfies $(-\Delta_0)^\gamma$ for each $\gamma < \sigma$. Does this also mean $Q_g^\gamma > 0$ for $\gamma < \sigma$? Proving positivity of Q_g^γ is equivalent to a Modica-type estimate.
- When is the Greens function of $(-\Delta_g)^\sigma$ positive? When can one prove asymptotic expansions near the pole?

References

- Compactness within the space of complete, constant Q-curvature metrics on the sphere with isolated singularities, joint with J. Andrade and J. do Ò, Int. Math. Res. Not. 2022, 17282–17302
- Compactness of singular solutions to the sixth order GJMS equation, joint with J. Andrade, J. do Ò, and J. Wei, arXiv:2302.06770

Thank you for your attention!

Supplements

The rest of these slides are supplemental, if people are interested.

Some basic formulas

We recall some basic formulates discussed this week. Let (M, g) be a Riemannian manifold of dimension $n \geq 3$. The GJMS operator $P_g^{2\sigma}$ of order 2σ is a conformally covariant operator satisfying

$$\widetilde{g} = u^{\frac{4}{n-2\sigma}} g \Rightarrow P_{\widetilde{g}}^{2\sigma}(v) = u^{-\frac{n+2\sigma}{n-2sigma}} P_g^{2\sigma}(uv).$$

The Q-curvature of order 2σ is (up to mulitplication by a normalizing constant) $P_g^{2\sigma}(1)$. The corresponding Yamabe-type problem is then equivalent to the PDE

$$P_g^{2\sigma}(u) = C u^{\frac{n+2\sigma}{n-2\sigma}}.$$

The round sphere is secretly your nemesis

As is usually the case with these conformally invariant problems, the round metric on the sphere is at the same time the easiest and most difficult example. The round metric has constant Q-curvature, so

$$(-\Delta)^\sigma u_{\mathrm{sph}} = c_{n,\sigma} u_{\mathrm{sph}}^{\frac{n+2\sigma}{n-2\sigma}}, \qquad u_{\mathrm{sph}}(x) = \left(\frac{1+|x|^2}{2} \right)^{\frac{2\sigma - n}{2}}.$$

On the other hand,

$$u_{\lambda, x_0} = \left(\frac{1+\lambda^2 |x-x_0|^2}{2\lambda} \right)^{\frac{2\sigma-n}{2}}$$

is also a solution for any $\lambda > 0$ and $x_0 \in \mathbf{R}^n$. Letting $\lambda \to 0$ we see u_{λ,x_0} blows up in a neighborhood of x_0 but converges uniformly to zero away from x_0. This blow-up behavior is exactly why we formulate the singular Yamabe problem in the first place!

Singular Yamabe problem in general

We give a general statement of the singular Yamabe problem for Q-curvature of oder 2σ. Let (M,g) be a compact Riemannian manifold and let $\Lambda \subset M$ be closed. We seek $\widetilde{g} \in [g]$ such that $Q^{2\sigma}_{\widetilde{g}}$ is constant and $\widetilde{g}$ is complete on $M\backslash\Lambda$. Writing $\widetilde{g} = u^{\frac{4}{n-2\sigma}} g$, we see this is equivalent to the boundary value problem

$$u : M\backslash\Lambda \to (0,\infty), \qquad \liminf_{x\to\Lambda} u(x) = \infty, \qquad P^{2\sigma}_g u = c_{n,\sigma} u^{\frac{n+2\sigma}{n-2\sigma}}.$$

Details of the proof

We start with a sequence $g_i \in \mathcal{M}_k^4$. Each $g_i = u_i^{\frac{4}{n-4}} g_0$ with a singular set $\Lambda_i = \{p_1^i, p_2^i, \dots, p_k^i\}$. Our goal is to extract a subsequence such that $\{u_i\}$ converges uniformly on compact subsets.

Reduction of the problem

As a first simplification we show that we may assume the singular sets Λ_i converge. Each Λ_i lies in the compact set

$$\{(q_1, \ldots, q_k) \in (\mathbf{S}^n)^k : \mathrm{dist}_{\overset{\circ}{g}}(q_j, q_l) \geq \delta_1 \text{ for each } j \neq l\},$$

and so we pass to a subsequence such that

$$p_j^i \to \bar{p}_j \text{ for each } j = 1, 2, \ldots, k.$$

We denote $\overline{\Lambda} = \{\bar{p}_1, \ldots, \bar{p}_k\}$.

To make some of the later analysis easier to follow, we choose the specific compact exhaustion of $\mathbf{S}^n \backslash \overline{\Lambda}$. For each $m \in \mathbf{N}$ let

$$K_m = \{q \in \mathbf{S}^n \backslash \overline{\Lambda} : \mathrm{dist}_{\overset{\circ}{g}}(q, \bar{p}_j) \geq 2^{-m} \text{ for each } j = 1, \ldots, k\}.$$

Observe that for each fixed m the function u_i is smooth on K_m once i is large enough.

Some tools

We need to use the following preliminary results:

- Balls with respect to the round metric are mean convex. This is proven for $\sigma = 1$ by Schoen, for $\sigma = 2$ by Chang, Han and Yang, and for $\sigma = 3$ by us. All three proofs use the method of moving planes.
- The Greens function blows up like $|x - x_0|^{2\sigma - n}$. This is proven for $\sigma = 2$ by Gursky and Malchiodi, and for $\sigma = 3$ by Chen and Hu.
- We use a Harnack-type inequality due to Caristi and Mitidieri.

A priori upper bounds

We begin with a uniform *a priori* upper bound for the possible conformal factor u.

Theorem (Andrade, do Ó, –)

Let $n \geq 5$ *and let* $\Lambda \subset \mathbf{S}^n$ *be a proper, closed subset and let* $g = u^{\frac{4}{n-4}} \mathring{g}$ *be a complete metric on* $\mathbf{S}^n \backslash \Lambda$ *such that*

$$Q_g = \frac{n(n^2-4)}{8}.$$

Then there exists $C > 0$ *depending only on* n *such that*

$$u(x) \leq C(\mathrm{dist}_{\mathring{g}}(x, \Lambda))^{\frac{4-n}{2}}. \tag{10}$$

Theorem (Andrade, do Ò, –, Wei)

The same sort of upper bound holds in the sixth order setting.

We prove the upper bound (10) using a fairly standard blow-up argument. Given $x_0 \notin \Lambda$ choose ρ small enough such that $\overset{\circ}{\mathbf{B}}_\rho(x_0) \subset \mathbf{S}^n \backslash \Lambda$ and define the function

$$f(x) = (\rho - \mathrm{dist}_{\overset{\circ}{g}}(x, x_0))^{\frac{n-4}{2}} u(x).$$

If $\rho = \frac{1}{2} \mathrm{dist}_{\overset{\circ}{g}}(x_0, \Lambda)$ then

$$f(x_0) = 2^{\frac{4-n}{2}} (\mathrm{dist}_{\overset{\circ}{g}}(x, \Lambda))^{\frac{n-4}{2}} u(x_0).$$

Therefore it will suffice to prove there exists a uniform C such that $f \leq C$ for all admissible choices.

If there does not exist such a C we may choose admissible singular sets Λ_i, conformal factors u_i, points $x_{0,i}$ and radii ρ_i such that

$$M_i = f(x_{1,i}) = \sup\{f(x) : x \in \mathring{\mathbf{B}}_{\rho_i}(x_{0,i})\} \to \infty.$$

We then rescale u_i by

$$\lambda_i = 2(u_i(x_{1,i}))^{\frac{4-n}{2}}, \qquad w_i(y) = \lambda_i^{\frac{n-4}{2}} u_i(\lambda_i y).$$

In the limit we obtain a function $\overline{w} : \mathbf{R}^n \to (0, \infty)$ satisfying

$$\overline{w}(0) = \sup \overline{w} = 2^{\frac{n-4}{2}}, \qquad \Delta_0^2 \overline{w} = \frac{n(n-2)(n^2-4)}{16} \overline{w}^{\frac{n+4}{n-4}}.$$

By the classification theorem of Lin we must have

$$\overline{w}(x) = \left(\frac{1+|x|^2}{2}. \right)^{\frac{4-n}{2}},$$

which in turn implies large coordinate spheres are geodesically concave with respect to the metric g_i. This contradicts a theorem of Chang, Han, and Yang stating that such spheres are always geodesically convex.

The limit is positive

Restricting to each K_m we use the *a priori* upper bound and the Arzela-Ascoli theorem to pass to a subsequence that converges uniformly to a limit function $\overline{u}$. The sets $\{K_m\}$ form a compact exhaustion of $\mathbf{S}^n \backslash \overline{\Lambda}$, so we now have a candidate conformal factor $\overline{u}$ defined on $\mathbf{S}^n \backslash \overline{\Lambda}$.

We would like $\overline{g} = \overline{u}^{\frac{4}{n-4}} \overset{\circ}{g}$ to be a conformal metric, and so we must show $\overline{u} > 0$ on $\mathbf{S}^n \backslash \overline{\Lambda}$. Supposing otherwise, there must be a point $q \in \mathbf{S}^n \backslash \overline{\Lambda}$ such that $\overline{u}(q) = 0$. We let

$$\epsilon_i = u_i(q) \to \overline{u}(q) = 0.$$

Next we rescale, defining

$$w_i : \mathbf{S}^n \backslash \Lambda_i \to (0, \infty), \qquad w_i(x) = \frac{1}{\epsilon_i} u_i(x).$$

By the scaling law for the Paneitz operator we have

$$P_{\overset{\circ}{g}}(w_i) = \epsilon_i^{\frac{8}{n-4}} \frac{n(n-4)(n^2-4)}{16} w_i^{\frac{n+4}{n-4}}, \tag{11}$$

and by our normalization

$$w_i(q) = 1. \tag{12}$$

Combining our *a priori* estimates and a Harnack-type inequality we then obtain a subsequence $w_i \to \overline{w}$ uniformly on the compact sets K_m, and so in the limit we have

$$\overline{w} : \mathbf{S}^n \backslash \overline{\Lambda} \to [0, \infty), \quad P_{\overset{\circ}{g}}(\overline{w}) = 0.$$

This must be a sum of Greens functions with poles at the punctures $\{\bar{p}_j\}$, *i.e.*

$$\overline{w} = \sum_{j=1}^{k} \alpha_j G_{\bar{p}_j}.$$

By the normalization (12) we know at least one of the coefficients $\{\alpha_j\}$ is nonzero, and so we assume $\alpha_1 \neq 0$.

A theorem of Gursky and Malchiodi states that the Greens function has the expansion

$$G_p(x) = \frac{1}{2n(n-2)(n-4)\omega_n} (\mathrm{dist}_{\overset{\circ}{g}}(x,p))^{4-n} + \mathcal{O}(1). \tag{13}$$

Here ω_n is the volume of a unit ball in n-dimensional Euclidean space. At this point we center our coordinates on $\bar{p}_1$ and apply the cylindrical transformation $t = -\log |x|$ and $\theta = x/|x|$ and let

$$v_i(t,\theta) = e^{\left(\frac{4-n}{2}\right)t} u_i(e^{-t}\theta) u_{\mathrm{sph}}(e^{-t}) = (\cosh t)^{\frac{4-n}{2}} u_i(e^{-t}\theta)$$

and let

$$z_i(t,\theta) = \frac{1}{\epsilon_i} v_i(t,\theta) = (\cosh t)^{\frac{4-n}{2}} w_i(t,\theta).$$

These functions v_i and z_i are all well defined for $t > T_1$, and they satisfy the PDEs

$$P_{\rm cyl} v_i = \frac{n(n-4)(n^2-4)}{16} v_i^{\frac{n+4}{n-4}}$$

and

$$P_{\rm cyl} z_i = \epsilon_i^{\frac{8}{n-4}} \frac{n(n-4)(n^2-4)}{16} z_i^{\frac{n+4}{n-4}}.$$

Our radial Pohozaev identities now tell us the integrals

$$\int_{\{t\}\times \mathbf{S}^{n-1}} \mathcal{H}(v_i)(t,\theta) d\theta$$

and

$$\int_{\{t\}\times \mathbf{S}^{n-1}} \mathcal{H}_{\rm cyl}(z_i) + \epsilon_i^{\frac{8}{n-4}} \frac{(n-4)^2(n^2-4)}{32} z_i^{\frac{2n}{n-4}} d\theta$$

do not depend on t.

By the convergence $w_i \to \overline{w}$ we have $z_i \to \overline{z}$. However, by the expansion of Gursky and Malchiodi we have

$$\overline{w}(x) \simeq \frac{\alpha_1}{2n(n-2)(n-4)\omega_n}|x|^{4-n} + \mathcal{O}(1)$$

near 0, which in turn implies

$$z_i \to \overline{z} = \frac{\alpha_1}{2n(n-2)(n-4)\omega_n} + \mathcal{O}(e^{(4-n)t}). \tag{14}$$

Evaluating the radial Pohozaev invariant on z_i and letting $i \to \infty$ we obtain

$$
\begin{aligned}
& \int_{\{t\} \times \mathbf{S}^{n-1}} \mathcal{H}_{\mathrm{cyl}}(z_i) + \epsilon_i^{\frac{8}{n-4}} \frac{(n-4)^2(n^2-4)}{32} z_i^{\frac{2n}{n-4}} d\theta \\
\to \quad & \int_{\{t\} \times \mathbf{S}^{n-1}} \mathcal{H}_{\mathrm{cyl}}(\overline{z}) d\theta \\
= \quad & -\frac{n\alpha_1^2}{128(n-2)^2\omega_n} + \mathcal{O}(e^{(4-n)t}).
\end{aligned}
$$

On the other hand,

$$\begin{aligned}\mathcal{P}_{\rm rad}(g_i, p_1^i) &= \int_{\{t\}\times \mathbf{S}^{n-1}} \mathcal{H}(v_i) d\theta \\ &= \int_{\{t\}\times \mathbf{S}^{n-1}} \epsilon_i^2 \mathcal{H}_{\rm cyl}(z_i) + \epsilon_i^{\frac{2n}{n-4}} \frac{(n^2-4)(n-4)^2}{32} z_i^{\frac{2n}{n-4}} d\theta \\ &\to 0,\end{aligned}$$

which then implies the Delaunay necksizes ϵ_1^i of g_i at the punctures p_1^i converge to 0. This contradicts the hypothesis that all the Delaunay necksizes remain bounded away from 0.

The limit metric lies in $\mathcal{M}_k$

We complete the proof by showing the limit metric $\overline{g} = \overline{u}^{\frac{4}{n-4}} \mathring{g} \in \mathcal{M}_k$. To this end, we already know the limit function $\overline{u}$ satisfies the PDE

$$P_{\mathring{g}} \overline{u} = \frac{n(n-4)(n^2-4)}{16} \overline{u}^{\frac{n+4}{n-4}},$$

and so $Q_{\overline{g}} = \frac{n(n^2-4)}{8}$. Similarly $R_{\overline{g}} < 0$ as well. It remains to show that $\overline{g}$ is complete on $\mathbf{S}^n \backslash \overline{\Lambda}$.

If this were not the case, then for some $j \in \{1, \ldots, k\}$ we would have

$$\mathcal{P}_{\mathrm{rad}}(\overline{g}, \bar{p}_j) = 0 \Rightarrow \mathcal{P}_{\mathrm{rad}}(g_i, p_j^i) \to 0,$$

but this contradicts the hypothesis that all the Delaunay necksizes are bounded away from zero.