Optimal Chang-Yang's inequality for axially symmetric functions on $\mathbb{S}^4$ and $\mathbb{S}^6$

Juncheng Wei

University of British Columbia

Joint work with Changfeng Gui, Tuoxin Li, Zikai Ye

June 30, 2023, Granada
Summer School on Conformal Geometry and Non-local Operators

Moser-Trudinger-Onofri inequality on $\mathbb{S}^2$

Moser-Trudinger: There exists a constant $C_1 \geq 0$, such that

$$\frac{1}{2}\int_{\mathbb{S}^2} |\nabla u|^2 + \int_{\mathbb{S}^2} u \mathrm{d}w - \frac{1}{2}\log \int_{\mathbb{S}^2} e^{2u}\mathrm{d}w \geq -C_1$$

Here $\mathrm{d}w$ denotes the Lebesgue measure on the unit sphere $\mathbb{S}^2$, normalized to make $\int_{\mathbb{S}^2} \mathrm{d}w = 1$.

Moser-Trudinger-Onofri inequality on $\mathbb{S}^2$

Moser-Trudinger: There exists a constant $C_1 \geq 0$, such that

$$\frac{1}{2}\int_{\mathbb{S}^2} |\nabla u|^2 + \int_{\mathbb{S}^2} u \mathrm{d}w - \frac{1}{2}\log \int_{\mathbb{S}^2} e^{2u}\mathrm{d}w \geq -C_1$$

Here $\mathrm{d}w$ denotes the Lebesgue measure on the unit sphere $\mathbb{S}^2$, normalized to make $\int_{\mathbb{S}^2} \mathrm{d}w = 1$.

Onofri: C_1 can be taken to be 0 (which is optimal).

Moser-Trudinger-Onofri inequality on $\mathbb{S}^2$

Moser-Trudinger: There exists a constant $C_1 \geq 0$, such that

$$\frac{1}{2}\int_{\mathbb{S}^2} |\nabla u|^2 + \int_{\mathbb{S}^2} u \mathrm{d}w - \frac{1}{2}\log \int_{\mathbb{S}^2} e^{2u} \mathrm{d}w \geq -C_1$$

Here $\mathrm{d}w$ denotes the Lebesgue measure on the unit sphere $\mathbb{S}^2$, normalized to make $\int_{\mathbb{S}^2} \mathrm{d}w = 1$.

Onofri: C_1 can be taken to be 0 (which is optimal).

Moser-Trudinger-Onofri Inequality:

$$\frac{1}{2}\int_{\mathbb{S}^2} |\nabla u|^2 + \int_{\mathbb{S}^2} u \mathrm{d}w - \frac{1}{2}\log \int_{\mathbb{S}^2} e^{2u} \mathrm{d}w \geq 0.$$

Chang-Yang's Inequality

Let $u \in H^1(\mathbb{S}^2)$. Define a functional

$$J_\alpha(u) = \frac{\alpha}{2} \int_{\mathbb{S}^2} |\nabla u|^2 + \int_{\mathbb{S}^2} u \mathrm{d}w - \frac{1}{2} \log \int_{\mathbb{S}^2} e^{2u} \mathrm{d}w.$$

Restrict J_α to the set of functions with the center of mass at the origin:

$$\mathcal{L} = \left\{ u \in H^1(\mathbb{S}^2) \ : \ \int_{\mathbb{S}^2} e^{2u} \vec{x} \mathrm{d}w = 0 \right\}.$$

Chang-Yang Inequality

Chang and Yang (1982) conjectured that for $\alpha \geq \frac{1}{2}$,

$$\frac{\alpha}{2}\int_{\mathbb{S}^2} |\nabla u|^2 + \int_{\mathbb{S}^2} u \mathrm{d}w - \frac{1}{2}\log\int_{\mathbb{S}^2} e^{2u}\mathrm{d}w \geq 0$$

$$\forall u \in H^1(\mathbb{S}^2),\ \int_{\mathbb{S}^2} e^{2u}\vec{x}\mathrm{d}w = 0.$$

Chang-Yang Inequality

- Chang-Yang (1982): true if $\alpha > 1 - \epsilon$;

Chang-Yang Inequality

- Chang-Yang (1982): true if $\alpha > 1 - \epsilon$;
- Feldman, Froese, Ghoussoub and Gui (1998): True for axially symmetric functions when $\alpha > 0.64 - \epsilon$.

Chang-Yang Inequality

- Chang-Yang (1982): true if $\alpha > 1 - \epsilon$;
- Feldman, Froese, Ghoussoub and Gui (1998): True for axially symmetric functions when $\alpha > 0.64 - \epsilon$.
- Gui-Wei (2000): True for axially symmetric case when $\alpha \geq \frac{1}{2}$.

Chang-Yang Inequality

- Chang-Yang (1982): true if $\alpha > 1 - \epsilon$;
- Feldman, Froese, Ghoussoub and Gui (1998): True for axially symmetric functions when $\alpha > 0.64 - \epsilon$.
- Gui-Wei (2000): True for axially symmetric case when $\alpha \geq \frac{1}{2}$.
- Ghoussoub-C.S. Lin (2008): All solutions are axially symmetric if $\alpha > \frac{2}{3} - \epsilon$.

Chang-Yang Inequality

- Chang-Yang (1982): true if $\alpha > 1 - \epsilon$;
- Feldman, Froese, Ghoussoub and Gui (1998): True for axially symmetric functions when $\alpha > 0.64 - \epsilon$.
- Gui-Wei (2000): True for axially symmetric case when $\alpha \geq \frac{1}{2}$.
- Ghoussoub-C.S. Lin (2008): All solutions are axially symmetric if $\alpha > \frac{2}{3} - \epsilon$.
- Gui-Moradifam (2018): All solutions are axially symmetric if $\alpha \geq \frac{1}{2}$—Complete solution to Chang-Yang Inequality.

Beckner's Inequality: from $\mathbb{S}^2$ to $\mathbb{S}^n$

Beckner's inequality is a high-order Moser-Trudinger-Onofri inequality. Consider the following functional J_α defined in $H^{\frac{n}{2}}(\mathbb{S}^n)$ by

$$J_\alpha(u) = \frac{\alpha}{2}\int_{\mathbb{S}^n}(P_n u)u\mathrm{d}w + (n-1)!\int_{\mathbb{S}^n} u\mathrm{d}w - \frac{(n-1)!}{n}\log\int_{\mathbb{S}^n} e^{nu}\mathrm{d}w,$$

where

$$P_n = \begin{cases} \prod_{k=0}^{\frac{n-2}{2}}(\Delta + k(n-k-1)), & \text{for } n \text{ even;} \\ (-\Delta + (\frac{n-1}{2})^2)^{1/2}\prod_{k=0}^{\frac{n-3}{2}}(\Delta + k(n-k-1)), & \text{for } n \text{ odd} \end{cases}$$

is the Paneitz (GJMS) operator on $\mathbb{S}^n$.

Beckner's Inequality

Beckner (1993) : for $\alpha = 1$:

$$\frac{1}{2}\int_{\mathbb{S}^n}(P_n u)u\mathrm{d}w + (n-1)!\int_{\mathbb{S}^n} u\mathrm{d}w - \frac{(n-1)!}{n}\log\int_{\mathbb{S}^n} e^{nu}\mathrm{d}w \geq 0$$

$$\forall u \in H^{\frac{n}{2}}(\mathbb{S}^n)$$

Higher order Moser-Trudinger-Onofri inequality

Higher Order Chang-Yang's Inequality

Restrict J_α to the set of functions with the center of mass at the origin:

$$\mathcal{L} = \left\{ u \in H^{\frac{n}{2}}(\mathbb{S}^n) \ : \ \int_{\mathbb{S}^n} e^{nu} \vec{x} \mathrm{d}w = 0 \right\}.$$

Higher Order Chang-Yang's Inequality: for $\alpha \geq \frac{1}{2}$, the Beckner's inequality on $\mathbb{S}^n$ still holds, i.e.

$$\frac{\alpha}{2} \int_{\mathbb{S}^n} (P_n u) u \mathrm{d}w + (n-1)! \int_{\mathbb{S}^n} u \mathrm{d}w - \frac{(n-1)!}{n} \ln \int_{\mathbb{S}^n} e^{nu} \mathrm{d}w \geq 0$$

$$\forall u \in H^{\frac{n}{2}}(\mathbb{S}^n), \ \int_{\mathbb{S}^n} e^{nu} \vec{x} \mathrm{d}w = 0$$

Euler-Lagrange equation

The Euler-Lagrange equation of J_α is the Q-curvature-type equation

$$\alpha P_n u + (n-1)!(1 - \frac{e^{nu}}{\int_{\mathbb{S}^n} e^{nu} \mathrm{d}w}) = 0 \text{ on } \mathbb{S}^n. \tag{1}$$

Euler-Lagrange equation

The Euler-Lagrange equation of J_α is the Q-curvature-type equation

$$\alpha P_n u + (n-1)!(1 - \frac{e^{nu}}{\int_{\mathbb{S}^n} e^{nu} \mathrm{d}w}) = 0 \text{ on } \mathbb{S}^n. \tag{1}$$

Higher Order Chang-Yang Conjecture: for $\alpha \geq \frac{1}{2}$ all solutions to

$$\alpha P_n u + (n-1)!(1 - \frac{e^{nu}}{\int_{\mathbb{S}^n} e^{nu} \mathrm{d}w}) = 0 \text{ on } \mathbb{S}^n$$

subject to

$$\int_{\mathbb{S}^n} \vec{x} e^{nu} = 0$$

are constants.

Progress

- Chang-Yang (1995): For general n and any $\alpha > \frac{1}{2}$, there exists a constant $C(\alpha) \geq 0$ such that $\inf_{u \in \mathcal{L}} J_\alpha(u) \geq -C(\alpha)$.

Progress

- Chang-Yang (1995): For general n and any $\alpha > \frac{1}{2}$, there exists a constant $C(\alpha) \geq 0$ such that $\inf_{u \in \mathcal{L}} J_\alpha(u) \geq -C(\alpha)$.
- Wei-Xu (2009): True if $\alpha > 1 - \epsilon_n$.

Progress

- Chang-Yang (1995): For general n and any $\alpha > \frac{1}{2}$, there exists a constant $C(\alpha) \geq 0$ such that $\inf_{u \in \mathcal{L}} J_\alpha(u) \geq -C(\alpha)$.
- Wei-Xu (2009): True if $\alpha > 1 - \epsilon_n$.
- General case is very difficult. On $\mathbb{S}^2$, Gui-Moradifam (2018) first used spherical covering inequality and moving plane method to prove that all solutions are axially symmetric. Gui-Wei (2000): all axially symmetric solutions are constants.

Progress

- Chang-Yang (1995): For general n and any $\alpha > \frac{1}{2}$, there exists a constant $C(\alpha) \geq 0$ such that $\inf_{u \in \mathcal{L}} J_\alpha(u) \geq -C(\alpha)$.
- Wei-Xu (2009): True if $\alpha > 1 - \epsilon_n$.
- General case is very difficult. On $\mathbb{S}^2$, Gui-Moradifam (2018) first used spherical covering inequality and moving plane method to prove that all solutions are axially symmetric. Gui-Wei (2000): all axially symmetric solutions are constants.
- Question: what about axially symmetric solutions?

Axially symmetric case

If u is axially symmetric about ξ_1-axis and denoting ξ_1 by x, then the Euler-Lagrange equation becomes (1) is then reduced to

$$\alpha(-1)^{\frac{n}{2}}[(1-x^2)^{\frac{n}{2}}u']^{(n-1)}+(n-1)!-\frac{(n-1)!\sqrt{\pi}\Gamma(\frac{n}{2})}{\Gamma(\frac{n+1}{2})\gamma}e^{nu}=0, \quad (2)$$

where

$$\gamma := \int_{\mathbb{S}^n} e^{nu}dw = \int_{-1}^{1}(1-x^2)^{\frac{n-2}{2}}e^{nu}$$

In axially symmetric case, the set $\mathcal{L}$ is replaced by

$$\mathcal{L}_r = \{u \in H^{\frac{n}{2}}(\mathbb{S}^n) : \ u = u(x) \text{ and } \int_{-1}^{1} x(1-x^2)^{\frac{n-2}{2}}e^{nu}dx = 0\}.$$

Axially symmetric case

- $\mathbb{S}^4$: Gui-Hu-Xie (2021): For any $\alpha \in [0.517, 1)$, (2) admits only constant solutions.

Axially symmetric case

- $\mathbb{S}^4$: Gui-Hu-Xie (2021): For any $\alpha \in [0.517, 1)$, (2) admits only constant solutions.
- $\mathbb{S}^6$: Gui-Hu-Xie (2022): When $n = 6$, for any $\alpha \in [0.6168, 1)$, (2) admits only constant solutions.

Axially symmetric case

- $\mathbb{S}^4$: Gui-Hu-Xie (2021): For any $\alpha \in [0.517, 1)$, (2) admits only constant solutions.
- $\mathbb{S}^6$: Gui-Hu-Xie (2022): When $n = 6$, for any $\alpha \in [0.6168, 1)$, (2) admits only constant solutions.
- $\mathbb{S}^8$: Gui-Hu-Xie (2022): When $n = 8$, for any $\alpha \in [0.8261, 1)$, (2) admits only constant solutions.

Axially symmetric case

- $\mathbb{S}^4$: Gui-Hu-Xie (2021): For any $\alpha \in [0.517, 1)$, (2) admits only constant solutions.
- $\mathbb{S}^6$: Gui-Hu-Xie (2022): When $n = 6$, for any $\alpha \in [0.6168, 1)$, (2) admits only constant solutions.
- $\mathbb{S}^8$: Gui-Hu-Xie (2022): When $n = 8$, for any $\alpha \in [0.8261, 1)$, (2) admits only constant solutions.
- $\mathbb{S}^n, n \geq 2$: Gui-Hu-Xie (2022): For general n and any $\frac{1}{n+1} < \alpha < \frac{1}{2}$, there exists non-constant solution to (2).

Main Results

Theorem 1

(Li-Wei-Ye 2022) Let $n = 4$. If $\alpha \geq \frac{1}{2}$, then the only critical point of the functional J_α restricted to $\mathcal{L}_r$ are constant functions.

Main Results

Theorem 1

(Li-Wei-Ye 2022) Let $n = 4$. If $\alpha \geq \frac{1}{2}$, then the only critical point of the functional J_α restricted to $\mathcal{L}_r$ are constant functions.

Theorem 2

(Gui-Li-Wei-Ye 2023) Let $n = 6$. If $\alpha \geq \frac{1}{2}$, then the only critical point of the functional J_α restricted to $\mathcal{L}_r$ are constant functions.

Nonlocal Operator

- Chang-Yang's inequality for general odd n. Nonlocal operator
$$P_n = \sqrt{-\Delta + (\frac{n-1}{2})^2} \Pi_{k=0}^{\frac{n-3}{2}} (-\Delta + k(n-k-1))$$
$n = 1$: Chang-Hang 2020
- On $\mathbb{S}^1$, the Lebedev-Milin inequality yields that for any $u \in H^1(D)$ with $\int_{\mathbb{S}^1} u \mathrm{d}\theta = 0$,
$$\log \left(\frac{1}{2\pi} \int_{\mathbb{S}^1} e^u \mathrm{d}\theta \right) \leq \frac{1}{4\pi} \|\nabla u\|^2_{L^2(D)} .$$

Chang-Yang's inequality in terms of Szego Limit Theorem on $\mathbb{S}^1$

Using Szego Limit Theorem $\mathbb{S}^1$, Chang-Hang (2020) proved: If e^u satisfies more orthogonality conditions, i.e. $\int_{\mathbb{S}^1} e^u e^{ik\theta} \mathrm{d}\theta = 0$, for $k = 1, \cdots, m$, then we have

$$\log \left(\frac{1}{2\pi} \int_{\mathbb{S}^1} e^u \mathrm{d}\theta \right) \leq \frac{1}{4\pi(m+1)} \|\nabla u\|^2_{L^2(D)} .$$

Equivalently, for $\alpha \geq \frac{1}{m+1}$

$$\frac{\alpha}{2} \int_{\mathbb{S}^1} (P_1 u) u \mathrm{d}w + (n-1)! \int_{\mathbb{S}^1} u \mathrm{d}w - \frac{(n-1)!}{n} \ln \int_{\mathbb{S}^1} e^u \mathrm{d}w \geq 0$$

$$\forall u \in H^{\frac{1}{2}}(\mathbb{S}^1), \int_{S^1} u e^{ik\theta} d\theta = 0, k = 1, ..., m$$

Szego Limit Theorem on $\mathbb{S}^2$

On $\mathbb{S}^2$, Chang-Hang (2020) showed that for any $u \in H^1(\mathbb{S}^2)$ with $\int_{\mathbb{S}^2} u \mathrm{d}w = 0$ and $\int_{\mathbb{S}^2} pe^u \mathrm{d}w = 0$ for any p being the eigenfunction of $-\Delta_{\mathbb{S}^2}$ of eigenvalue $k(k+1)$, $k = 1, \cdots, m$, then

$$\log\left(\int_{\mathbb{S}^2} e^u \mathrm{d}w\right) \leq \left(\frac{1}{4\pi N_m} + \epsilon\right) \|\nabla u\|^2_{L^2(\mathbb{S}^2)} + c_\epsilon,$$

where N_m is an integer and c_ϵ is a constant.
It is unknown that whether or not ϵ can be chosen to be 0. Also, analogous results remain open for $\mathbb{S}^n$.

Proofs

- Proof on $\mathbb{S}^2$
- Proof on S^6

Proofs

- Proof on $\mathbb{S}^2$
- Proof on S^6
- Need to prove: for $\alpha \geq \frac{1}{2}$ all solutions

$$\alpha(-1)^{\frac{n}{2}}[(1-x^2)^{\frac{n}{2}}u']^{(n-1)} + (n-1)! - \frac{(n-1)!\sqrt{\pi}\Gamma(\frac{n}{2})}{\Gamma(\frac{n+1}{2})\gamma}e^{nu} = 0,$$

are constants.

Theorem in axially symmetric case on $\mathbb{S}^2$

On $\mathbb{S}^2$, the Euler-Lagrange equation (2) becomes

$$\alpha((1-x^2)u')' - 1 + \frac{2}{\gamma}e^{2u} = 0. \tag{3}$$

Theorem 3 (Gui-Wei 2000)

If $\frac{1}{2} \leq \alpha < 1$, then (3) admits only constant solutions.

Key Quantity G

Let $G(x) = (1 - x^2)u'(x)$. Then

$$\alpha G' - 1 + \frac{2}{\gamma} e^{2u} = 0. \tag{4}$$

$$(1 - x^2)G'' + \frac{2}{\alpha} G - 2GG' = 0. \tag{5}$$

Key Quantity G

Let $G(x) = (1 - x^2)u'(x)$. Then

$$\alpha G' - 1 + \frac{2}{\gamma}e^{2u} = 0. \tag{4}$$

$$(1 - x^2)G'' + \frac{2}{\alpha}G - 2GG' = 0. \tag{5}$$

Idea: Use Eigenfunction expansions to show that (5) (which is a nonlinear equation) has only zero solution.

Legendre polynomial expansion

Axially symmetric eigenfunctions on $\mathbb{S}^2$: Legendre polynomials $P_n(x)$

$$((1-x^2)P_k')' + \lambda_k P_k = 0, \lambda_k = k(k+1).$$

Moreover,

$$|P_k'(x)| \leq |P_k'(1)| = \frac{1}{2}\lambda_k, \int_{-1}^{1} P_m P_n = \frac{2\delta_{mn}}{2n+1}.$$

We have the orthogonal decomposition

$$G(x) = a_0 + \beta x + \sum_{k=2}^{\infty} a_k P_k(x).$$

Aim: show that

$$a_0 = a_1 = a_2 = \cdots = a_k = \ldots = 0$$

About a_0

Since the center of mass equals zero,

$$\int_{-1}^{1} xe^{2u} = 0$$

we derive that

$$a_0 = 0$$

$$G(x) = \beta x + \sum_{k=2}^{\infty} a_k P_k(x).$$

Some useful identities

Let $b_k^2 = a_k^2 \int_{-1}^1 P_k^2$, then by orthogonality,

$$\int_{-1}^1 G^2 = \frac{2}{3}\beta^2 + \sum_{k=2}^{\infty} b_k^2.$$

$$\int_{-1}^1 (1-x^2)(G')^2 = \frac{4}{3}\beta^2 + \sum_{k=2}^{\infty} \lambda_k b_k^2.$$

By the equation of P_k and integration by parts, we have

$$\int_{-1}^1 P_k G = -\frac{2}{\alpha\lambda_k} \int_{-1}^1 (1-x^2) P_k' \frac{e^{2u}}{\gamma}, k \geq 2. \tag{6}$$

By (4), we obtain

$$\int_{-1}^1 (1-x^2)\frac{e^{2u}}{\gamma} = \frac{2}{3}(1-\alpha\beta). \tag{7}$$

Some useful identities

The following two identities play key roles in the proof. Multiplying (5) by x and integrating by parts yields

$$\int_{-1}^{1} G^2 = \frac{4}{3}(3 - \frac{1}{\alpha})\beta. \tag{8}$$

Similarly, multiplying (5) by G, we get

$$\int_{-1}^{1}(1-x^2)(G')^2 = (\frac{2}{\alpha} - 1)\int_{-1}^{1} G^2. \tag{9}$$

We remark that in the last integral, the cubic term $\int_{-1}^{1} G^2 G' = 0$, which makes the proof very easy. This is also the main difference between $\mathbb{S}^2$ and $\mathbb{S}^n$, $n \geq 4$.

A rough estimate

We will show $\beta = 0$, which implies $G = 0$ by (8). The basic strategy is to show that if $\beta \neq 0$, then

$$\beta = \frac{1}{\alpha},$$

which contradicts to (7).

$$\int_{-1}^{1}(1 - x^2)\frac{e^{2u}}{\gamma} = \frac{2}{3}(1 - \alpha\beta).$$

Now we assume $\beta \neq 0$, then by (7), $\frac{1}{\alpha} - \beta > 0$.
Rest of the idea: derive estimates of the rest coefficients in terms of

$$\frac{1}{\alpha} - \beta$$

and do iterations.

We first derive an estimate on b_k^2. For $k \geq 2$, by (6) and (7), we have

$$b_k^2 = \frac{2k+1}{2}\left(\frac{2}{\alpha\lambda_k}\int_{-1}^{1}(1-x^2)|P_k'|\frac{e^{2u}}{\gamma}\right)^2$$
$$\leq \frac{2k+1}{2}\left(\frac{2}{\alpha\lambda_k}\frac{\lambda_k}{2}\frac{2}{3}(1-\alpha\beta)\right)^2$$
$$= \frac{2(2k+1)}{9}(\frac{1}{\alpha}-\beta)^2. \qquad (10)$$

Here we used uniform estimate

$$|P_k'| \leq |P_k'(1)| = \frac{\lambda_k}{2}$$

Rough estimates

Now we define the key semi-norm:

$$D := \sum_{k=3}^{\infty} (\lambda_k - 6) b_k^2.$$

On the one hand, $D \geq 0$ since $\lambda_k = k(k+1)$. On the other hand,

$$\begin{aligned} D &= \int_{-1}^{1} (1-x^2)(G')^2 - 6\int_{-1}^{1} G^2 + \frac{4}{3}\beta^2 \\ &= \frac{2}{3}\beta\left(4\beta + (7 - \frac{2}{\alpha})(\frac{2}{\alpha} - 6)\right) \end{aligned} \tag{11}$$

In view of the fact that $0 < \beta < \frac{1}{\alpha}$, we have the following rough estimates

$$\beta \geq 1.5, \ \ \alpha < 0.537.$$

Lower bound of D

To obtain better estimates, we need to estimate the lower bound of D more carefully. We fix an integer $n \geq 3$, then

$$
\begin{aligned}
D &= \sum_{k=3}^{n} (\lambda_k - 6) b_k^2 + \sum_{k=n+1}^{\infty} (\lambda_k - 6) b_k^2 \\
&\geq \sum_{k=3}^{n} (\lambda_k - 6) b_k^2 + \frac{\lambda_{n+1} - 6}{\lambda_{n+1}} \sum_{k=n+1}^{\infty} \lambda_k b_k^2 \\
&= \sum_{k=2}^{n} (\lambda_k - 6 - \frac{\lambda_{n+1} - 6}{\lambda_{n+1}} \lambda_k) b_k^2 - \frac{\lambda_{n+1} - 6}{\lambda_{n+1}} \sum_{k=2}^{\infty} \lambda_k b_k^2 \\
&= \sum_{k=2}^{n} 6 \frac{\lambda_k - \lambda_{n+1}}{\lambda_{n+1}} b_k^2 + \frac{\lambda_{n+1} - 6}{\lambda_{n+1}} (\int_{-1}^{1} (1 - x^2)(G')^2 - \frac{4}{3} \beta^2) \\
&= \sum_{k=2}^{n} 6 \frac{\lambda_k - \lambda_{n+1}}{\lambda_{n+1}} b_k^2 + \frac{\lambda_{n+1} - 6}{\lambda_{n+1}} \left(\frac{2}{3} \beta (\frac{2}{\alpha} - 1)(6 - \frac{2}{\alpha}) - \frac{4}{3} \beta^2 \right).
\end{aligned}
\tag{12}
$$

Combining (11) and (12), after some simple computation, we obtain

$$12\beta(\frac{1}{\alpha}-2)+\frac{4\beta}{\lambda_{n+1}}\left((\frac{2}{\alpha}-1)(6-\frac{2}{\alpha})-\frac{2}{\alpha}\right) \tag{13}$$
$$\geq 4\beta(1-\frac{2}{\lambda_{n+1}})(\frac{1}{\alpha}-\beta)-\sum_{k=2}^{n}6\frac{\lambda_k-\lambda_{n+1}}{\lambda_{n+1}}b_k^2.$$

Since $\frac{1}{2}\leq\alpha<1$,

$$12\beta(\frac{1}{\alpha}-2)+\frac{4\beta}{\lambda_{n+1}}\left((\frac{2}{\alpha}-1)(6-\frac{2}{\alpha})-\frac{2}{\alpha}\right)\leq\frac{8\beta}{\lambda_{n+1}}, \tag{14}$$

which, together with estimates of b_k (10), yields the inequality

$$\frac{8\beta}{\lambda_{n+1}}\geq(4\beta(1-\frac{2}{\lambda_{n+1}})-\frac{20}{3}\frac{\lambda_{n+1}-6}{\lambda_{n+1}}(\frac{1}{\alpha}-\beta)-\frac{4}{3}c_n(\frac{1}{\alpha}-\beta))(\frac{1}{\alpha}-\beta), \tag{15}$$

Induction procedure

where

$$c_n = \sum_{k=3}^{n} \frac{\lambda_{n+1} - \lambda_k}{\lambda_{n+1}}(2k+1) = \frac{1}{2}\lambda_{n+1} - 9 + \frac{36}{\lambda_{n+1}}. \qquad (16)$$

We claim

$$\frac{1}{\alpha} - \beta \leq \frac{4}{\lambda_n}, \forall n \geq 4. \qquad (17)$$

This is proved by induction procedure. Two key ingredients

- semi-norm D
- decaying estimates of b_k

$$b_k^2 \leq \frac{2(2k+1)}{9}(\frac{1}{\alpha} - \beta)^2.$$

Finally, letting $n \to +\infty$ in (17), we obtain

$$\frac{1}{\alpha} - \beta = 0,$$

which is a contradiction. From the discussion in the beginning, we know $G \equiv 0$, which implies that u is a constant. Thus we complete the proof of Theorem 3.

Statement of theorems on $\mathbb{S}^4$ and $\mathbb{S}^6$

On $\mathbb{S}^4$, (2) becomes

$$\alpha((1-x^2)^2 u')''' + 6 - \frac{8}{\gamma} e^{4u} = 0 \tag{18}$$

On $\mathbb{S}^6$, (2) becomes

$$-\alpha[(1-x^2)^3 u']^{(5)} + 120 - 128\frac{e^{6u}}{\gamma} = 0, \;\; x \in (-1,1). \tag{19}$$

Theorem 4 (Li-Wei-Ye 2022, Gui-Li-Wei-Ye 2023)

If $\frac{1}{2} \leq \alpha < 1$, then (18) and (19) admit only constant solutions.

Key ingredients

- Obtain the optimal semi-norm estimates
- Use the decaying properties of Gegenbauer polynomials to obtain sharp estimates of the coefficents b_k
- Use the cancellation properties of Gegenbauer polynomials to proceed with the induction steps.

Axially symmetric eigenfunctions for the Paneitz operator P_n: Gegenbauer polynomials

Gegenbauer polynomials, order ν and degree k, are given by

$$C_k^\nu(x) = \frac{(-1)^k}{2^k k!}\frac{\Gamma(\nu+\frac{1}{2})\Gamma(k+2\nu)}{\Gamma(2\nu)\Gamma(\nu+k+\frac{1}{2})}(1-x^2)^{-\nu+\frac{1}{2}}\frac{d^k}{dx^k}(1-x^2)^{k+\nu-\frac{1}{2}}.$$

C_k^ν is an even function if k is even and it is odd if k is odd. The derivative of C_k^ν satisfies

$$\frac{d}{dx}C_k^\nu(x) = 2\nu C_{k-1}^{\nu+1}(x). \tag{20}$$

Let F_k^ν be the normalization of C_k^ν such that $F_k^\nu(1) = 1$, i.e.

$$F_k^\nu = \frac{k!\Gamma(2\nu)}{\Gamma(k+2\nu)}C_k^\nu. \tag{21}$$

Decaying properties of Gegenbauer polynomials

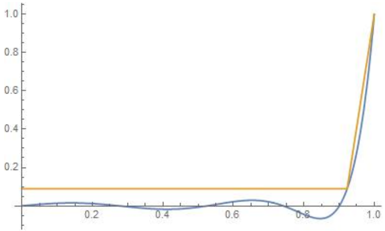

Figure: Graph of $\tilde{F}'_{10}$

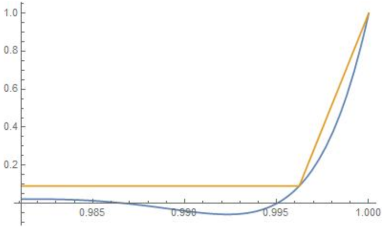

Figure: Graph of $\tilde{F}'_{50}$ near 1

Cancellation of consecutive Gegenbauer polynomials

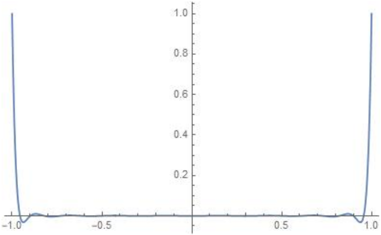

Figure: Graph of $\tilde{F}'_{19}$

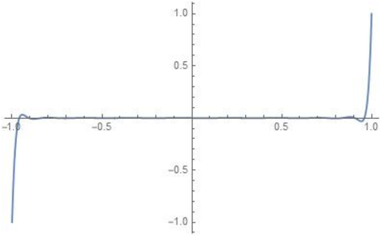

Figure: Graph of $\tilde{F}'_{20}$

In the rest of the talk, I will discuss the proof of $\mathbb{S}^6$:

Theorem 5 (Gui-Li-Wei-Ye 2023)

For $\alpha \geq \frac{1}{2}$, all solutions to

$$\alpha[(1-x^2)^3 u']^{(5)} + 120 - 128\frac{e^{6u}}{\int (1-x^2)^2 e^{6u}} = 0$$

must be constants.

Gegenbauer polynomials

On $\mathbb{S}^6$, the corresponding Gegenbauer polynomial is $C_k^{\frac{5}{2}}$. For notational simplicity, in what follows we will write F_k for
$F_k^{\frac{5}{2}} = \frac{k!4!}{(k+4)!} C_k^{\frac{5}{2}}$.
It turns out that F_k satisfies

$$(1-x^2)F_k'' - 6xF_k' + \lambda_k F_k = 0 \tag{22}$$

and

$$\int_{-1}^{1} (1-x^2)F_k F_l = \frac{128}{(2k+5)(\lambda_k+4)(\lambda_k+6)}\delta_{kl}, \tag{23}$$

where $\lambda_k = k(k+5)$.

Gegenbauer expansion

Similarly, we define $G = (1-x^2)u'$. Then G satisfies the equation

$$\alpha[(1-x^2)^2 G]^{(5)} + 120 - 128\frac{e^{6u}}{\gamma} = 0 \tag{24}$$

and

$$(1-x^2)^3[(1-x^2)^2 G]^{(6)} + \frac{720}{\alpha}(1-x^2)^2 G \tag{25}$$
$$-6(1-x^2)^2 G[(1-x^2)^2 G]^{(5)} = 0.$$

Expand G in terms of Gegenbauer polynomials

$$G = \beta x + a_2 F_2(x) + \sum_{k=3}^{\infty} a_k F_k(x). \tag{26}$$

Integral Identities

Denote

$$g = (1 - x^2)^2 \frac{e^{6u}}{\gamma}, \;\; a := \int_{-1}^{1} (1 - x^2) g. \tag{27}$$

Testing the equations of G by F_1, $\int_{-1}^{x} (1 - s^2)^{\frac{n-2}{2}} F_k(s) ds$, x respectively and integrating by parts, we obtain the following integral identities

$$\int_{-1}^{1} (1 - x^2) F_1 G = \frac{16}{105} \beta, \tag{28}$$

$$a = \int_{-1}^{1} (1 - x^2) g = \frac{6}{7} (1 - \alpha\beta), \tag{29}$$

$$\int_{-1}^{1} (1 - x^2) F_k G = - \frac{128}{\alpha(\lambda_k + 4)(\lambda_k + 6)} \int_{-1}^{1} (1 - x^2) g F_k', \;\; k \geq 2, \tag{30}$$

$$\int_{-1}^{1} |[(1 - x^2)^2 G]''|^2 = \frac{256}{35} (7 - \frac{1}{\alpha}) \beta. \tag{31}$$

Semi-norm

To get a rough estimate of β and $a = \frac{6}{7}(1 - \alpha\beta)$, we need an estimate of $\lfloor G \rfloor^2$ defined as following

$$\lfloor G \rfloor^2 = -\int_{-1}^{1} (1 - x^2)^2 [(1 - x^2)^3 G']^{(5)} G. \tag{32}$$

By integrating by parts and applying the equation of G, we obtain

$$\lfloor G \rfloor^2 = -15 \int_{-1}^{1} |[(1 - x^2)^2 G]''|^2 + \frac{720}{\alpha} \int_{-1}^{1} (1 - x^2)^2 G^2$$
$$+30 \int_{-1}^{1} (1 - x^2)^4 G'(G'')^2 + 160 \int_{-1}^{1} (1 - x^2)^3 (G')^3.$$

We need to estimate the last two cubic terms.

Gui-Hu-Xie's estimates of $\lfloor G \rfloor^2$

To estimate $\lfloor G \rfloor^2$, Gui-Hu-Xie applied the following lemma

Lemma 6 (Lemma 3.2 in Gui-Hu-Xie 2022)

For all $x \in (-1, 1)$, we have

$$G_j := (-1)^j [(1 - x^2)^j G]^{(2j+1)} \leq \frac{(2j+1)!}{\alpha}, \ 0 \leq j \leq 2. \tag{33}$$

to obtain

$$G' \leq \frac{1}{\alpha}$$

Applying it directly to the last two integrals, they obtained

$$30\int_{-1}^{1}(1-x^2)^4 G'(G'')^2 + 160\int_{-1}^{1}(1-x^2)^3(G')^3$$

$$\leq \frac{30}{\alpha}\int_{-1}^{1}(1-x^2)^4(G'')^2 + \frac{160}{\alpha}\int_{-1}^{1}(1-x^2)^3(G')^2$$

$$\lfloor G \rfloor^2 \leq (\frac{30}{\alpha} - 15)\int_{-1}^{1}|[(1-x^2)^2 G]''|^2 - \frac{320}{\alpha}\int_{-1}^{1}(1-x^2)^3(G')^2. \tag{34}$$

However, this estimate is not enough to obtain a rough bound for β and we need more refined estimates.

Refined estimates of semi-norms $\lfloor G \rfloor^2$

We claim that in fact,

$$30\int_{-1}^{1}(1-x^2)^4 G'(G'')^2 + 160\int_{-1}^{1}(1-x^2)^3(G')^3 \leq \frac{160}{\alpha}\int_{-1}^{1}(1-x^2)^3(G')^2.$$

Compared with Gui-Hu-Xie's estimate, our estimates can be viewed formally as dropping the first integral and applying $G' \leq \frac{1}{\alpha}$ to the second integral.

As a consequence, we obtain refined estimates of $\lfloor G \rfloor^2$.

Proposition 1 (Gui-Li-Wei-Ye 2023)

$$\lfloor G \rfloor^2 \leq -15\int_{-1}^{1}|[(1-x^2)^2 G]''|^2 + \frac{720}{\alpha}\int_{-1}^{1}(1-x^2)^2 G^2 + \frac{160}{\alpha}\int_{-1}^{1}(1-x^2)^3(G')^2$$

Proof of Proposition 1

Integrating (34) by parts, we get

$$\lfloor G \rfloor^2 = -15\int_{-1}^{1} |[(1-x^2)^2 G]''|^2 + \frac{720}{\alpha}\int_{-1}^{1}(1-x^2)^2 G^2 + \int_{-1}^{1}(1-x^2)^3 \tilde{G}(G')^2,$$

where

$$\tilde{G} = -15(1-x^2)G''' + 120xG'' + 160G'. \tag{35}$$

Let

$$\hat{G} = -15(1-x^2)G''' + 120xG'' + 150G'. \tag{36}$$

Direct calculation yields that $\hat{G}$ satisfies

$$(1-x^2)\hat{G}'' - 8x\hat{G}' - 12\hat{G} = -15[(1-x^2)^2 G]^{(5)} \geq -\frac{1800}{\alpha}.$$

By Maximum Principle

$$\hat{G} \leq \frac{150}{\alpha}.$$

Proof of the main theorem

We claim that $\beta = 0$, which yields that $(1-x^2)^2 G$ is a linear function by (31). Since G is bounded on $(-1,1)$, we get $G \equiv 0$ and we are done.

So it suffices to show that $\beta = 0$. We will argue by contradiction. If $\beta \neq 0$, then $0 < \beta < \frac{1}{\alpha}$ since

$$a = \int_{-1}^{1} (1-x^2) g = \frac{6}{7}(1-\alpha\beta) > 0.$$

It then suffices to show $a = 0$. We will achieve this by proving

$$a = \frac{6}{7}(1-\alpha\beta) \leq \frac{16}{\lambda_n}, \ \forall n \geq 5 \text{ odd}. \tag{37}$$

Rough estimates

We first derive rough estimates on β and a. To begin with, we define $b_k^2 = a_k^2 \int_{-1}^1 (1-x^2) F_k^2$ and introduce the quantity

$$D = \sum_{k=3}^{\infty} \Big[\lambda_k(\lambda_k+4)(\lambda_k+6) - (14 - \frac{74}{9\alpha})(\lambda_k+4)(\lambda_k+6) - \frac{160}{\alpha}\lambda_k - \frac{720}{\alpha} \Big] b_k^2 .$$

Recalling the estimates of $\lfloor G \rfloor^2$ and the integral identities, we get

$$\begin{aligned} D =& \lfloor G \rfloor^2 - (14 - \frac{74}{9\alpha}) \int_{-1}^1 |[(1-x^2)^2 G]''|^2 - \frac{160}{\alpha} \int_{-1}^1 (1-x^2)^3 (G')^2 \\ & - \frac{720}{\alpha} \int_{-1}^1 (1-x^2)^2 G^2 + \frac{16}{105}(\frac{2080}{3\alpha} + 960)\beta^2 \\ \le& (\frac{74}{9\alpha} - 29) \int_{-1}^1 |[(1-x^2)^2 G]''|^2 + \frac{16}{105}(\frac{2080}{3\alpha} + 960)\beta^2 \\ =& \frac{256}{35}(\frac{74}{9\alpha} - 29)(7 - \frac{1}{\alpha})\beta + \frac{512}{7}(\frac{13}{9\alpha} + 2)\beta^2 . \end{aligned} \tag{38}$$

Rough estimates

Since $D \geq 0$, $\alpha \geq \frac{1}{2}$ and $0 < \beta < \frac{1}{\alpha}$, we obtain

$$\beta \geq \frac{9}{440}(29 - \frac{74}{9\alpha})(7 - \frac{1}{\alpha}) \geq \frac{113}{88}, \tag{39}$$

and

$$\frac{256}{35}(\frac{74}{9\alpha} - 29)(7 - \frac{1}{\alpha}) + \frac{512}{7}(\frac{13}{9\alpha} + 2)\frac{1}{\alpha} \geq 0, \tag{40}$$

which implies that

$$\alpha < 0.578. \tag{41}$$

Lower bound of D

On the other hand, fix any integer $n \geq 3$, we have

$$D = \sum_{k=3}^{\infty} \left[\lambda_k(\lambda_k + 4)(\lambda_k + 6) - (14 - \frac{74}{9\alpha})(\lambda_k + 4)(\lambda_k + 6) \right.$$

$$\left. -\frac{160}{\alpha}\lambda_k - \frac{720}{\alpha} \right] b_k^2$$

$$\geq \sum_{k=n+1}^{\infty} \left[\lambda_{n+1} - 14 + \frac{74}{9\alpha} - \frac{160\lambda_{n+1} + 720}{(\lambda_{n+1} + 4)(\lambda_{n+1} + 6)\alpha} \right]$$

$$\cdot (\lambda_k + 4)(\lambda_k + 6) b_k^2$$

$$\sum_{k=3}^{n} \left[\lambda_k - 14 + \frac{74}{9\alpha} - \frac{160\lambda_k + 720}{(\lambda_k + 4)(\lambda_k + 6)\alpha} \right] (\lambda_k + 4)(\lambda_k + 6) b_k^2$$

$$\geq (\lambda_{n+1} - 14 + \frac{275}{63\alpha}) \sum_{k=n+1}^{\infty} (\lambda_k + 4)(\lambda_k + 6) b_k^2$$

$$+ \sum_{k=3}^{n} (\lambda_k - 14 + \frac{176}{63}\alpha)(\lambda_k + 4)(\lambda_k + 6) b_k^2.$$

Bounds of D

The right hand side of the inequality above is equal to

$$\sum_{k=3}^{n}(\lambda_k - \lambda_{n+1} - \frac{11}{7\alpha})(\lambda_k + 4)(\lambda_k + 6)b_k^2$$
$$+(\lambda_{n+1} - 14 + \frac{275}{63\alpha})\left[\frac{256}{35}(7 - \frac{1}{\alpha})\beta - \frac{128}{7}\beta^2 - 360b_2^2\right].$$

Combining the lower bound above and the upper bound (38) of D, we get

$$0 \leq \frac{256}{35}(7 - \frac{1}{\alpha})(\frac{27}{7\alpha} - 15 - \lambda_{n+1})\beta + \frac{128}{7}(\lambda_{n+1} - 6 + \frac{71}{7\alpha})\beta^2$$
$$+ \frac{176}{63\alpha}(\lambda_2 + 4)(\lambda_2 + 6)b_2^2$$
$$+ \sum_{k=2}^{n}(\lambda_{n+1} - \lambda_k + \frac{11}{7\alpha})(\lambda_k + 4)(\lambda_k + 6)b_k^2. \tag{42}$$

Gegenbauer coefficients b_k

Then we need to estimate b_k. In Gui-Hu-Xie, they used (30) and the following uniform estimate

$$|F_k'(x)| \leq |F_k'(1)| = \frac{\lambda_k}{6} \tag{43}$$

to estimate b_k as follows

$$\begin{aligned} b_k^2 &= a_k^2 \int_{-1}^{1} (1-x^2) F_k^2 = \frac{1}{\int_{-1}^{1}(1-x^2)F_k^2} \left[\frac{128}{\alpha\lambda_k} \int_{-1}^{1} (1-x^2) g F_k' \right]^2 \\ &\leq \frac{(2k+5)(\lambda_k+4)(\lambda_k+6)}{128} \left[\frac{128}{\alpha\lambda_k(\lambda_k+4)(\lambda_k+6)} \frac{\lambda_k}{6} a \right]^2 \\ &= \frac{32(2k+5)}{9\alpha^2(\lambda_k+4)(\lambda_k+6)} a^2. \end{aligned}$$

Refined estimates on b_k

However, this estimate is not strong enough to deduce the induction

$$a = \frac{6}{7}(1 - \alpha\beta) \leq \frac{d_0}{\lambda_n}. \tag{44}$$

Likewise, we need a refined estimate on b_k, which follows from the following refined estimate on Gegenbauer polynomials. For simplicity, we denote

$$\tilde{F}'_k = \frac{6}{\lambda_k} F'_k = \frac{720}{\lambda_k(\lambda_k + 4)(\lambda_k + 6)} C^{\frac{7}{2}}_{k-1} \tag{45}$$

so that $\tilde{F}'_k(1) = 1$. We split the integral in the right hand side of b_k into two parts. To this end, we define

$$a_+ := \int_0^1 (1 - x^2) g, \; a_- := \int_{-1}^0 (1 - x^2) g,$$

$$A_k^+ := \int_0^1 (1 - x^2) \tilde{F}'_k g, \; A_k^- := \int_{-1}^0 (1 - x^2) \tilde{F}'_k g.$$

$$a = a_+ + a_-, a_+ = \lambda a$$

Refined estimates on b_k

The following theorem gives a refined estimate on $A_k^{\pm}$, hence on b_k.

Theorem 7 (Gui-Li-Wei-Ye 2023)

Let $d = 8$, $b = 0.33$. Suppose $a \leq \frac{16}{\lambda_n}$ for some $n \geq 3$. Then for all even k, we have

$$\max\{|A_k^+|, |A_{k+1}^+|\} \leq \mathcal{A}_k^+ := \begin{cases} a_+ - \frac{1-b}{d}\lambda_k a_+^2, & \text{if } \lambda_k \leq \frac{\lambda_n}{4}, \\ b a_+ + (1-b)\frac{d}{4\lambda_k}, & \text{if } \frac{\lambda_n}{4} < \lambda_k \leq \lambda_n, \end{cases}$$

$$\max\{|A_k^-|, |A_{k+1}^-|\} \leq \mathcal{A}_k^- := \begin{cases} a_- - \frac{1-b}{d}\lambda_k a_-^2, & \text{if } a_- \leq \frac{4}{\lambda_n}, \\ b a_- + (1-b)\frac{d}{4\lambda_k}, & \text{if } \frac{4}{\lambda_n} < a_- \leq \frac{8}{\lambda_n}. \end{cases}$$

The proof relies on pointwise estimates of Gegenbauer polynomials.

Decaying properties of Gegenbauer polynomials

Lemma 8 (Gui-Li-Wei-Ye 2023)

For all $k \geq 8$, *we have*

$$\widetilde{F}_k' \geq -0.04, \quad 0 \leq x \leq 1.$$

Lemma 9 (Gui-Li-Wei-Ye 2023)

Let $d = 8$ *and* $b = 0.33$. *Then for all* $k \geq 6$,

$$\widetilde{F}_k' \leq \begin{cases} b, & 0 \leq x \leq 1 - \dfrac{d}{\lambda_k}, \\ 1 - \dfrac{\lambda_k}{d}(1-b)(1-x), & 1 - \dfrac{d}{\lambda_k} \leq x \leq 1. \end{cases}$$

Behavior of $\tilde{F}'_k$

The above two lemmas can be illustrated in the following figures.

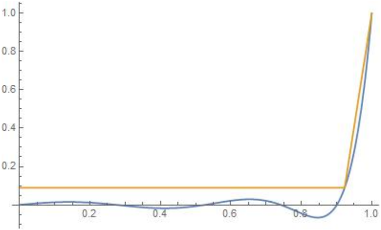

Figure: Graph of $\tilde{F}'_{10}$

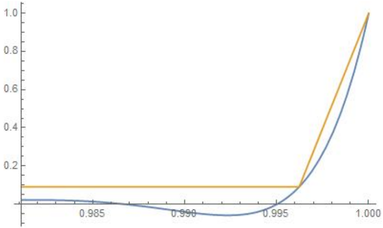

Figure: Graph of $\tilde{F}'_{50}$ near 1

The above two lemmas can be proved using the following point-wise estimates:

Lemma 10 (Corollary 5.3 in Nemes-Olde Daalhuis 2019)

Let $0 < \zeta < \pi$, $\nu > 0$ *and* $N \geq \nu - 1$ *be an integer. Then*

$$C^{\nu}_{k-1}(\cos\zeta) = \frac{2}{\Gamma(\nu)(2\sin\zeta)^{\nu}} \left(\sum_{n=0}^{N-1} t_n(\nu - \frac{1}{2}) \frac{\Gamma(k-1+2\nu)}{\Gamma(k+n+\nu)} * \frac{\cos\left(\delta_{\nu,k-1,n}\right)}{\sin^n\zeta} + R_N(\nu,\zeta,k-1) \right),$$

where $\delta_{k,n} = (k+n+\nu)\zeta - (\nu - n)\frac{\pi}{2}$, $t_n(\mu) = \frac{(\frac{1}{2}-\mu)_n(\frac{1}{2}+\mu)_n}{(-2)^n n!}$, *and* $(x)_n = \frac{\Gamma(x+n)}{\Gamma(x)}$. *The remainder term* R_N *satisfies the estimate*

$$|R_N(\nu,\zeta,k)| \leq \frac{|t_N(\nu-\frac{1}{2})|\Gamma(k-1+2\nu)}{\Gamma(k+N+\nu)\sin^N\zeta} \cdot \begin{cases} |\sec\zeta| & \text{if } 0 < \zeta \leq \frac{\pi}{4} \\ & \text{or } \frac{3\pi}{4} \leq \zeta < \pi, \\ 2\sin\zeta & \text{if } \frac{\pi}{4} < \zeta < \frac{3\pi}{4}. \end{cases}$$

Proof of Theorem 7

With the help of above two lemmas, we are able to prove Theorem 7. In the following argument, we may assume $k \geq 6$ and omit the details for $3 \leq k \leq 5$. Define $I = (0, 1 - \frac{d}{\lambda_k})$, $II = (1 - \frac{d}{\lambda_k}, 1)$, and $a_I = \int_I (1-x^2)g$, $a_{II} = \int_{II}(1-x^2)g$. Then by Lemma 9, we have

$$\begin{aligned}
\int_0^1 (1-x^2)\widetilde{F}_k' g &= \int_I (1-x^2)\widetilde{F}_k' g + \int_{II} (1-x^2)\widetilde{F}_k' g \\
&\leq \int_I (1-x^2) bg + \int_{II} (1-x^2)(1 - \frac{\lambda_k}{d}(1-b)(1-x))g \\
&= ba_I + a_{II} - \frac{\lambda_k}{d}(1-b)\int_{II}(1-x^2)(1-x)g \\
&\leq ba_I + a_{II} - \frac{\lambda_k}{d}(1-b)\frac{(\int_{II}(1-x^2)g)^2}{\int_{II}(1+x)g} \\
&\leq ba_I + a_{II} - \frac{\lambda_k}{d}(1-b)a_{II}^2 \\
&= ba_+ + (1-b)(a_{II} - \frac{\lambda_k}{d}a_{II}^2).
\end{aligned}$$

If $\lambda_k \le \frac{\lambda_n}{4}$, we have $a_{II} \le a_+ \le a \le \frac{16}{\lambda_n} \le \frac{d}{2\lambda_k}$. Hence,

$$\int_0^1 (1-x^2)\widetilde{F}_k' g \le a_+ + (1-b)(a_+ - \frac{\lambda_k}{d} a_+^2) = a_+ - \frac{\lambda_k}{d}(1-b)a_+^2.$$

For the case when $\lambda_k > \frac{\lambda_n}{4}$, we get directly

$$\int_0^1 (1-x^2)\widetilde{F}_k' g \le b a_+ + (1-b)\frac{d}{4\lambda_k}.$$

On the other hand, Lemma 8 yields

$$\int_0^1 (1-x^2)\widetilde{F}_k' g \ge -0.04 \int_0^1 (1-x^2) g = -0.04 a_+.$$

Combining the above three estimates, we obtain the desired estimate on A_k^+. The estimate on A_{k+1}^+ is similar.
Similarly, on estimating A_k^- and A_{k+1}^-, just note that $a_- \le \frac{a}{2} \le \frac{8}{\lambda_n}$.
We can go through analogous proof. We omit the details. $\square$

Induction procedure

Now we can start the induction procedure to prove $a \leq \frac{16}{\lambda_n}$, for all odd $n \geq 5$. Note that from our rough estimates of β (39) and α (41), we already have $a \leq 0.221 \leq \frac{16}{\lambda_5}$.
By induction, now we assume $a \leq \frac{16}{\lambda_n}$ for some $n \geq 5$ odd. Then we will show that $a \leq \frac{16}{\lambda_{n+2}}$. We argue by contradiction and suppose $a > \frac{16}{\lambda_{n+2}}$ on the contrary.

Induction procedure

Now we estimate the summation in (42).
Let $B_k = \frac{9\alpha^2}{32}(\lambda_{n+1} - \lambda_k + \frac{11}{7\alpha})(2k+5)$, then for every even k, we have

$$\frac{9\alpha^2}{32}\left[(\lambda_{n+1} - \lambda_k + \frac{11}{7\alpha})(\lambda_k + 4)(\lambda_k + 6)b_k^2 \right.$$
$$\left. + (\lambda_{n+1} - \lambda_{k+1} + \frac{11}{7\alpha})(\lambda_{k+1} + 4)(\lambda_{k+1} + 6)b_{k+1}^2\right]$$
$$= B_k(\int_{-1}^{1}(1-x^2)\tilde{F}_k' g)^2 + B_{k+1}(\int_{-1}^{1}(1-x^2)\tilde{F}_{k+1}' g)^2.$$

Induction procedure

Then we split the right hand side into three parts as follows.

$$
\begin{aligned}
R_{k,1} :=& B_k \left[\left(\int_0^1 (1-x^2)\tilde{F}_k' g \right)^2 + \left(\int_{-1}^0 (1-x^2)\tilde{F}_k' g \right)^2 \right] \\
&+ B_{k+1} \left[\left(\int_0^1 (1-x^2)\tilde{F}_{k+1}' g \right)^2 + \left(\int_{-1}^0 (1-x^2)\tilde{F}_{k+1}' g \right)^2 \right], \\
R_{k,2} :=& 2B_k \int_0^1 (1-x^2)\tilde{F}_k' g \int_{-1}^0 (1-x^2)(\tilde{F}_k' + \tilde{F}_{k+1}') g \\
&+ 2B_{k+1} \int_0^1 (1-x^2)(\tilde{F}_{k+1}' - \tilde{F}_k') g \int_{-1}^0 (1-x^2)\tilde{F}_{k+1}' g, \\
R_{k,3} :=& \ 2(B_{k+1} - B_k) \int_0^1 (1-x^2)\tilde{F}_k' g \int_{-1}^0 (1-x^2)\tilde{F}_{k+1}' g.
\end{aligned}
$$

Estimates of $R_{k,1}$ and $R_{k,3}$

By point-wise estimates of Gegenbauer polynomials, we get

$$\begin{aligned} R_{k,1} &= B_k \left[\left(\int_0^1 (1-x^2)\tilde{F}_k' g\right)^2 + \left(\int_{-1}^0 (1-x^2)\tilde{F}_k' g\right)^2 \right] \\ &\quad + B_{k+1} \left[\left(\int_0^1 (1-x^2)\tilde{F}_{k+1}' g\right)^2 + \left(\int_{-1}^0 (1-x^2)\tilde{F}_{k+1}' g\right)^2 \right] \\ &\leq B_k \left(|\mathcal{A}_k^+|^2 + |\mathcal{A}_k^-|^2\right) + B_{k+1} \left(|\mathcal{A}_k^+|^2 + |\mathcal{A}_k^-|^2\right), \end{aligned} \tag{46}$$

and

$$R_{k,3} \leq \begin{cases} 2(B_{k+1} - B_k)\lambda(1-\lambda)a^2, & \text{if } B_k \leq B_{k+1}, \\ 2(B_k - B_{k+1})m_0(1-\lambda)a^2, & \text{if } B_{k+1} < B_k. \end{cases} \tag{47}$$

Cancellation of consecutive Gegenbauer polynomials

To estimate $R_{k,2}$, we need the cancellation property of consecutive Gegenbauer polynomials.

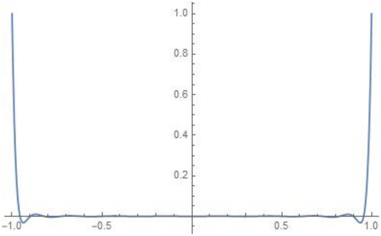

Figure: Graph of $\tilde{F}'_{19}$

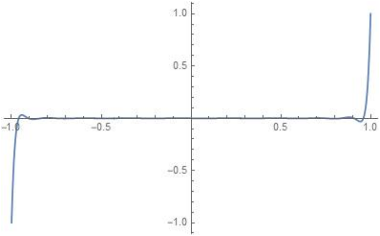

Figure: Graph of $\tilde{F}'_{20}$

From the graphs of $\widetilde{F}'_{19}$ and $\widetilde{F}'_{20}$, we can see that they almost cancel when $x < 0$, which is equivalent to say that they are almost equal when $x > 0$.

Cancellation of consecutive Gegenbauer polynomials

Lemma 11 (Gui-Li-Wei-Ye 2023)

Let $c_n = \max_{0 \le x \le 1} |\tilde{F}'_{n+1} - \tilde{F}'_n|$, *then* $c_n \le 0.12$ *if* $6 \le n \le 17$ *and* $c_n < 0.04$ *if* $n \ge 18$.

Remark 1

Numerically, we can find that in fact that we have better estimates $c_n \le \frac{0.84}{n}$, *but it is hard to prove. In contrast, there exists good estimate for the difference of consecutive Legendre polynomials.*

From Lemma 11, we have

$$|\int_{-1}^{0} (1-x^2)(\tilde{F}'_k + \tilde{F}'_{k+1})g| \le c_k a_- = c_k(1-\lambda)a,$$

$$|\int_{0}^{1} (1-x^2)(\tilde{F}'_{k+1} - \tilde{F}'_k)g| \le c_k a_+ = c_k \lambda a.$$

Estimate of $R_{k,2}$

Hence we obtain the estimate of $R_{k,2}$

$$
\begin{aligned}
R_{k,2} &= 2B_k \int_0^1 (1-x^2)\tilde{F}'_k g \int_{-1}^0 (1-x^2)(\tilde{F}'_k + \tilde{F}'_{k+1})g \\
&\quad + 2B_{k+1} \int_0^1 (1-x^2)(\tilde{F}'_{k+1} - \tilde{F}'_k)g \int_{-1}^0 (1-x^2)\tilde{F}'_{k+1} g \\
&\quad \leq 2(B_k + B_{k+1})c_k \lambda(1-\lambda)a^2.
\end{aligned}
$$

Combining the estimates of $R_{k,i}$, $i=1,2,3$, we obtain

$$\begin{aligned}
&\frac{9\alpha^2}{32}[(\lambda_{n+1}-\lambda_k+\frac{11}{7\alpha})(\lambda_k+4)(\lambda_k+6)b_k^2\\
&+(\lambda_{n+1}-\lambda_{k+1}+\frac{11}{7\alpha})(\lambda_{k+1}+4)(\lambda_{k+1}+6)b_{k+1}^2]\\
&\leq B_k\left(|\mathcal{A}_k^+|^2+|\mathcal{A}_k^-|^2\right)+B_{k+1}\left(|\mathcal{A}_k^+|^2+|\mathcal{A}_k^-|^2\right)\\
&+2(B_k+B_{k+1})c_k\lambda(1-\lambda)a^2\\
&+\begin{cases}2(B_{k+1}-B_k)\lambda(1-\lambda)a^2, & \text{if } B_k\leq B_{k+1},\\ 2(B_k-B_{k+1})m_0(1-\lambda)a^2, & \text{if } B_{k+1}<B_k.\end{cases}
\end{aligned}$$

The right hand side above can be viewed as a function $f_{k,a}(\lambda)$ of $\lambda=\frac{a_+}{a}$.

The worst case

The following proposition yields that the worst case is $\lambda = 1$, which is expected. In particular, in this case, we can drop the small terms $R_{k,2}$ and $R_{k,3}$.

Proposition 2 (Gui-Li-Wei-Ye 2023)

Suppose a satisfies $a \leq \frac{16}{\lambda_n}$ for some odd $n \geq 3$. Let $f_{k,a}(\lambda)$ be defined as above. Then for any k even, for $n \geq 41$, we have
(1) If $\lambda_k \leq \frac{1}{4}\lambda_n$, then

$$f_{k,a}(\lambda) \leq f_{k,a}(1) = (B_k + B_{k+1})(a - \frac{1-b}{d}\lambda_k a^2)^2. \tag{48}$$

(2) If $\frac{1}{4}\lambda_n < \lambda_k \leq \lambda_n$, then

$$f_{k,a}(\lambda) \leq f_{k,a}(1) = (B_k + B_{k+1})(ba + (1-b)\frac{d}{4\lambda_k})^2. \tag{49}$$

Final version of (42)

From the proposition above and $\alpha \geq \frac{1}{2}$, we obtain from (42) that

$$
\begin{aligned}
0 \leq & -\frac{512}{7}(\lambda_{n+1}+\frac{51}{7})(1-\frac{7}{6}a)+\frac{512}{7}(\lambda_{n+1}+\frac{100}{7})(1-\frac{7}{6}a)^2 \\
& +\frac{22528}{63\alpha}a^2 \\
& +\frac{128}{9}\sum_{m=1}^{\frac{n-5}{4}}[(\lambda_{n+1}-\lambda_{2m}+\frac{22}{7})(4m+5) \\
& \quad +(\lambda_{n+1}-\lambda_{2m+1}+\frac{22}{7})(4m+7)](1-\frac{1-b}{d}\lambda_{2m}\frac{16}{\lambda_{n+2}})^2a^2 \\
& +\frac{128}{9}\sum_{m=\frac{n-1}{4}}^{\frac{n-1}{2}}[(\lambda_{n+1}-\lambda_{2m}+\frac{22}{7})(4m+5) \\
& \quad +(\lambda_{n+1}-\lambda_{2m+1}+\frac{22}{7})(4m+7)](ba+(1-b)\frac{d}{4\lambda_{2m}})^2.
\end{aligned}
$$

Deriving a contradiction

To get a contradiction, it suffices to show that the right hand side, denoted by $g_n(a)$, is negative for $\frac{16}{\lambda_{n+2}} < a \leq \frac{16}{\lambda_n}$.
Direct computation yields that for $n > 10000$, $g_n(a)$ can be decomposed into three parts $g_{n,i}(a)$, $i = 1, 2, 3$ with estimates

$$g_{n,1}(a) := -\frac{512}{7}(\lambda_{n+1} + \frac{51}{7})(1 - \frac{7}{6}a) + \frac{512}{7}(\lambda_{n+1} + \frac{100}{7})(1 - \frac{7}{6}a)^2 + \frac{22528}{63\alpha}a^2$$
$$\leq -853.33,$$

$$g_{n,2}(a) := \frac{128}{9}\sum_{m=1}^{\frac{n-5}{4}}[(\lambda_{n+1} - \lambda_{2m} + \frac{22}{7})(4m+5) + (\lambda_{n+1} - \lambda_{2m+1} + \frac{22}{7})(4m+7)](1 - \frac{1-b}{d}\lambda_{2m}\frac{16}{\lambda_{n+2}})^2 a^2$$
$$\leq 571.1095.$$

Deriving a contradiction

$$
\begin{aligned}
g_{n,3}(a) :=& \frac{128}{9} \sum_{m=\frac{n-1}{4}}^{\frac{n-1}{2}} [(\lambda_{n+1} - \lambda_{2m} + \frac{22}{7})(4m+5) \\
&+ (\lambda_{n+1} - \lambda_{2m+1} + \frac{22}{7})(4m+7)](ba + (1-b)\frac{d}{4\lambda_{2m}})^2 \\
\leq& 280.95.
\end{aligned}
$$

Combining three estimates above, we found

$$0 \leq g_n(a) \leq -853.33 + 571.1095 + 280.95 < -1.27 < 0,$$

for all $n > 10000$ and $\frac{16}{\lambda_{n+2}} < a \leq \frac{16}{\lambda_n}$, which is a contradiction. For $n < 10000$, $g_n(a) < 0$ is checked by Matlab. Thus, we finish the proof of the main Theorem in $\mathbb{S}^6$.

Results on $\mathbb{S}^n n \geq 8$

In the end, we briefly discuss some partial results on $\mathbb{S}^8$. On $\mathbb{S}^8$, (2) becomes

$$\alpha[(1-x^2)^4 u']^{(7)} + 7! - 9 * 2^9 \frac{e^{8u}}{\gamma} = 0. \tag{50}$$

Theorem 12 (Theorem 1.1 in Gui-Hu-Xie 2022)

If $0.827 \leq \alpha < 1$, *then* (50) *admits only constant solutions.*

Theorem 13

If 0.54 $\leq \alpha < 1$, *then* (50) *admits only constant solutions.*

Summary

The proof of our theorems on $\mathbb{S}^4$, $\mathbb{S}^6$ and $\mathbb{S}^8$ mainly consist of two parts. Firstly, we derive an estimates about the seminorm

$$\lfloor G \rfloor^2 = (-1)^{\frac{n}{2}} \int_{-1}^{1} (1-x^2)^{\frac{n-2}{2}} G[(1-x^2)^{\frac{n}{2}} G']^{(n-1)}.$$

The difficulty is in the simplification of the integral

$$(-1)^{\frac{n}{2}} \int_{-1}^{1} (1-x^2)^{\frac{n-2}{2}} G^2[(1-x^2)^{\frac{n-2}{2}} G]^{(n-1)},$$

which seems too complicated to deal with as n increases. Secondly, the estimates of Gegenbauer coefficients of G heavily rely on the estimates of Gegenbauer polynomials C_n^{ν} (decaying, cancellations). However, as far as we know, there is no satisfactory formula to characterize the behavior of Gegenbauer polynomials in general.

Thanks for your attention!