xavier.cabre@upc.edu

International Doctoral Summer School in
Conformal Geometry and Non-local operators

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Xavier Cabré (ICREA & UPC)

On the nonlocal mean curvature

Or:

Michael-Simon type Sobolev inequalities and applications to local and nonlocal mean curvature problems.

• PLAN (detailed:)

1. The Euclidean isoperimetric inequality
2. The Michael-Simon (MS) and Allard Sobolev ineq.
3. The fractional Sobolev inequality (and a geometric version)
4. Fractional perimeter and nonlocal mean curvature (NMC)
5. Density estimate and application: fract. MS ineq. on nonl. minimal surf.
6. Appl'n 1: gradient estimate for nonl. minimal graphs.
7. Appl'n 2 (of MS Sobolev ineq's): extinction time for (nonlocal) mean curvature flow (7 not taught)

(8. Foliations by nonlocal minimal surfaces: minimality and viscosity property. Calibration for the fract. perimeter
9. Calibration for the fract. Laplacian.) 8 & 9 not covered

• General plan: A- Local & nonlocal Michael-Simon sob. ineq.
B- Appl'n: gradient estimate for nonl. min. graphs
C- Appl'n: extinction time for (nonl.) MCFlow.
D- Foliations and nonlocal calibrations.

1. The Euclidean isop. ineq.

Thm 1 $E \subset \mathbb{R}^n$ smooth & bdd open $\Rightarrow$

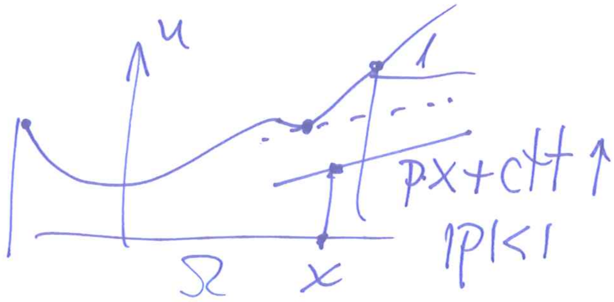
$$(1) \quad \boxed{\frac{|\partial E|}{|E|^{\frac{n-1}{n}}} \geq \frac{|\partial B_1|}{|B_1|^{\frac{n-1}{n}}} \left(= n|B_1|^{1/n}\right) \text{ \& eq. iff } E \text{ is a ball}}$$

• Proof (Cabré 1996. See [Cabré, Isoperimetric ... survey, Chin. Ann. Math. 2017])

$$\begin{cases} \Delta u = \dfrac{|\partial E|}{|E|} \\ u_\nu = 1 \end{cases}$$

Claim $B_1(0) \subset \nabla u(\Gamma_u)$ (Exercise 1)

where $\Gamma_u = \{x \in E : u(y) \geq u(x) + \nabla u(x)\cdot(y-x) \ \forall y \in \overline{E}\}$



$$|B_1| \leq |\nabla u(\Gamma_u)| \leq \int_{\Gamma_u} |\mathrm{Jac}\, \nabla u(x)|\, dx = \int_{\Gamma_u} \underbrace{\det D^2 u(x)}_{\geq 0}\, dx$$

$$|\nabla u(\Gamma_u)| = \int_{\nabla u(\Gamma_u)} dp$$

area formula: $p = \nabla u(x)$

$$\leq \int_{\Gamma_u} \left(\frac{\Delta u}{n}\right)^n dx$$

$$\leq \left(\frac{|\partial E|}{n|E|}\right)^n |E|. \quad \square$$

Note: $E = B_1 \rightarrow u(x) = \frac{|x|^2}{2} + ct$

Analyze $\leq$ in proof $\Rightarrow$ iff. $\square$

2. The Michael-Simon (MS) & Allard Sob. ineq.

Classical perimeter functional

$\downarrow$ 1st variation

Mean curvature $= H$.

Minimal surfaces : $H = 0$

• 1967, M. Miranda proved

$M = M_n \subset \mathbb{R}^{n+1}$ is a min. surface &

$E \subset M$ smooth, open bdd $\Rightarrow$ $|\partial E| \geq c(n) \cdot |E|^{\frac{n-1}{n}}$

for some dimn'l cst. $c(n)$.

• Extension : Michael-Simon & Allard, independ. 1972

Thm 2 (MS & Allard, '72)

$M \subset \mathbb{R}^{n+1}$ smooth hypersurface, $E \subset M$ smooth open bdd set. $\Rightarrow$

(2) $$|E|^{\frac{n-1}{n}} \leq C(n) \left\{ |\partial E| + \int_E |H| \right\}.$$

Before proving it,

Corol 3 (MS Sobolev ineq) $M \subset \mathbb{R}^{n+1}$ smooth hyper.

$u \in C^1_c(M)$, $1 \leq p < n \Rightarrow$ $\left(p^* = \frac{np}{n-p}\right)$

(3) $$\left(\int_M |u|^{p^*} \right)^{1/p^*} \leq C(n) \left(\int_M |\nabla u|^p + \int_M |H|^p |u|^p \right)^{1/p}$$

tangential gradient.

For the following ① [Cab-Miraglio] suffices. More on ② [Cabré-Poggesi, LNM, 2017]

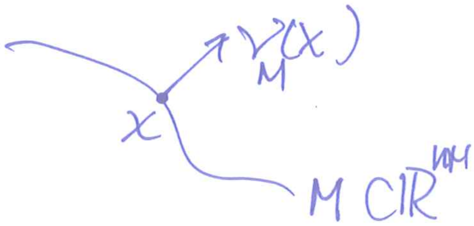
- Proof of Thm 2, following

① [Cabré & Miraglio, Universal Hardy-Sobolev... Comm. Cont. Math 2022]

Exercise 2 Learn basics of Riemannian Geometry or, if not, tangential derivatives as in [Giusti, Min. surf...]. ← same approach as ① ②

$x \in M, \lambda > 0 \to \lambda x \in \underbrace{\lambda M}_{}$ has volume $= \lambda^{n} \mathrm{vol}(M)$

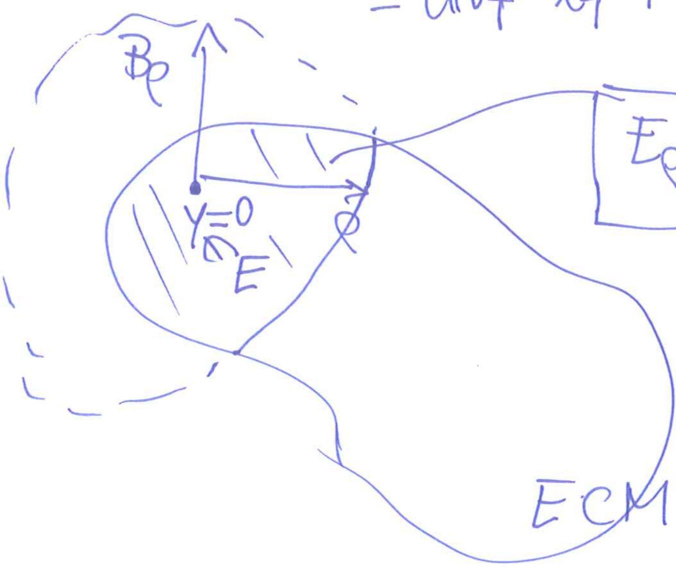
Exercise 3: $\boxed{\mathrm{div}_T\, x = n}$



Also $H = \mathrm{div}_T\, \nu_M$. So,

$$n = \mathrm{div}_T\, x = \mathrm{div}_T\left(x_T + (x\cdot\nu_M)\,\nu_M\right)$$

$$= \mathrm{div}_T\, x_T + \underbrace{\nabla_T(x\cdot\nu_M)\cdot\nu_M}_{0} + (x\cdot\nu_M)\,H$$



$\boxed{E_\rho = E\cap B_\rho,\ B_\rho = B_\rho^{n+1}}$

$$n|E_\rho| = \int_{E_\rho} \mathrm{div}_T\, x \, dV \overset{\text{div thm}}{=}$$

$$= \int_{\partial E_\rho} \underbrace{x_T\cdot\nu_{E_\rho}}_{|\cdot|\le\rho}\, dA + \int_{E_\rho} (x\cdot\nu_M) H \, dV$$

$$\le \rho|\partial E_\rho| + \rho\int_{E_\rho} |H| \, dV.$$

$$|\partial E_\rho| = |\partial B_\rho \cap E| + |\partial E \cap B_\rho|$$
$$\leq \frac{d}{d\rho}|E_\rho| + |\partial E \cap B_\rho| \quad \Rightarrow$$

$$\frac{d}{d\rho}\left(-\rho^{-n}|E_\rho|\right) \leq \rho^{-n}\left\{|\partial E \cap B_\rho| + \int_{E_\rho}|H|\,dV\right\} \quad \Leftrightarrow$$

$$\frac{d}{d\rho}\left(\rho^{-n}|E_\rho| \exp\int_0^\rho \frac{|\partial E \cap B_\sigma| + \int_{E_\sigma}|H|\,dV}{|E_\sigma|}\,d\sigma\right) \geq 0$$

$$\Downarrow$$

$$\rho^{-n}|E_\rho| \exp\int_0^\rho \cdots \geq \lim_{\rho\downarrow 0}\frac{|E_\rho|}{\rho^n} = |B_1^n| = \omega_n$$

Choose $\rho_0 := (2|E|\,\omega_n^{-1})^{1/n} \longrightarrow$

$$\exp\int_0^{\rho_0} \frac{|\partial E \cap B_\sigma(y)| + \int_{E_\sigma}|H|\,dV}{|E_\sigma|}\,d\sigma \geq \rho_0^n\,\omega_n|E_{\rho_0}|^{-1}$$
$$\geq \rho_0^n\,\omega_n|E|^{-1} = 2.$$

Hence:

$\forall y\in E\ \exists r(y) \in (0,\rho_0)$ st.

$$\rho_0\left(|\partial E \cap B_{r(y)}(y)| + \int_{E_{r(y)}}|H|\,dV\right) \geq |E_{r(y)}|\log 2.$$

$$|E_{r(y)}| \leq c(n)|E|^{1/n}\left\{|\partial E \cap B_{r(y)}(y)| + \int_{E_{r(y)}}|H|\,dV\right\}$$

Finally, Besicovitch subcovering of $\bigcup_{y\in E} B_{r(y)}(y) \supset E \rightsquigarrow$ countable subcollection $\{B(y_i)\}_i$ st

$E \subset \bigcup_i B(y_i)$ & st. every pt of E belongs at most to $N(n)$ balls $B(y_i)$ □

$B(y_i) = B_{r(y_i)}(y_i)$

Theorem of [Simon Brendle, JAMS, 2021]

The Euclidean constant in the isop. ineq. is also OK (and thus it is the optimal constant) in the isop. ineq. on any minimal surface (and also in the MS Sobolev ineq. for $p=1$ on any hypersurf. $M_n \subset \mathbb{R}^{n+1}$)

The proof is a very clever extension of the one of Cabré in Thm 1 above. One starts solving

$$\begin{cases} \Delta u = \dfrac{|\partial E|}{|E|} & \text{in } E \subset M \\ u_\nu = 1 & \text{on } \partial E. \end{cases}$$

add artificial variable

One then takes $M \subset \mathbb{R}^{n+1} \subset \mathbb{R}^{n+2}$

Let $T_x^\perp M$ be the normal bundle to $T_x M$ in $\mathbb{R}^{n+2}$. Thus $\dim(T_x^\perp M) = 2$

One considers the map

$$(x,y) \in E \times T^\perp M \longrightarrow \mathbb{R}^{n+2}$$
$$(x,y) \longmapsto \phi(x,y) = \nabla u(x) + y$$

and the set $\{(x,y) : |\nabla u(x)|^2 + |y|^2 < 1\}$

instead of $\{x : |\nabla u(x)| = |p| < 1\}$.

1st lecture 1 1/2 h

3. The fract. Sobolev ineq. (and a geometric version).

Prop 4 (Fractional Sob. ineq. on hypersurfaces "with density")

$n \geq 1$, $s \in (0,1)$, $p \geq 1$ st $\boxed{sp < n}$.

$M \subset \mathbb{R}^{n+1}$ hypersurface st $\exists\, C_* > 0$ (a cst) &

$$|M \cap B_\rho^{n+1}(x_0)| \geq C_* \rho^n \qquad \forall \rho > 0 \ \forall x_0 \in M. \tag{4}$$

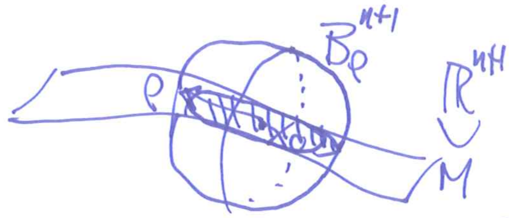
Then, for all $u \in C_c^1(M)$, $\left(p_s^* = \dfrac{np}{n-sp}\right)$

$$\left(\int_M |u|^{p_s^*}\right)^{1/p_s^*} \leq C \left(\int_M \int_M \frac{|u(x)-u(y)|^p}{|x-y|^{n+sp}}\, dV(x)\, dV(y)\right)^{1/p} \tag{5}$$

with $C = C(n, s, p, C_*)$.

In particular (4) and (5) hold if $\boxed{M = \mathbb{R}^n}$ $(\subset \mathbb{R}^{n+1})$ & $C_* = C_*(n)$.

(Note: C_* depends on the geom. of M ⚠)



• Proof by Brezis, unpub. 2001

↳ adapted to hypersurf. in [Cabré & Cozzi, A gradient estimate... Duke M.J. 2019]

$\boxed{B_r = B_r^{n+1} \text{ NOTATION}}$

see also [Cabré-Cozzi-Csató, A fract. MS... AIHP 2022]

$|u(x)| \leq |u(x)-u(y)| + |u(y)|$; $\forall r > 0 \to$

$$|u(x)| \leq \fint_{M\cap B_r(x)} |u(x)-u(y)|\, dV(y) + \fint_{M \cap B_r(x)} |u(y)|\, dV(y)$$

① ② Jensen →

$$① \leq \left(\fint_{M\cap B_r(x)} |u(x)-u(y)|^p\, dV(y)\right)^{1/p} \leq \left(r^{n+sp} \fint_{M\cap B_r(x)} \frac{|u(x)-u(y)|^p}{|x-y|^{n+sp}}\, dV(y)\right)^{1/p}$$

$1 - p_s^*$

NOTE: If $p=1$ & $E\subset\mathbb{R}^n$ bdd & $u=\chi_E$ (by density), then (5) says $|E|^{\frac{n-s}{n}} \leq C(n,s)\int_E dx\int_{\mathbb{R}^n\setminus E} dy\, \frac{1}{|x-y|^{n+s}}$

fract. perimeter as in section 4 below

$$|u(x)| \leq C\left\{ r^s\left(\int_M \frac{|u(x)-u(y)|^p}{|x-y|^{n+sp}}\,dV(y)\right)^{1/p} + r^{-\frac{n}{p_s^*}}\left(\int_M |u(y)|^{p_s^*}\,dV(y)\right)^{1/p_s^*}\right\}$$

optimize in r

$$\boxed{r(x)} := \left(\int_M |u(y)|^{p^*}\,dV(y)\right)^{\frac{p}{np_s^*}}\left(\int_M \frac{|u(x)-u(y)|^p}{|x-y|^{n+sp}}\,dV(y)\right)^{-\frac{1}{n}}$$

$\Downarrow$

$$|u(x)| \leq C\left(\int_M \frac{|u(x)-u(y)|^p}{|x-y|^{n+sp}}\,dV(y)\right)^{1/p_s^*}\left(\int_M |u(y)|^{p^*}\,dV(y)\right)^{\frac{sp}{np_s^*}}$$

Now, $^{1/p_s^*}$ & $\int_M dV(x)$ □. $\xrightarrow{\text{Def}} \boxed{[u]_{W^{s,p}(M)}}$ $\boxed{0<s<1,\ p\geq 1}$

$\to$ NOTE

4. Fractional perimeter & NMC (nonl. mean curv.)

As introduced in the seminal paper

- [Caffarelli - Roquejoffre - Savin, Nonlocal min. surfaces, CPAM 2010]

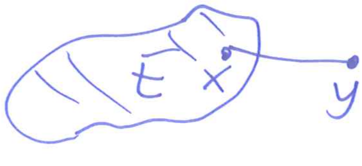
Def 5 Fractional perimeter $E\subset\mathbb{R}^N$ (sometimes we will take $N=n+1$; others $N=n$)
open set

$\boxed{0<\alpha<1}$

$$P_{N,\alpha}(E) = C_{N,\alpha}\int_E dx\int_{E^c} dy\,\frac{1}{|x-y|^{N+\alpha}} \quad ; \quad \boxed{E^c=\mathbb{R}^N\setminus E}$$

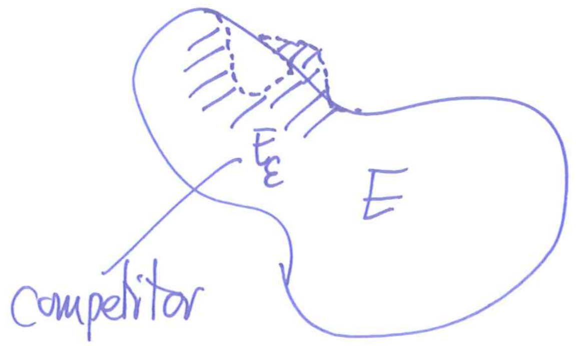
$C_{N,\alpha}\approx C\cdot(1-\alpha)$ as $\alpha\uparrow 1 \Rightarrow P_{N,\alpha}(E) \xrightarrow[\alpha\uparrow 1]{} |\partial E|_{N-1} = Per(E)$

Finite if E open bdd smooth



$$A,B\subset\mathbb{R}^N \rightsquigarrow \boxed{L(A,B):=\int_A dx\int_B dy\,\frac{1}{|x-y|^{N+\alpha}}} = L(B,A)$$

Euler-Lagrange (or 1st var'n) of fract. perimeter under volume constraint: fractional isop. pb



compare fract. perimeters:

4 disjoint sets:

balls min. fract. per. under given volume: [Garsia-Rodemich 1974] – rearrangement
[Frank-Seiringer '08] equality

$$P(E)=L(E_1,E_3)+L(E_1,E_4)+\cancel{L(E_2,E_3)}+L(E_2,E_4)$$

& $\overset{=}{}$ at $O(\varepsilon)$ $\to \wedge$

$$P(E_\varepsilon)=L(E_1,E_2)+L(E_1,E_4)+\cancel{L(E_3,E_2)}+L(E_3,E_4)$$

$$0\geq L(E_3,E_1)-L(E_3,E_4)-\big(L(E_2,E_1)-L(E_2,E_4)\big)=\int_{E_3}dx\int_{\mathbb{R}^N}dy\,\frac{\chi_{E_1}(y)-\chi_{E_4}(y)}{|x-y|^{N+\alpha}}$$

$$(\chi_E-\chi_{E^c})(y)$$

$$-\int_{E_2}dx\int_{\mathbb{R}^N}dy\,\frac{\chi_{E_1}(y)-\chi_{E_4}(y)}{|x-y|^{N+\alpha}}$$

$|E_2|=|E_3|$

E_2 & E_3 $\downarrow$ $x\in\partial E$

$\forall x\in\partial E$

Def 6 $\boxed{H_{\alpha,E}(x)} := c_{N,\alpha}\,\mathrm{PV}\int_{\mathbb{R}^N}\frac{\chi_{E^c}(y)-\chi_E(y)}{|x-y|^{N+\alpha}}\,dy$ = NMC at x

$\boxed{\text{CNMC surfaces}} = \boxed{\text{ctt}}$ for minimizers with vol. constraint sometimes we do not write cst

$$\boxed{H_{\alpha,E}(x)} = \int_{\mathbb{R}^N} \frac{\chi_{E^c}(y) - \chi_E(y)}{|x-y|^{N+\alpha}}\, dy$$

<u>Exercise 4</u>: div thm

Prove

$$= -\frac{2}{\alpha} \int_{\partial E} \frac{(x-y)\cdot \nu_E(y)}{|x-y|^{N+\alpha}}\, dV(y)$$

[balls have cst NMC
cylinders " " "]

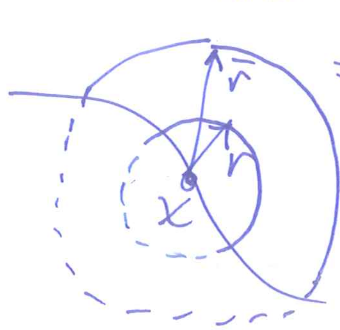
$$= \underbrace{(-\Delta)^{\alpha/2}(\chi_E - \chi_{E^c})(x)}_{\text{fract. Lapl. of order } \alpha}$$

$\underline{\underline{\alpha/2 \in (0, 1/2)}}$!!

think in spherical coordinates around x:

$$= \int_0^\infty r^{-1-\alpha} \int_{\partial B_r(x)}$$

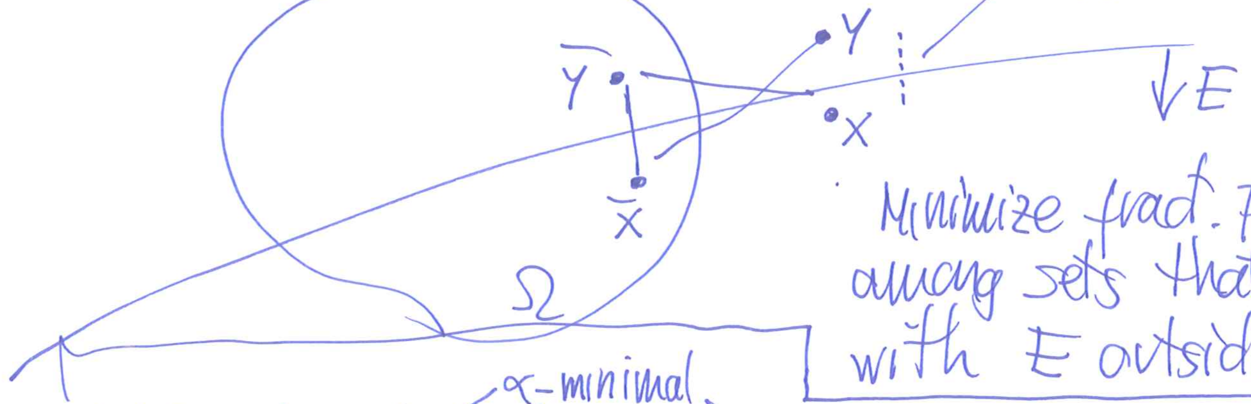
$x-x$



- NMC appeared in Caff-Souganidis as limit $\varepsilon \downarrow 0$ of fractional (Allen-Cahn heat eq'n) / mean curv. flow.

- Minimizers with no volume constraint. The Dirichlet (or Plateau) pb in $\Omega \subset \mathbb{R}^N$.

forbidden ∞ term. But: common to everybody



Minimize fract. perimeter among sets that agree with E outside Ω:

<u>Note</u>: In these notes, "nonl α-minimal minimal" surface $\equiv$ minimizing α-min. surface.

$F \subset \mathbb{R}^N$ st $F \setminus \Omega = E \setminus \Omega$ (given $E \subset \mathbb{R}^N$ "fixed")

← Plateau or Dir. exterior pb.

Def 7 Relative fract. perimeter (relative to Ω):

$$\boxed{\begin{aligned} P_{N,\alpha}(E;\Omega) = {} & L(\Omega\cap E, \Omega\setminus E) \\ & + L(\Omega\cap E, \Omega^c\cap E^c) \\ & + L(\Omega^c\cap E, \Omega\setminus E). \end{aligned}}$$

They solve $NMC = H(E) = 0$ $\forall x \in \partial E$ weak visc. sense: nonlocal min. surf.

[CRS, 2010] prove:
- ∃ of minimizer (l.s.c)
- Density estimate (see section 5)
- EL equ in visc. sense
- Improvement of flatness
- Ext. pb & monotonicity formula
- Dimension reduction

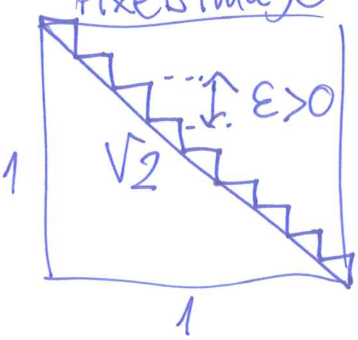
Main later contrib by Valdinoci, Savin, Figalli, Serra, Cinti, Cabre, ...

Thm [CRS] NMS minimizing are $C^{1,\alpha}$ except for a set of $\mathcal{H}^{N-2}$ dimension.

- Nonlocal or fract. perimeter methods are also very relevant in Image Processing

2nd lect. 3h

Pixels image



$\mathrm{Per}^{(\varepsilon)}(\text{staircase}) = 2 > \sqrt{2} \quad \forall \varepsilon > 0$

$\mathrm{Per}^{(\varepsilon)}_\alpha(\text{staircase}) \xrightarrow[\varepsilon\downarrow 0]{} \mathrm{Per}_\alpha(\text{diagonal}) \quad \forall \alpha \in (0,1)$

$\downarrow \alpha \uparrow 1$

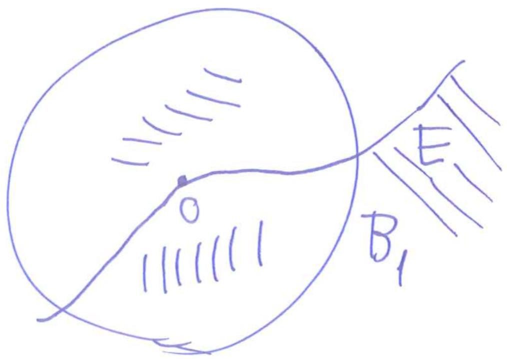
$\sqrt{2}$

5. Density estimate and an application: fract. MS ineq. on nonl. min. surfaces.

Thm 8 (density estimate, both for min. surf & nonl. min surf. ← [CRS '10])

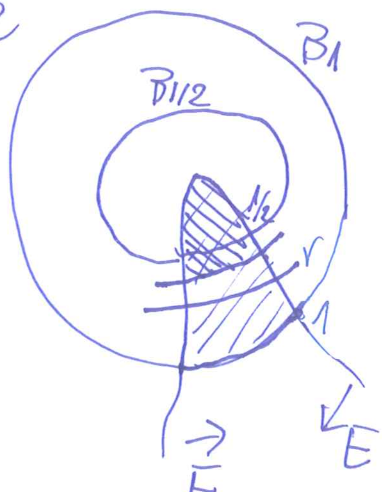
If the minimizing surface ∂E in B_1 passes through $0 \in \mathbb{R}^N$, i.e., $0 \in \partial E$, then

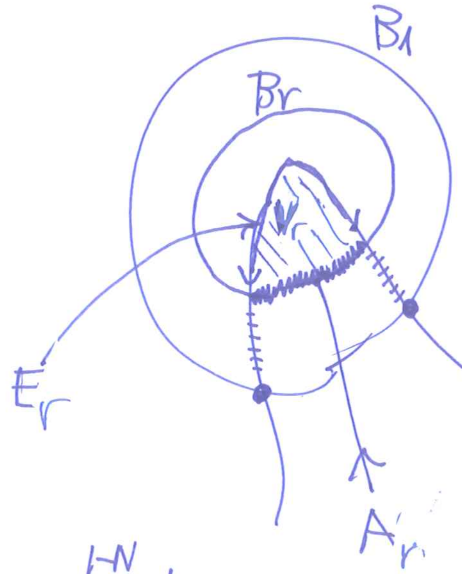
$$|E \cap B_1| \geq c(N,\alpha) > 0 \quad \& \quad |E^c \cap B_1| \geq c(N,\alpha) > 0.$$



Proof (see [CRS '10] for clear detailed proof in nonl. case)

By contradiction assume $|E \cap B_1|$ small. Then we prove E does not arrive to 0.





Isop. ineq. ⊕ Minimality of E

$$\frac{1}{C} V_r^{\frac{N-1}{N}} \leq |\partial V_r| = |E_r| + |A_r| \leq 2|A_r| = 2\frac{d}{dr}V_r$$

$$c \leq V_r^{\frac{1-N}{N}} V_r' = N\left(V_r^{1/N}\right)'$$

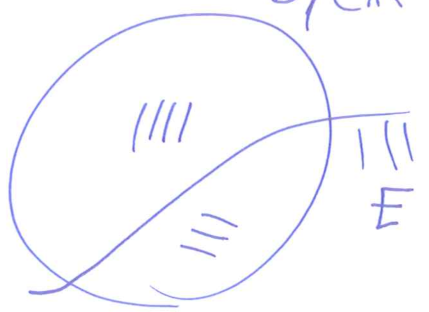
$$\frac{C}{N}\frac{1}{2} = \frac{C}{N}\int_{1/2}^{1} dr \leq \int_{1/2}^{1} \left(V_r^{1/N}\right)' dr = V_1^{1/N} - V_{1/2}^{1/N}$$

Nonlinear ODE that gives

$$|E \cap B_1| = V_1 \text{ small } \left(= \varepsilon(n)\right) \Longrightarrow V_{1/2} = 0 \iff E \cap B_{1/2} = \emptyset \quad \square$$

Now, from Thm 8 (density est) and the relative isop ineq.

$$|\partial E \cap B_1|^{\frac{N}{N-1}} \geq c(N) \min\left(|E \cap B_1|, |E^c \cap B_1|\right) \geq c(N) \quad (\text{or } c(N,\alpha))$$



and thus the density estimate (4) for the fract. Sob. ineq. holds (after rescaling). Hence:

Corol 9. The fract. Sobolev ineq. (5) holds on every minimizing nonlocal α-minimal surface $\forall \alpha \in (0,1)$ $\forall s \in (0,1)$ (take $M = \partial E \subset \mathbb{R}^{n+1}$, $\widehat{\mathbb{R}^{n+1}}\ \widehat{\mathbb{R}^N}$; $N = n+1$)

[Cabré-Cozzi, Agrad..... DMJ 2019]

6. Appl'n 1 (of fract MS ineq): a gradient estimate for nonl. min. graphs.

Theorem 10 [C-Cozzi DMJ 2019] $u:\mathbb{R}^n \to \mathbb{R}$

$E = \{(x', x_{n+1}) \in \mathbb{R}^n \times \mathbb{R} : x_{n+1} < u(x')\} \subset \mathbb{R}^{n+1}$, $n \geq 1$, $\alpha \in (0,1)$.

∂E nonl. α-min. surface in $B'_{2r} \times \mathbb{R}$ $\Longrightarrow$

$\Rightarrow u \in C^\infty(B'_r)$ &

$$\|\nabla_{x'} u\|_{L^\infty(B'_r)} \leq C(n,\alpha)\left(1 + \frac{\operatorname{osc}_{B'_r} u}{r}\right)^{n+1+\alpha}.$$

smoothness was only known for $n=1,2$ (Savin, Figalli, Vald.)

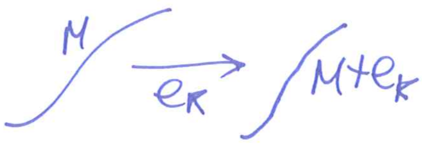
- In the local case, the r.h.s is $\exp(\operatorname{osc})$ instead of power: $n=2$ [Finn, 1963]
  $n \geq 3$ [Bombieri-De Giorgi-Miranda, 1969].

- Main ideas of the proof

(*) LOCAL CASE: $H = 0$ (1^{st} var. of perim) $M = \partial E$ min. surf.

Lineariz.

Jacobi op'r: $\underbrace{Jv = (\Delta_{LB} + |A|^2)v}$ (2^{nd} var of perim)



Second. fund. form of the min. surf $M = k_1^2 + \ldots + k_n^2$.

$H=0$ is invariant by translations & $e_k \cdot \nu_E = \nu_E^k$ $\Rightarrow$ Every component ν_E^k of the normal to M solves the Jacobi eqn: $\boxed{J\nu_E^k = 0 \text{ in } M \quad \forall k \in \{1,\ldots,n+1\}}$

⊛ NONLOCAL CASE $(M=\partial E)$

Jacobi operator : $J_\alpha v = \underbrace{\Big(\mathcal{L}_{\frac{1+\alpha}{2}} + \overbrace{c^2_{\frac{1+\alpha}{2}}}^{\text{frac. 2nd fund. form.}}\Big)}_{\text{where}} v = \left(\mathcal{L}_s + c_s^2\right) v$ $\quad \boxed{s=\frac{1+\alpha}{2}\in\left(\tfrac{1}{2},1\right)}$

$$\begin{cases} \mathcal{L}_{\frac{1+\alpha}{2}} v(x) := \displaystyle\int_M \frac{v(y)-v(x)}{|x-y|^{n+1+\alpha}}\, dV(y) \\ c^2_{\frac{1+\alpha}{2}}(x) := \displaystyle\int_M \frac{\langle \nu_E(x)-\nu_E(y), \nu_E(x)\rangle}{|x-y|^{n+1+\alpha}}\, dV(y). \end{cases}$$

Translation invariance of $H_\alpha = 0 \Rightarrow$

Lemma 11 (see Thm 1.3(i), and the key computation in Prop 4.1, both of [C-Cozzi, 2019])

$$\boxed{J_\alpha \nu_E^k = 0 \quad \text{in } M=\partial E \quad \forall k\in\{1,\dots,n+1\}.}$$

Now $0 = J_\alpha \nu_E^{n+1} = \mathcal{L}_s \nu_E^{n+1} + c_s^2(x)\, \nu_E^{n+1} \geq \mathcal{L}_s \nu_E^{n+1}$

"kind of" frac. Lapl $-(-\Delta_M)^s$

$$\left| \begin{array}{l} M \text{ is a graph } x_{n+1}=u(x') \\ \Rightarrow \nu_E^{n+1} = \dfrac{1}{\sqrt{1+|\nabla_{x'}u|^2}} > 0 \quad \text{in } \mathbb{R}^n \end{array} \right|$$

Hence, $0<\nu_E^{n+1}$ is a positive supersolution of the frac. eqn $\mathcal{L}_s \nu_E^{n+1} = 0$ in M.
$(\leq)$

Now, since we have a fractional Sobolev ineq. on the graph $M = \partial E$, a fractional version of the Moser iteration technique (first done in $\mathbb{R}^n$ by M. Kassmann,) to get a weak Harnack inequality (see [GT] for local case)
[Gilbarg-Trud.]
& show

Prop 12 [C-Cozzi 2019]

$$\boxed{\inf_{\partial E \cap B_1} \nu_E^{n+1} \geq C_{n\alpha} \left\{ \int_{\partial E \cap B_1} \nu_E^{n+1}(y)\, dV(y) + \int_{\partial E \setminus B_1} \frac{\nu_E^{n+1}(y)}{|y|^{n+1+\alpha}}\, dV(y) \right\}.}$$

Since $\underbrace{\inf \nu_E^{n+1}}_{\text{easy lower bdd.}} = \dfrac{1}{\sup \sqrt{1+|\nabla_x u|^2}}$ } gives immediately the grad. estimate.

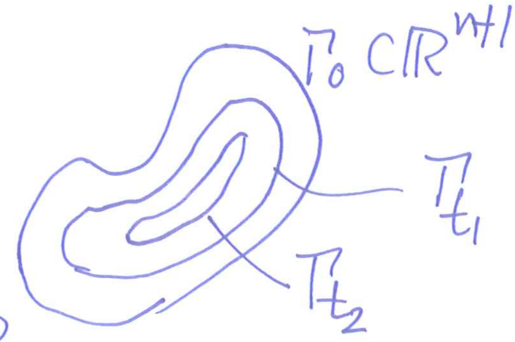
3rd & last lect. 4½h

7. Appl'n 2 (of fract MS inequalities): extinction times for (nonlocal) mean curvature flow.

- The local case: see [Evans, Regularity... LNM 1660, (1997)]

Let $\{T_t\}_{0 \leq t \leq T^*}$ be a smooth flow of hypersurfaces of $\mathbb{R}^{n+1}$ flowing by MCF (mean curvature flow):

"the surface flows with speed in the normal direction equal to $-H = -H(T_t)$" (shrinking)



this is a parabolic flow enjoying comparison principle. Thus,

if $\Gamma_0 \subset B_R = B_R(0)$, then $\Gamma_t \subset B_{R(t)}$

where

$R(t) = (R^2 - 2nt)^{1/2}$.

Since $R(t) = 0$ for $t = R^2/(2n)$ $\rightsquigarrow$

$\underline{\text{Extinction time} = T^* \leq C(n)\, \mathrm{diam}(\Gamma_0)^2}$

A better estimate, got using the MS ineq:

<u>Prop'n 13</u> $\{\Gamma_t\}_{0 \leq t \leq T^*}$ smooth MC flow in $\mathbb{R}^{n+1}$.

Then, $T^* \leq C(n)\, |\Gamma_0|^{\frac{2}{n}}$.

<u>Proof</u>: $\dfrac{d}{dt} |\Gamma_t| = - \displaystyle\int_{\Gamma_t} \langle \vec{v}, \vec{H} \rangle \, dV_t$

MC vector $= H \cdot \nu_{\Gamma_t}$

velocity of flow $(= H \nu_{\Gamma_t})$

$= - \displaystyle\int_{\Gamma_t} |H|^2 \, dV_t \leq$

$\leq -C(n)\, |\Gamma_t|^{\frac{n-2}{n}}$.

General result (see [Ecker], [Guan-Li] in refs of [Cabré-Cozzi-Csató])

$\hookrightarrow \dfrac{d}{dt} |\Gamma_t|^{2/n} \leq -\tilde{C}(n) = C \longrightarrow$

[use MS in $M = \Gamma_t$ compact hypers. with $u \equiv 1$ & $p = 2$]

$$\left.\begin{array}{l} |\Gamma_t|^{2/n} - |\Gamma_0|^{2/n} \leq -ct \\ \downarrow \text{ as } t \uparrow T^* = \text{extinct. time.} \\ 0 \end{array}\right\} \text{ Thus, } T^* \leq c|\Gamma_0|^{2/n}. \ \square$$

[Cabré-Cozzi-Csató, A fract. MS...., AIHP 2022] shows the analogue estimate:

Prop'n 14 $T^* \leq C(n,\alpha)\, |\Gamma_0|^{\frac{1+\alpha}{n}}$ for fractional MCF ($\alpha \in (0,1)$)

if Ω_0 is a C^2 bdd open convex set & $\Gamma_0 = \partial\Omega_0$.

It was known that convexity is preserved under the fract. MCF.

The proof, as before, relies in a now fractional MS Sobolev ineq. in convex sets of $\mathbb{R}^{n+1}$. The only other fract. MS ineq. known is the one on nonl. min. surf of [C-Cozzi].

Open pb: fract MS in general hypersurfaces.

Thm 15 [C-Cozzi-Csató] $n \geq 1$, $\alpha \in (0,1)$, $s \in (0,1)$, $p \geq 1$ st $sp < n$.

$\Omega \subset \mathbb{R}^{n+1}$ open convex set (perhaps unbdd). Then, $M = \partial\Omega$

$$\|u\|_{L^{p_s^*}(M)} \leq C(n,\alpha,s,p) \left\{ \int_M dV(x) \int_M dV(y) \frac{|u(x)-u(y)|^p}{|x-y|^{n+sp}} + \int_M H_\alpha(x)^{sp/\alpha} |u(x)|^p \, dV(x) \right\}^{1/p}$$

$\forall u \in W^{s,p}(M)$, ($p_s^* = np/(n-sp)$).

end