The space of properly embedded minimal surfaces with finite total curvature

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Abstract.

It is showed that the set of non degenerate properly embedded minimal surfaces with finite total curvature and fixed topology in $\mathbb{R}^3$ has a structure of finite dimensional real analytic manifold —the *non degeneration* is defined in terms of the space of Jacobi functions on the surface which have logarithmic growth at the ends—. As application we show that if a non degenerate minimal surface has a symmetry which fixes its ends, then any nearby minimal surface has the same kind of symmetry. Finally, we construct a natural Lagrangian immersion of the space of non degenerate minimal surfaces, quotiented by certain group of rigid motions, into a Euclidean space with its standard symplectic structure.

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Introduction.

It is known, see R. Bohme, F. Tomi, J. Tromba and B. White, [1,27,28,30,31] and references there in, that several natural families of compact minimal surfaces admit a structure

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of —infinite dimensional— smooth manifold. This fact is one of the hits of the theory of minimal surfaces and allows the authors above to apply differential topology —Morse theory, degree theory,...— on this space. In this way they obtain important results about existence, number of solutions and qualitative description for these kind of surfaces. In this paper we want to introduce this point of view in a class of *non compact* minimal surfaces *without boundary* : the class of properly embedded minimal surfaces in $\mathbb{R}^3$ with finite total curvature. The simplest examples are the plane and the Catenoid. Some non trivial existence results are obtained in Costa [4], Hoffman and Karcher [9] and Hoffman and Meeks [10]. Known non existence results say that in this family there are neither surfaces with positive genus and one or two ends, Schoen [25], nor surfaces of genus zero and more than two ends, Lopez, Perez and Ros [15,22]. Costa [5] shows that the space of surfaces with genus one and three ends is parametrized by a halfline. Finally, Ros [24] describes the weak compactness of the space for arbitrary genus and number of ends, and shows the strong compactness —up to homotheties— of the space of surfaces of genus one and more than four ends. For more details about the actual situation of the theory for this family, see the survey [9].

Denote by $\mathcal{M}$ the space of minimal surfaces of finite total curvature, genus k and r ends —to avoid trivial cases we will suppose throughout this paper that $k \geq 1$ and $r \geq 3$—, properly immersed in $\mathbb{R}^3$ and with embedded horizontal ends —such an end must be asymptotic to a plane or to a half-Catenoid—. These restrictions are verified, in particular, for the *embedded* surfaces. We first show that *the different natural topologies on $\mathcal{M}$ are equivalent.* Given $M \in \mathcal{M}$, the infinitesimal deformations of M in $\mathcal{M}$ are given by the space $\mathcal{J}(M)$ of the Jacobi functions on M, i.e. functions u on M such that $Lu = 0$, L being the Jacobi operator of M, which have logarithmic growth at the ends. We will show that $\dim \mathcal{J}(M) \geq r + 3$ for any M. If we denote by $\mathcal{M}^* = \{M \in \mathcal{M} : \dim \mathcal{J}(M) = r + 3\}$ the subspace of *non degenerate* surfaces, then we will see that $\mathcal{M}^*$ is an open subset of $\mathcal{M}$ and

> $\mathcal{M}^*$ *is an $(r+3)$-dimensional real analytic manifold.*

If $M \in \mathcal{M}^*$, the tangent space at M is given by $\mathcal{J}(M)$. The unique embedded surfaces where the dimension of $\mathcal{J}(M)$ is known are the Hoffman-Meeks [10] surfaces M_k of genus $k \leq 37$ and three ends. In this case, using a result of Nayatani [20], we have that $\dim \mathcal{J}(M) = 6$. Thus the one-parameter deformation of these surfaces given in Hoffman and Karcher [9] contains all the surfaces nearby M_k, up to dilatations preserving the vertical direction.

As application of the above result we show that

> *If M is non degenerate and invariant by an isometry of $\mathbb{R}^3$ preserving the upper halfspace, then all the surfaces nearby M have the same kind of symmetry.*

Note that, as it was observed in [9], all known embedded examples admit symmetries of the above type.

If $\mathcal{G}$ denotes the subgroup of direct rigid motions of $\mathbb{R}^3$ preserving the upper halfspace, we define a natural continuous map $f : \mathcal{M}/\mathcal{G} \longrightarrow \mathbb{R}^{2r}$ depending only on information about the ends of the surface, and we show that

$f : \mathcal{M}^*/\mathcal{G} \longrightarrow \mathbb{R}^{2r}$ *is a real analytic Lagrangian immersion.*

In addition, we provide the second fundamental form of this natural Lagrangian immersion.

The basic idea of the arguments we will develope is the following: We want to parametrize the set of minimal surfaces near a given one. So, we consider the constraint of vanishing the mean curvature on deformations of the original surface. As we are working on non compact surfaces with catenoid or planar type ends, we are obliged to take deformations with logarithmic growing at infinity. Then the results are obtained via the Implicit Function Theorem after a careful study of the mean curvature operator for graphs defined outside a disk in the horizontal plane $\{x_3 = 0\}$. In order to get the surjectivity of the differential of the mean curvature operator, we need to control the space of logarithmically growing Jacobi fields. Fortunately, when the analysis is transferred to the underlying compact surface obtained from the finite total curvature assumption, the allowed singularities turn out to have a nice behaviour in terms of the above operator.

Also note that, thank to the results of Meeks and Rosenberg [18] and Collin [3], when the number of ends is greater than one, the finite total curvature assumption is equivalent to the finiteness of the topology.

Finally we remark that Mazzeo, Pollack and Uhlenbeck [16], and also Kusner, Mazzeo and Pollack [12] have obtained recently finite dimensional smoothness results for the moduli spaces of others noncompact geometric objects: the space of solutions to the singular Yamabe problem on a finitely punctured n-dimensional sphere, and the space of properly embedded *non minimal* constant mean curvature surfaces in $\mathbb{R}^3$ with finite topology. Our general strategy and some of our statements are of the same type that those in [16] and [12] —they also have a Lagrangian structure—. ¿From the analytic machinery point of view, the main difference between both approaches is that they use weighted Sobolev spaces while we are able to work with functional spaces on the compactified surface whose elements are regular, up to a finite dimensional subspace of functions with given singularities.

1 Embedded minimal ends.

First of all, we will expose some well-known facts about minimal surfaces, for more details, see Osserman [21] and Hoffman and Karcher [9], and as a direct consequence we introduce the graph coordinate for properly embedded minimal ends with finite total curvature. Consider a conformal minimal immersion $\psi : M \longrightarrow \mathbb{R}^3$ of an orientable surface M into the three-dimensional euclidean space. Denote by g and ω the meromorphic function and the holomorphic one-form determinated by the Weierstrass representation of ψ,

$$\psi = \left(\frac{1}{2} \left(\overline{\int \omega} - \int g^2 \omega \right), \text{Real} \int g\omega \right) \in \mathbb{C} \times \mathbb{R} \equiv \mathbb{R}^3. \tag{1}$$

Recall that g is the stereographic projection from the North Pole of the Gauss map of ψ. If ψ is complete and its total Gaussian curvature $\int_M K dA$ is finite, then M has the conformal type of a finitely punctured compact surface $\overline{M} - \{p_1, \ldots, p_r\}$, and the Weierstrass data extend in a meromorphic way to $\overline{M}$. In particular, the Gauss map of ψ is well-defined at each p_i. The points $p_1, \ldots, p_r$ will be identified with the ends of ψ. We remark that in this family, completeness is equivalent to the properness of ψ —recall that ψ is said to be proper if $\{\|\psi\| \leq R\}$ is compact for any $R > 0$—. Assume that ψ has parallel embedded ends —these conditions are verified, in particular, when ψ is an embedding—. Suppose also that, after a rotation in $\mathbb{R}^3$ if necessary, its Gauss map N takes vertical values at the punctures p_i. Properly embedded minimal ends with finite total curvature are characterized in terms of their Weierstrass Representation: such an end must be asymptotic to an end of a Catenoid or of a plane. If we suppose that the normal limit vector at p_i is $(0, 0, 1)$, the Weierstrass data of ψ in terms of a conformal coordinate z centered at the end p_i are given by

$$g(z) = \frac{g_1(z)}{z^k}, \qquad \omega = f(z) z^{2(k-1)} dz, \qquad |z| < \varepsilon,$$

where $k \in \mathbb{N}$ and g_1, f are holomorphic functions such that $g_1(0) \neq 0, f(0) \neq 0$. So, (1) yields

$$\psi(z) = \left(\frac{1}{2} \left(\overline{f_1(z)} - \int \frac{g_1^2 f}{z^2} dz \right), \text{Real} \int g_1 f z^{k-2}\, dz \right),$$

where f_1 is holomorphic in $|z| < \varepsilon$. We should impose $(g_1^2 f)'(0) = 0$ and, when $k = 1$, $(g_1 f)(0) \in \mathbb{R}$ in order to insure that the last expression is well-defined. Thus we can write

$$\psi(z) = \left(\frac{t(z)}{z}, -a \log |z| + t_1(z) \right), \tag{2}$$

where t is a smooth non vanishing complex valued function in $\{|z| < \varepsilon\}$, $a \in \mathbb{R}$ and t_1 is harmonic in $\{|z| < \varepsilon\}$. When $k = 1$, a does not vanishes and ψ is asymptotic to an end of a vertical Catenoid with logarithmic growth a. If $k \geq 2$, the third coordinate function is bounded and ψ is asymptotic to the end of a horizontal plane. These cases are usually referred as Catenoid end or planar end, respectively. Symmetrically, if the limit normal at the end is $(0, 0, -1)$, the Weierstrass data are $g(z) = g_1(z)z^k$, $\omega = f(z)z^{-2}dz$ and a similar argument gives $\psi(z) = \left(\frac{t(z)}{\bar{z}}, -a \log |z| + t_1(z)\right)$, with t, a, t_1 as above.

Take an end p_i of ψ where N points to $(0, 0, 1)$. We have from (2) that, with the notation $\psi = (x_1 + ix_2, x_3)$, after inversion in $\mathbb{C}$ of $x_1 + ix_2$ we obtain a new —non conformal— coordinate w on $\overline{M}$, satisfying $w = \frac{1}{x_1+ix_2} = \frac{z}{t(z)}$ such that

$$\psi(w) = \left(\frac{1}{w}, -a \log |w| + h(w)\right), \tag{3}$$

with h a smooth real valued function in $|w| < \delta$. We will say that w is the **graph coordinate** at the end p_i. Symmetrically, if $N(p_i) = (0, 0, -1)$ we had obtained $\psi(z) = \left(\frac{t(z)}{\bar{z}}, -a \log |z| + t_1(z)\right)$, and in terms of the graph coordinate $w = \frac{1}{x_1+ix_2} = \frac{\bar{z}}{t(z)}$, the same expression (3) holds. We can associate to each end p_i of M the **logarithmic growth** a and the **height** $h(0)$: If p_i is a Catenoid end, its logarithmic growth coincides with the one of the asymptotic Catenoid, and its height differs of the third coordinate of the center of the asymptotic Catenoid in the term $a \log |\frac{a}{2}|$. When p_i is a planar end, its logarithmic growth vanishes and its height coincides with the height of the asymptotic plane. Also note that if γ is a closed curve around p_i, the logarithmic growth can be obtained by $\int_\gamma \eta \, ds = 2\pi a \langle N(p_i), e_3\rangle e_3$, η being the unit conormal vector along γ and $e_3 = (0, 0, 1)$. We will denote by $\mathbf{log}(\psi) = (a_1, \ldots, a_r)$, $\mathbf{height}(\psi) = (b_1, \ldots, b_r)$, the lists of logarithmic growths and heights of the ends of ψ.

We remark that any minimal surface which is a graph on the domain $\{|x_1 + ix_2| > \frac{1}{\delta}\}$ has finite total curvature at the end —this follows, for instance, from Fischer-Colbrie [7] because the end is complete and stable—, so it admits the representation given in (3).

If $M \subset \mathbb{R}^3$ is a properly embedded minimal surface with finite total curvature, then its ends $p_1, \ldots, p_r$ are naturally ordered with respect to the x_3 linear coordinate. This order coincides with the lexicographical order of the pairs (logarithmic growth, height) at the punctures. The maximum principle at infinity, see Langevin and Rosenberg [14] or Meeks and Rosenberg [17], implies that for different ends of M the above pairs are different.

2 The mean curvature operator at an end.

Let $\varepsilon > 0$. Consider a not necessarily minimal immersion $\psi : \{0 < |w| < \varepsilon\} \longrightarrow \mathbb{R}^3$ of the type

$$\psi(w) = \left(\frac{1}{w}, -a\log|w| + h(w)\right), \tag{4}$$

where $a \in \mathbb{R}$ and $h \in C^{2,\alpha}(\{|w| < \varepsilon\})$, the usual Hölder space. Put $\rho = |w|$ and $u = -a\log\rho + h$. The induced metric $ds^2 = (g_{ij})$ is given by

$$g_{ij}(w) = \frac{1}{\rho^4}\left(\delta_{ij} + \rho^4 D_i u D_j u\right), \tag{5}$$

where $\delta_{ij} = 1$ if $i = j$, 0 otherwise, and $D_i u$ denotes the corresponding first derivative of u. If we denote by dA —resp dA_0— the measure associated to the metric ds^2 —resp. $|dw|^2$—, (5) implies that

$$dA = \frac{Q^{\frac{1}{2}}}{\rho^4} dA_0, \qquad \text{where} \qquad Q = 1 + \rho^4\|\nabla_0 u\|^2, \tag{6}$$

and the subindex $\bullet_0$ denotes that the corresponding object is computed respect to the flat metric of the w-plane. As $u = -a\log\rho + h$, we get that $\rho^2 D_i u = -a\langle w, e_i\rangle + \rho^2 D_i h$ lies in $C^{1,\alpha}(\{|w| < \varepsilon\})$ and depends analytically on its three variables a, w and $\nabla_0 h$. Also, $Q = 1 + \rho^2 Q_1$, where

$$Q_1 = \rho^2\|\nabla_0 u\|^2 = a^2 + \rho^2\|\nabla_0 h\|^2 - 2a\langle w, \nabla_0 h\rangle.$$

Hence, $Q, Q_1 \in C^{1,\alpha}$, and Q, Q_1, g_{ij}, dA are analytic in its three variables.

The Gauss map of ψ is given by

$$N = Q^{-\frac{1}{2}}\left(\overline{w}^2\nabla_0 u, 1\right) = Q^{-\frac{1}{2}}\left(-a\overline{w} + \overline{w}^2\nabla_0 h, 1\right), \tag{7}$$

where $\overline{w}^2\nabla_0 u$ means the product of the complex numbers $\overline{w}^2$ and $\nabla_0 u$.

Lemma 1 *In the above situation,*

1. *The Gauss map N is a $C^{1,\alpha}(\{|w| < \varepsilon\}, \mathbb{R}^3)$-valued map which depends analytically on its variables $(a, w, \nabla_0 h) \in \mathbb{R} \times \{|w| < \varepsilon\} \times \mathbb{C}$.*

2. *If $\{e_1, e_2, e_3\}$ is the usual basis in $\mathbb{R}^3$, the functions $\langle N, e_1\rangle, \langle N, e_2\rangle, \det(\psi, N, e_3)$ lie in $C^{1,\alpha}(\{|w| < \varepsilon\})$, depend analytically on its variables a, w and $\nabla_0 h$ and vanish at the puncture $w = 0$.*

3. *The support function of* ψ *is given by* $\langle \psi, N\rangle = -a\langle N, e_3\rangle \log \rho + f(a, w, h, \nabla_0 h)$, *where* $f \in C^{1,\alpha}(\{|w| < \varepsilon\})$, *takes the value* $-a + h(0)$ *at* $w = 0$ *and depends analytically on its variables* $(a, w, h, \nabla_0 h) \in \mathbb{R} \times \{|w| < \varepsilon\} \times \mathbb{R} \times \mathbb{C}$.

Proof. *1* is a direct consequence of (7) and the above comment about Q. Concerning *2*, we only have to compute $\det(\psi, N, e_3)$ from (4),(7):

$$\det(\psi, N, e_3) = \det\left((\frac{1}{w}, 0), Q^{-\frac{1}{2}}(-a\overline{w} + \overline{w}^2 \nabla_0 h, 0), (0, 1) \right) =$$

$$= Q^{-\frac{1}{2}} \langle \frac{i}{w}, -a\overline{w} + \overline{w}^2 \nabla_0 h\rangle = -Q^{-\frac{1}{2}} \text{Real } (i\overline{w}\nabla_0 h).$$

Finally, from (4) and (7) it follows that

$$\langle \psi, N\rangle = -a\langle N, e_3\rangle \log \rho + \langle N, e_3\rangle \left(-a + \text{Real}(\overline{w}\nabla_0 h) + h\right).$$

This concludes the proof.

On the other hand, as ψ can be viewed as the graph of the function $u(x_1 + ix_2)$ with $x_1 + ix_2 = \frac{1}{w}$, its mean curvature is given by

$$2H(x_1 + ix_2) = \text{ div}_1 \left(\frac{\nabla_1 u}{\sqrt{1 + \|\nabla_1 u\|^2}} \right),$$

where the subindex $\bullet_1$ means that the object is computed in the Euclidean geometry of the parameters $x_1 + ix_2$. Using that

$$\rho^4 (\nabla_0 u)(w) = (\nabla_1 u)(x_1 + ix_2), \qquad \text{and} \qquad \text{div}_1(\rho^4 Y) = \rho^4 \text{div}_0 Y$$

for any vector field Y on $\{0 < |w| < \varepsilon\}$, we conclude that the mean curvature of ψ in terms of the parameter w is given by

$$2H = \rho^4 \text{ div}_0 \left(\frac{\nabla_0 u}{\sqrt{1 + \rho^4 \|\nabla_0 u\|^2}} \right) = \rho^4 \text{ div}_0 \left(Q^{-\frac{1}{2}} \nabla_0 u \right). \tag{8}$$

The next structure result for the mean curvature operator at one end will play a key role in what follows.

Proposition 1 *Let $\varepsilon > 0, a \in \mathbb{R}$ and $h \in C^{2,\alpha}(\{|w| < \varepsilon\})$. Denote by H the mean curvature function of the immersion $\psi : \{0 < |w| < \varepsilon\} \longrightarrow \mathbb{R}^3$ given by $\psi(w) = \left(\frac{1}{w}, -a \log \rho + h(w)\right)$, where $\rho = |w|$. Then, $\frac{1}{\rho^4} H$ is a operator of the type*

$$\frac{1}{\rho^4} H = \sum_{i,j} a_{ij}(w, a, \nabla_0 h) D_{ij} h + b(w, a, \nabla_0 h), \tag{9}$$

which depends analytically on its variables $a \in \mathbb{R}$, $|w| < \varepsilon$, $\nabla_0 h \in \mathbb{C}$ and $\nabla_0^2 h \in S_2(\mathbb{R})$, the space of real symmetric matrices of order 2. In particular, it is $C^\alpha(\{|w| < \varepsilon\})$-valued.

Proof. ¿From (8) we have

$$\frac{2}{\rho^4} H = \ \mathrm{div}_0 \left(Q^{-\frac{1}{2}} \nabla_0 u\right) = Q^{-\frac{1}{2}} \Delta_0 u + \langle \nabla_0 (Q^{-\frac{1}{2}}), \nabla_0 u \rangle = Q^{-\frac{1}{2}} \Delta_0 h - \frac{1}{2} Q^{-\frac{3}{2}} \langle \nabla_0 Q, \nabla_0 u \rangle. \tag{10}$$

To prove the analyticity of $\frac{1}{\rho^4} H$ we only need to check this fact for the last inner product of the expression above. But

$$\langle \nabla_0 Q, \nabla_0 u \rangle = \langle \nabla_0 (\rho^2 Q_1), \nabla_0 (-a \log \rho + h) \rangle = \langle 2 Q_1 w + \rho^2 \nabla_0 Q_1, -\frac{aw}{\rho^2} + \nabla_0 h \rangle =$$

$$= -2a Q_1 - a \langle \nabla_0 Q_1, w \rangle + \langle 2w Q_1 + \rho^2 \nabla_0 Q_1, \nabla_0 h \rangle, \tag{11}$$

which is analytic on its variables, as we claimed. Hence, $\frac{1}{\rho^4} H$ lies in C^α, and the expression (9) follows directly from (10), (11). This completes the proof.

Remark 1 *By simple computation we can show that given $C > 0$, the operator $\rho^{-4} H$ is uniformly elliptic on the pairs (a, h) such that $|a|$, $\rho^2 \|\nabla_0 h\| \leq C$, and that $b(w, a, \nabla_0 h)$ is uniformly bounded on the pairs (a, h) with $|a|$, $\rho \|\nabla_0 h\| \leq C$.*

Lemma 2 *Consider $\varepsilon' > \varepsilon > 0$, $a \in \mathbb{R}$ and $h \in C^\infty(\{|w| < \varepsilon'\})$ such that the immersion $\psi : \{0 < |w| < \varepsilon'\} \longrightarrow \mathbb{R}^3$ given by $\psi(w) = \left(\frac{1}{w}, -a \log \rho + h(w)\right)$ is minimal. Then, the map*

$$\begin{array}{rccc} E: & \mathbb{R} \times C^{k,\alpha}(\{|w| \leq \varepsilon\}) & \longrightarrow & \mathbb{R} \times C^{k-2,\alpha}(\{|w| \leq \varepsilon\}) \times C^{k,\alpha}(\{|w| = \varepsilon\}) \\ & (b, f) & \longmapsto & \left(b, \frac{1}{\rho^4} H(b, f), f|_{\{|w| = \varepsilon\}}\right), \end{array}$$

where $H(b, f)$ is the mean curvature of the immersion $w \mapsto \left(\frac{1}{w}, -b \log \rho + h(w) + \frac{f(w)}{\langle N, e_3 \rangle}\right)$, $0 < |w| \leq \varepsilon$, is a real analytic local diffeomorphism around $(a, 0)$, for any $k \geq 2$.

Proof. From Proposition 1 we conclude directly that E is real analytic. In order to compute the differential of E at $(a,0)$, we need the following fact that will be proved in section 5:

$$D\left(\frac{1}{\rho^4}H\right)_{(a,0)}(b,f)=\frac{1}{2}\overline{L}\left(-b\langle N,e_3\rangle\log\rho+f\right), \tag{12}$$

for any $(b,f)\in\mathbb{R}\times C^{k,\alpha}\left(\{|w|\leq\varepsilon\}\right)$, where $\overline{L}=\overline{\Delta}+\|\overline{\nabla}N\|^2$ is the Schrödinger operator associated to the extended Gauss map $N:\{|w|<\varepsilon'\}\longrightarrow S^2(1)$ of ψ, $\overline{\Delta},\|\overline{\nabla}N\|^2$ being the laplacian and the square length of the gradient of N respect to the Riemannian metric $d\overline{s}^2=\rho^4ds^2$ obtained from (5). Hence, (12) yields

$$DE_{(a,0)}(b,f)=\left(b,\frac{1}{2}\overline{L}\left(-b\langle N,e_3\rangle\log\rho+f\right),f|_{\{|w|=\varepsilon\}}\right).$$

If $(b,f)\in\text{Kernel}\left(DE_{(a,0)}\right)$, then $b=0$ and f is a solution of the problem

$$\begin{cases}\overline{L}f=0 & \text{in} \quad \{|w|\leq\varepsilon\}\\ f=0 & \text{in} \quad \{|w|=\varepsilon\}.\end{cases}$$

As $\langle N,e_3\rangle$ is positive in $\{|w|\leq\varepsilon\}$ and $\overline{L}\langle N,e_3\rangle=0$, it is a standard fact that the first eigenvalue for the Dirichlet problem

$$\begin{cases}\overline{L}u+\lambda u=0 & \text{in} \quad \{|w|\leq\varepsilon\}\\ u=0 & \text{in} \quad \{|w|=\varepsilon\}\end{cases}$$

is positive. Hence f vanishes identically and so, $DE_{(a,0)}$ is injective. ¿From Proposition 1 and (12) we have that $\overline{L}\left(-b\langle N,e_3\rangle\log\rho\right)\in C^{k-2,\alpha}\left(\{|w|\leq\varepsilon\}\right)$. Hence, given $b\in\mathbb{R}$, $v\in C^{k-2,\alpha}\left(\{|w|\leq\varepsilon\}\right)$ and $\varphi\in C^{k,\alpha}\left(\{|w|=\varepsilon\}\right)$, linear elliptic theory [8] insures that there exists $f\in C^{k,\alpha}\left(\{|w|\leq\varepsilon\}\right)$ satisfying

$$\begin{cases}\overline{L}f=2v-\overline{L}\left(-b\langle N,e_3\rangle\log\rho\right) & \text{in} \quad \{|w|\leq\varepsilon\}\\ f=\varphi & \text{in} \quad \{|w|=\varepsilon\},\end{cases}$$

hence $DE_{(a,0)}(b,f)=(b,v,\varphi)$. Now lemma 2 follows directly from the Inverse function theorem.

Also note that the maximum principle at infinity, see [14], implies that given $b\in\mathbb{R}$, $\varphi\in C^{k,\alpha}\left(\{|w|=\varepsilon\}\right)$, there is at most one function $f\in C^{k,\alpha}\left(\{|w|\leq\varepsilon\}\right)$ such that $E(b,f)=(b,0,\varphi)$.

3 The topology of the space of minimal surfaces.

Let $\mathcal{E}$ the space of properly embedded minimal surfaces in $\mathbb{R}^3$ with finite total curvature, horizontal ends and fixed topology —genus $k \geq 1$ and r ends, $r \geq 3$—. The absolute total curvature of the surfaces in $\mathcal{E}$ is $4\pi(k + r - 1)$, [11]. In this section, we will show that the different natural notions of convergence in the space $\mathcal{E}$ are equivalent. Let $M_n \in \mathcal{E}$, $n = 1, 2, \ldots$, and $M_\infty \in \mathcal{E}$.

1. We say that $\{M_n\}_n$ *converges* to M_∞ *as subsets* if $M_\infty = \{p \in \mathbb{R}^3 :$ there exists a sequence $p_n \in M_n$ with $p_n \to p\}$. This is the same that the convergence for the Hausdorff distance on compact subsets of $\mathbb{R}^3$.

2. We say that $\{M_n\}_n$ *converges* to M_∞ *smoothly* if for any relatively compact subdomain Ω of M_∞, for any embedded tubular neighbourhood of Ω in $\mathbb{R}^3$, $\Omega(\varepsilon) = \{p + tN_\infty(p) : p \in \Omega,\ |t| < \varepsilon\}$, $\varepsilon > 0$ small enough, N_∞ being the Gauss map of M_∞, and for each n large enough we have that $\Omega_n = M_n \cap \Omega(\varepsilon)$ consists in a —single— graph over Ω, $p \in \Omega \longmapsto p + u_n(p)N_\infty(p)$, and the sequence $\{u_n\}_n$ converges to zero in $C^k(\Omega)$, for any $k \geq 0$.

3. Let $\overline{M}_n, \overline{M}_\infty$ be the compactified surfaces obtained from M_n, M_∞, respectively. Consider the r-dimensional vector space, r being the number of ends of M_∞, spanned by r functions $f_1, \ldots, f_r$ defined as follows:

   3.a. $f_i \in C^\infty(\overline{M}_\infty - \{p_i\})$, where $p_1, \ldots, p_r$ are the ends of M_∞.

   3.b. In the graph coordinate w for M_∞ around p_i, $f_i(w) = \log |w|$, and f_i vanishes in a neighbourhood of each p_j, for each $j \neq i$.

   We say that the sequence $\{\overline{M}_n\}_n$ *converges* to $\overline{M}_\infty$ *smoothly* if there exists a vector field $\widetilde{N} \in C^\infty(\overline{M}_\infty, \mathbb{R}^3)$ transverse to M_∞ which takes the values $\pm e_3$ outside some compact subset of M_∞, such that for each n large enough, M_n is a global $\widetilde{N}$-graph over M_∞, $p \in M_\infty \longmapsto p + u_n(p)\widetilde{N}(p)$, where $u_n = \varphi_n + v_n$, $\varphi_n \in \mathrm{Span}\{f_1, \ldots, f_r\}$, $v_n \in C^\infty(\overline{M}_\infty)$, $\varphi_n \to 0$ in $\mathrm{Span}\{f_1, \ldots, f_r\}$ and $v_n \to 0$ in $C^k(\overline{M}_\infty)$, $k \geq 0$. We remark that this convergence is independent of the choice of $\widetilde{N}$.

The relation beetween these three types of convergence is stated at the following result:

Theorem 1 *Given $\{M_n\}_n \subset \mathcal{E}$ and $M_\infty \in \mathcal{E}$, the following assertions are equivalent:*

1. $M_n \to M_\infty$ as subsets.

2. $M_n \to M_\infty$ *smoothly.*

3. $\overline{M}_n \to \overline{M}_\infty$ *smoothly.*

Proof. Clearly *3* $\Longrightarrow$*2* $\Longrightarrow$*1*. Suppose we have *1*. It follows from the arguments in Choi and Schoen [2], and White [29], see also Ros [24], that there exists a subsequence $\{M_{n'}\}_{n'} \subset \{M_n\}_n$ such that $M_{n'}$ converges smoothly with finite multiplicity on compact subsets of $\mathbb{R}^3 - X$, X being a finite subset, to a properly embedded minimal surface M'_∞. It is clear that we must have $M'_\infty = M_\infty$. Moreover, if $X \neq \emptyset$, we know that given a small neighbourhood U of X, the total curvature on $U \cap M_{n'}$ is near a positive multiple of 4π for n' large enough. As $C(M_{n'}) = C(M_\infty)$, where $C(\bullet)$ denotes the absolute total curvature enclosed in the corresponding domain, we conclude that $X = \emptyset$ and $M_{n'}$ converges to M_∞ uniformly on compact sets of $\mathbb{R}^3$ with multiplicity one, i.e. $M_{n'} \longrightarrow M_\infty$ smoothly. Note that this argument also proves that every subsequence of $\{M_n\}_n$ has a smoothly convergent subsequence to M_∞. This fact is equivalent to $M_n \to M_\infty$ smoothly, so we have proved that *1* implies *2*.

Assume now that assertion *2* holds. Note that all the surfaces M_n, M_∞ have the same absolute total curvature. Take $\delta > 0$ small and consider the relatively compact subdomain $\Omega \subset M_\infty$ obtained by cutting M_∞ with a vertical solid cylinder $\{|x_1+ix_2| \leq R\} \times \mathbb{R}$, with R large enough in such a way that $C(M_\infty) - C(\Omega) < \frac{\delta}{2}$. Hence, for n large enough, the domain Ω_n obtained by the same procedure with M_n instead of M_∞ will verify $C(M_n) - C(\Omega_n) < \delta$, and Ω_n is a graph over Ω. We can also suppose that along $\partial\Omega_n$, the relation $|\langle N_n, e_3\rangle| \geq 1-\delta$ holds, where N_n is the Gauss map of M_n, and that each component of $\partial\Omega_n$ projects injectively into the (x_1+ix_2)-plane. As N_n is an open map, it follows that either $|\langle N_n, e_3\rangle| \geq 1-\delta$ on $M_n - \Omega_n$ or the spherical image $N_n(M_n - \Omega_n)$ has area near 4π. As the second possibility cannot happen, we conclude that the projection of each component of $M_n - \Omega_n$ into $\{|x_1 + ix_2| > R\}$ is a proper local diffeomorphism, so this projection is necessarily globally injective on each component, or in other words, for n large enough, M_n is a global $\widetilde{N}$-graph over M_∞ for a suitable chosen transversal section $\widetilde{N}$ verifiying the conditions of definition 3. Let u_n be the function defined by this graph. From the smooth convergence of M_n to M_∞ we get the convergence of the logarithmic growths of the ends of M_n to the logarithmic growths of the ends of M_∞ —this holds because that growths can be computed as the integral along each component of $\partial\Omega_n$ of its conormal vector—. Hence, with the notation of definition 3, the component of u_n in $\mathrm{Span}\{f_1, \ldots, f_r\}$ converges to zero. In the notation of lemma 2, this implies that for n large enough, if M_n is written as $w \mapsto \left(\frac{1}{w}, -b_n \log\rho + h(w) + \frac{f_n(w)}{\langle N, e_3\rangle}\right)$ around an end p_i, where $b_n \in \mathbb{R}$ and $f_n \in C^\infty(\{|w| \leq \varepsilon\})$, then $\{b_n\}_n \to 0$ and $f_n|_{\{|w|=\varepsilon\}} \to 0$ in $C^k(\{|w| = \varepsilon\})$ for any $k \geq 0$. Thus, $E(b_n, f_n) = \left(b_n, 0, f_n|_{\{|w|=\varepsilon\}}\right)$ lies in a neighbourhood of

$(a, 0, 0)$ in $\mathbb{R} \times C^{k-2,\alpha}(\{|w| \leq \varepsilon\}) \times C^{k,\alpha}(\{|w| = \varepsilon\})$ where the map E in lemma 2 is bijective. As the maximum principle at infinity [14] insures that b_n, $f_n|_{\{|w|=\varepsilon\}}$ determine uniquely to $f_n(w)$, $|w| \leq \varepsilon$, it follows from lemma 2 that $\{f_n\}_n$ converges to zero in $C^{k,\alpha}(\{|w| \leq \varepsilon\})$ for any $k \geq 2$. This concludes the proof.

Although we have stated theorem 1 for the case of embedded minimal surfaces for the sake of clarity, the result extends to the space $\mathcal{M}$ of simple proper minimal immersions with finite total curvature, embedded horizontal ends and fixed topology. An immersion is called simple, following [31], if it gives an embedding when restricted to some open dense subset —for proper minimal immersions this is the same that to assume that the immersion does not factorizes through a non trivial covering map—. We will consider on $\mathcal{M}$ the above topology. So, $\mathcal{E} \subset \mathcal{M}$ and it can be shown, use for instance theorem 1 of [24], that $\mathcal{E}$ is closed in $\mathcal{M}$. In spite of we are mainly interested in the embedded case, it could happen that arbitrary nearby surfaces to an embedded one were not embedded but only simply immersed —this is possible only if two different ends of the embedded surface have the same logarithmic growth—. So, we will also consider this kind of immersions. For the remainder of the paper, by a *proper minimal immersion* we will mean a *simple proper minimal immersion*, and this surface will be represented by $M \hookrightarrow \mathbb{R}^3$ or $\psi : M \longrightarrow \mathbb{R}^3$ indistinctly. As there is not a natural order for the ends of a surface in $\mathcal{M}$ we will consider all possible orders and we will view different orders as different surfaces. The topology of $\mathcal{M}$ will be modified, correspondly, in an obvious way to be sensitive to the order of the ends.

4 The Jacobi operator.

Let $M \in \mathcal{M}$, and $\overline{M} = M \cup \{p_1, \ldots, p_r\}$ the associated compact Riemann surface. Put $N : \overline{M} \longrightarrow S^2(1)$ the extended Gauss map and ds^2 the induced metric on M. Take a function $\lambda \in C^\infty(M)$ such that around each end p_i, $\lambda(w) = \rho^{-4}$ in terms of the graph coordinate at p_i. Then (5) implies that $d\overline{s}^2 = \frac{1}{\lambda} ds^2$ is a Riemannian metric on $\overline{M}$ compatible with its complex structure. The Jacobi operator of M is $L = \Delta + \|\sigma\|^2 = \Delta + \|\nabla N\|^2$, where Δ is the Laplacian of ds^2 and $\|\sigma\|^2$ the square length of the second fundamental form of the embedding. In certain sense, this operator is the Hessian of the Area functional. A way to understand the geometric role of L is the following: take $\{M_t\}_{|t|<\varepsilon}$, $M_0 = M$ a smooth deformation of M and denote by $H(t)$ the mean curvature of M_t. Calling $u = \langle \frac{d}{dt}\Big|_{t=0} \psi_t, N\rangle$, where $\psi_t : M \longrightarrow \mathbb{R}^3$ is a smooth family of immersions such that ψ_t represents the surface

M_t, we have —see for instance [30]—

$$\left.\frac{d}{dt}\right|_{t=0} H(t) = \frac{1}{2} Lu. \tag{13}$$

In particular, the equation $Lu = 0$ corresponds to an infinitesimal deformation of M by minimal surfaces. A function u satisfying the last equation is called a Jacobi function on M. The operator L can be "compactified" to a Schrödinger operator $\overline{L} = \lambda L = \overline{\Delta} + \|\overline{\nabla} N\|^2$ on $\overline{M}$, where $\overline{\bullet}$ means that the corresponding object is computed respect to $d\overline{s}^2$.

Let $\mathcal{B} = \mathcal{B}(M) \subset C^{2,\alpha}(M)$ the space of functions v such that in the graph coordinate w around the end p_i, $i = 1, \ldots, r$, are given by

$$v(w) = -a_i \langle N, e_3 \rangle \log \rho + u(w), \tag{14}$$

with $a_i \in \mathbb{R}$ and $u \in C^{2,\alpha}(\{|w| < \varepsilon\})$. Consider the r-dimensional space $V = V(M) = \{u_{\mathbf{a}} : \mathbf{a} \in \mathbb{R}^r\} \subset \mathcal{B}$ whose functions are given by $u_{\mathbf{a}} = -\langle N, e_3 \rangle \sum_{i=1}^r a_i f_i$, where $\mathbf{a} = (a_1, \ldots, a_r) \in \mathbb{R}^r$ and $f_1, \ldots, f_r$ are the functions which appear in the definition 3 of section 3. Thus around each p_i, $u_{\mathbf{a}}$ is given by $u_{\mathbf{a}}(w) = -a_i \langle N, e_3 \rangle \log \rho$. Clearly $\mathcal{B} = V \oplus C^{2,\alpha}(\overline{M}) = \{u_{\mathbf{a}} + u : \mathbf{a} \in \mathbb{R}^r, u \in C^{2,\alpha}(\overline{M})\}$. Hence, $\mathcal{B}$ becomes a Banach space with the natural norm on each component, and the topology of $\mathcal{B}$ is independent of the choice of $f_1, \ldots, f_r$. By identification of $\mathcal{B}$ with $\mathbb{R}^r \oplus C^{2,\alpha}(\overline{M})$, we will see in section 5 that $\overline{L}$ is the differential of a real analytic operator defined in a neighbourhood of $(\mathbf{log}(M), 0)$ in $\mathbb{R}^r \oplus C^{2,\alpha}(\overline{M})$ with values in $C^{\alpha}(\overline{M})$, hence $\overline{L} : \mathcal{B} \longrightarrow C^{\alpha}(\overline{M})$ is a bounded linear operator. In particular, $u\overline{L}v$ —resp. uLv— is $d\overline{s}^2$-integrable —resp. ds^2-integrable—, for any pair of functions $u, v \in \mathcal{B}$.

Given a function $v \in \mathcal{B}, v = u_{\mathbf{a}} + u \in V \oplus C^{2,\alpha}(\overline{M})$ we define

$$\mathbf{Log}(v) = \mathbf{a}, \qquad \mathbf{Height}(v) = (u(p_1)\langle N(p_1), e_3 \rangle, \ldots, u(p_r)\langle N(p_r), e_3 \rangle).$$

Hence we have two bounded linear operators **Log**, **Height**: $\mathcal{B} \longrightarrow \mathbb{R}^r$. Lemma 1 says that $\langle N, e_1 \rangle, \langle N, e_2 \rangle, \det(p, N, e_3)$, p being the position vector on M, lie in $\mathcal{B}$ and **Log**, **Height** vanish on these functions. It is clear that also $\langle N, e_3 \rangle \in \mathcal{B}$ and $\mathbf{Log}(\langle N, e_3 \rangle) = (0, \ldots, 0)$, $\mathbf{Height}(\langle N, e_3 \rangle) = (1, \ldots, 1) \in \mathbb{R}^r$. Furthermore, it follows also that the support function $\langle p, N \rangle \in \mathcal{B}$ and $\mathbf{Log}(\langle p, N \rangle) = \mathbf{log}(M)$, $\mathbf{Height}(\langle p, N \rangle) = \mathbf{height}(M) - \mathbf{log}(M)$, where **log**, **height** are defined in section 1. These linear operators **Log**, **Height** be related with the differential operator $\overline{L}$ by means of a skew-symmetric bilinear form:

Lemma 3 *Given $u, v \in \mathcal{B}$, we have*

$$\int_M (uLv - vLu)\, dA = \int_{\overline{M}} (u\overline{L}v - v\overline{L}u)\, d\overline{A} =$$

$$= -2\pi \left[\langle \mathbf{Log}(u), \mathbf{Height}(v) \rangle - \langle \mathbf{Log}(v), \mathbf{Height}(u) \rangle \right], \tag{15}$$

Proof. Assume that in the graph coordinate around the end p_i, $i = 1, \ldots, r$, the functions u and v are given by $u(w) = -a_i\langle N, e_3\rangle \log\rho + u_i'(w)$, $\quad v(w) = -b_i\langle N, e_3\rangle \log\rho + v_i'(w)$, u_i', v_i' being $C^{2,\alpha}$-functions around $w = 0$. So,

$$\begin{aligned}
&\mathbf{Log}(u) = (a_1, \ldots, a_r), \quad \mathbf{Height}(u) = \left(u_1'(0)\langle N(p_1), e_3\rangle, \ldots, u_r'(0)\langle N(p_r), e_3\rangle\right), \\
&\mathbf{Log}(v) = (b_1, \ldots, b_r), \quad \mathbf{Height}(v) = \left(v_1'(0)\langle N(p_1), e_3\rangle, \ldots, v_r'(0)\langle N(p_r), e_3\rangle\right).
\end{aligned}$$

Consider a conformal coordinate z centered at p_i. Thus, $w = t(z)z$ or $w = t(z)\overline{z}$ for some smooth complex valued function t with $t(0) \neq 0$. In the new parameter we have around p_i, $u(z) = -a_i\langle N, e_3\rangle \log|z| + u_i''(z)$, $v(z) = -b_i\langle N, e_3\rangle \log|z| + v_i''(z)$, where $u_i'' = u_i' - a_i\langle N, e_3\rangle \log|t(z)|$ and $v_i'' = v_i' - b_i\langle N, e_3\rangle \log|t(z)|$. Note that the sum in the right-hand side of (15) is equal to

$$-2\pi \sum_{i=1}^{r} \left(a_i v_i''(0) - b_i u_i''(0)\right) \langle N(p_i), e_3\rangle.$$

Denote by $D(p_i, r)$ the conformal disk $\{|z| < r\}$ around p_i, and $M(r) = M - \cup_{i=1}^{r} D(p_i, r)$. Then

$$\int_{M(r)} (uLv - vLu)\, dA = \int_{\partial M(r)} \left(u\frac{\partial v}{\partial \eta} - v\frac{\partial u}{\partial \eta}\right) ds,$$

where dA, ds are measured in the induced metric by ψ, and η is the exterior conormal field to ψ along $\partial M(r)$. As the last integral is conformal-invariant, it can be written on each disk as $-\int_{\{|z|=r\}} \left(u\frac{\partial v}{\partial r} - v\frac{\partial u}{\partial r}\right)|dz|$, where $z = re^{i\theta}$. By using the above local expression of v we have

$$\frac{\partial v}{\partial r} = -b_i\langle N, e_3\rangle \frac{1}{r} + \text{ lower order terms},$$

where the not specified terms grow at most logarithmically. A similar expression can be obtained for u. Thus,

$$-\int_{\{|z|=r\}} \left(u\frac{\partial v}{\partial r} - v\frac{\partial u}{\partial r}\right)|dz| =$$

$$= -\int_{\{|z|=r\}} \left(-u(z)b_i\langle N, e_3\rangle\frac{1}{r} + v(z)a_i\langle N, e_3\rangle\frac{1}{r} + \text{lower order terms}\right)|dz| =$$

$$= \frac{1}{r}\int_{\{|z|=r\}} \left(u_i''(z)b_i\langle N, e_3\rangle - v_i''(z)a_i\langle N, e_3\rangle\right)|dz| + o(1) = 2\pi\left(u_i''(0)b_i - v_i''(0)a_i\right)\langle N, e_3\rangle + o(1),$$

$o(1)$ being an expression whis converges to zero as r goes to zero. This completes the proof of the lemma.

We are now interested in the Kernel and Image of $\overline{L} : \mathcal{B} \longrightarrow C^{\alpha}(\overline{M})$. For any $X \subset C^{\alpha}(\overline{M})$ we will represent its L^2-orthogonal in $C^{\alpha}(\overline{M})$ respect to the metric $d\overline{s}^2$ by $X^{\perp} \subset C^{\alpha}(\overline{M})$. Note that for any finite dimensional subspace $W \subset C^{\alpha}(\overline{M})$ we have that $C^{\alpha}(\overline{M}) = W \oplus W^{\perp}$ and $W^{\perp\perp} = W$. Moreover, if $W' \subset C^{\alpha}(\overline{M})$ is a subspace with $W^{\perp} \subset W'$, then $C^{\alpha}(\overline{M}) = W' \oplus W'^{\perp}$ and $W'^{\perp\perp} = W'$ are also true. Consider in the Banach space $\mathcal{B}$ the subspaces

$$\mathcal{J} = \mathcal{J}(M) = \text{ Kernel}(\overline{L}), \qquad \mathcal{K} = \mathcal{K}(M) = C^{2,\alpha}(\overline{M}) \cap \mathcal{J}, \quad \text{and} \quad \mathcal{K}_0 = \mathcal{K}_0(M) = \overline{L}(\mathcal{B})^{\perp}.$$

By linear elliptic theory [8] we know that $\mathcal{K} = \text{Kernel}\left(\overline{L} \mid_{C^{2,\alpha}(\overline{M})}\right) \subset C^{\infty}(\overline{M})$ has finite dimension and that $\overline{L}(C^{2,\alpha}(\overline{M})) = \mathcal{K}^{\perp}$. Thus, $\mathcal{K}_0 = \overline{L}(\mathcal{B})^{\perp} \subset \overline{L}(C^{2,\alpha}(\overline{M}))^{\perp} = \mathcal{K}^{\perp\perp} = \mathcal{K}$. Moreover, as $\overline{L}(\mathcal{B})$ contains $\mathcal{K}^{\perp}$, it follows that $\overline{L}(\mathcal{B}) = \overline{L}(\mathcal{B})^{\perp\perp} = \mathcal{K}_0^{\perp}$. The space $\mathcal{K}$ consists of the Jacobi functions on M which are bounded at the ends.

Lemma 4 *In the above situation,*

1. $\mathcal{K}_0 = \{v \in \mathcal{K} : v(p_i) = 0,\ 1 \leq i \leq r\}$,
2. $\dim \mathcal{J} = r + \dim \mathcal{K}_0$.

Proof. Given $v \in \mathcal{K}$, we have $v \in \mathcal{K}_0$ if and only if $\int_{\overline{M}} v\overline{L}u d\overline{A} = 0$, $\forall u \in \mathcal{B}$. Using (15), the last equation is equivalent to

$$0 = \int_{\overline{M}} u\overline{L}v\ d\overline{A} + 2\pi \left[\langle \mathbf{Log}(u), \mathbf{Height}(v)\rangle - \langle \mathbf{Log}(v), \mathbf{Height}(u)\rangle\right] =$$

$$= 2\pi\langle \mathbf{Log}(u), \mathbf{Height}(v)\rangle,$$

for each $u \in \mathcal{B}$. This gives *1.* Concerning *2*, consider on the r-dimensional space V defined just after lemma 3, the decomposition $V = V_1 \oplus V_2$, where $V_1 = \{u_{\mathbf{a}} : \overline{L}u_{\mathbf{a}} \in \overline{L}(C^{2,\alpha}(\overline{M}))\}$ and V_2 is a suplementary subspace. Hence, $\overline{L}$ is injective on V_2 and $\overline{L}(\mathcal{B}) = \overline{L}(V_2) \oplus \overline{L}(C^{2,\alpha}(\overline{M}))$. In particular, $\dim \mathcal{K}_0 = \text{codim } \overline{L}(\mathcal{B}) = \text{codim } \overline{L}(C^{2,\alpha}(\overline{M})) - \dim \overline{L}(V_2) = \dim \mathcal{K} - \dim V_2$, that is

$$\dim \mathcal{K}_0 = \dim \mathcal{K} - r + \dim V_1. \tag{16}$$

Now consider the natural projection $\mathcal{B} \longrightarrow V$ restricted to $\mathcal{J}$, $\pi : \mathcal{J} \longrightarrow V$. It is clear that $\text{Kernel}(\pi) = \mathcal{K}$. Furthermore, given $v \in \mathcal{J}$, $v = u_{\mathbf{a}} + u$, $u_{\mathbf{a}} \in V$, $u \in C^{2,\alpha}(\overline{M})$, it follows that $0 = \overline{L}v = \overline{L}u_{\mathbf{a}} + \overline{L}u$ and so, $\pi(v) = u_{\mathbf{a}} \in V_1$. Also, for any $u_{\mathbf{a}} \in V_1$ there exists $u \in C^{2,\alpha}(\overline{M})$ such that $\overline{L}u = \overline{L}u_{\mathbf{a}}$, that is, $u_{\mathbf{a}} - u \in \mathcal{J}$. Thus $\pi(\mathcal{J}) = V_1$ and then

$$\dim \mathcal{J} = \dim \mathcal{K} + \dim V_1. \tag{17}$$

¿From (16) and (17) we conclude the proof of the lemma.

It is well-known that vector fields in $\mathbb{R}^3$ whose flow consists on isometries —Killing Fields— or dilatations induce on M Jacobi functions. As our surface has horizontal ends, this insures the following set of bounded Jacobi functions, see lemma 1:

$$\text{Killing} \;=\; \text{Span}\{\langle N, e_1\rangle, \langle N, e_2\rangle, \langle N, e_3\rangle, \det(p, N, e_3)\} \subset \mathcal{K},$$

p being the position vector on M. We will call $\text{Killing}_0 = \text{Killing} \cap \mathcal{K}_0$. Then, it follows that

$$\text{Killing}_0 = \;\text{Span}\{\langle N, e_1\rangle, \langle N, e_2\rangle, \det(p, N, e_3)\} \subset \mathcal{K}_0.$$

Moreover, the support function $\langle p, N\rangle$ lies in $\mathcal{J}$. If $\text{Killing}_0 = \mathcal{K}_0$, we will say that M is a *non degenerate* minimal surface. This condition insures a nice behavior of the set of minimal immersions near M, as we will see in theorem 2. As we are assuming that the genus of M is as least one and the number of ends is as least three, it follows that the symmetry group of M is always finite and thus, dim Killing $= 4$ and dim $\text{Killing}_0 = 3$. Note also that the assumption $\mathcal{K} = \text{Killing}$ implies that $\mathcal{K}_0 = \text{Killing}_0$. As a direct consequence of lemma 4 we have

Proposition 2 *With the notation above,* $\dim \mathcal{J}(M) \geq r + 3$, *and the equality holds if and only if M is non degenerate.*

Soret [26] has shown that $\dim \mathcal{J}(M) \geq r - 5 - 2k$, k being the genus of the surface.

5 The smoothness of the space of minimal surfaces.

Let us return to our orientable proper minimal immersion $\psi : M = \overline{M} - \{p_1, \ldots, p_r\} \longrightarrow \mathbb{R}^3$ with finite total curvature and embedded horizontal ends, $M \in \mathcal{M}$. Put $\mathbf{c} = (c_1, \ldots, c_r) = \mathbf{log}(M)$ the list of logarithmic growths of ψ. As we saw in section 1, ψ can be written in the graph coordinate w around each end p_i as

$$\psi(w) = \left(\frac{1}{w}, -c_i \log|w| + h(w)\right), \tag{18}$$

with $h \in C^\infty(\{|w| < \varepsilon\})$. Consider a family $\psi_{\mathbf{a}} : M \longrightarrow \mathbb{R}^3$, where $\mathbf{a} = (a_1, \ldots, a_r)$ varies in a neighbourhood $\mathcal{A} \subset \mathbb{R}^r$ of $\mathbf{c}$, of immersions such that

1. The family is smooth in $\mathcal{A} \times M$,

2. for each end p_i, the expression of $\psi_{\mathbf{a}}$ in terms of the graph coordinate $w, |w| < \varepsilon$, is given by $\psi_{\mathbf{a}}(w) = (\frac{1}{w}, -a_i \log |w| + h(w))$.

3. $\psi_{\mathbf{c}} = \psi$.

Let $\widetilde{N} : \overline{M} \longrightarrow \mathbb{R}^3$ be a smooth map such that $\langle N, \widetilde{N} \rangle = 1$ on $\overline{M}$, and $\widetilde{N} = \frac{1}{\langle N, e_3 \rangle} e_3$ in $\{|w| < \varepsilon\}$ for each end. Theorem 1 insures that given $M \in \mathcal{M}$, there exists a neighbourhood of M in $\mathcal{M}$ such that each minimal surface in this neighbourhood is represented by $\psi_{\mathbf{a}} + u\widetilde{N}$ for suitable $\mathbf{a} \in \mathbb{R}^r, u \in C^\infty(\overline{M})$ near $\mathbf{c} = \mathbf{log}(M)$, zero, respectively.

Given $u \in C^{2,\alpha}(\overline{M})$ small enough and $\mathbf{a}$ near $\mathbf{c}$, consider the immersion $\psi_{\mathbf{a}} + u\widetilde{N}$. Let $\overline{H}(\mathbf{a}, u) = \lambda H(\psi_{\mathbf{a}} + u\widetilde{N})$, where $H(\psi_{\mathbf{a}} + u\widetilde{N}) \in C^\alpha(M)$ is the mean curvature of $\psi_{\mathbf{a}} + u\widetilde{N}$, and $\lambda \in C^\infty(M)$ is defined at the beginning of section 4. Note that around an end p_i of ψ, $\psi_{\mathbf{a}} + u\widetilde{N}$ can be written in terms of the graph coordinate w as

$$(\psi_{\mathbf{a}} + u\widetilde{N})(w) = \left(\frac{1}{w}, -a_i \log \rho + h + \frac{u}{\langle N, e_3 \rangle} \right). \tag{19}$$

As $\overline{H}(\mathbf{a}, u) = \rho^{-4} H(\psi_{\mathbf{a}} + u\widetilde{N})$, proposition 1 yields

Lemma 5 *There exist neighbourhoods $\mathcal{U}$ of the origin in $C^{2,\alpha}(\overline{M})$ and $\mathcal{A}$ of $\mathbf{c}$ in $\mathbb{R}^r$ such that the map $\overline{H} : \mathcal{A} \times \mathcal{U} \longrightarrow C^\alpha(\overline{M})$ is a real analytic operator.*

We will denote by $M_{\mathbf{a},u}$ the surface determinated by the immersion $\psi_{\mathbf{a}} + u\widetilde{N} : M \longrightarrow \mathbb{R}^3$, $(\mathbf{a}, u) \in \mathcal{A} \times \mathcal{U}$. Nearby minimal surfaces to M in $\mathcal{M}$ can be represented by points of $\mathcal{A} \times \mathcal{U}$, see section 3, and in this neighbourhood, which will be denoted by $\mathcal{M} \cap (\mathcal{A} \times \mathcal{U})$, both topologies coincide. Given $M_{\mathbf{a},u} \in \mathcal{M} \cap (\mathcal{A} \times \mathcal{U})$ we can realize the r-dimensional space $V(M_{\mathbf{a},u}) = \{u_{\dot{\mathbf{a}}} : \dot{\mathbf{a}} \in \mathbb{R}^r\}$ in section 4 as follows: consider the curve $\mathbf{a}(t) = \mathbf{a} + t\dot{\mathbf{a}}$ with $\dot{\mathbf{a}} = (\dot{a}_1, \ldots, \dot{a}_r) \in \mathbb{R}^r$, and the function on M, $u_{\dot{\mathbf{a}}} = \langle \frac{d}{dt}\big|_{t=0} \psi_{\mathbf{a}(t)}, N_{\mathbf{a},u} \rangle$, where $N_{\mathbf{a},u}$ is the Gauss map of $M_{\mathbf{a},u}$. By using the form of the deformation $\psi_{\mathbf{a}}$ at the ends we deduce that in the graph coordinate for M around p_i,

$$\left.\frac{d}{dt}\right|_{t=0} \psi_{\mathbf{a}(t)} = (0, 0, -\dot{a}_i \log \rho).$$

As consequence, $u_{\dot{\mathbf{a}}}(w) = -\dot{a}_i \langle N_{\mathbf{a},u}, e_3 \rangle \log \rho$ has the required expression in terms of the w-coordinate, see section 4, and so, $\mathcal{B}(M_{\mathbf{a},u}) = V(M_{\mathbf{a},u}) \oplus C^{2,\alpha}(\overline{M}_{\mathbf{a},u})$. If we consider $\dot{u} \in C^{2,\alpha}(\overline{M})$ we have from (13) that

$$\left.\frac{d}{dt}\right|_{t=0} H\left(\psi_{\mathbf{a}(t)} + (u + t\dot{u})\widetilde{N}\right) = \frac{1}{2} L_{\mathbf{a},u}\left(u_{\dot{\mathbf{a}}} + \dot{u}\langle \widetilde{N}, N_{\mathbf{a},u}\rangle\right),$$

$L_{\mathbf{a},u}$ being the Jacobi operator of $M_{\mathbf{a},u}$. By means of the isomorphisms $\dot{\mathbf{a}} \mapsto u_{\dot{\mathbf{a}}}$ and $\dot{u} \mapsto \dot{u}\langle \widetilde{N}, N_{\mathbf{a},u}\rangle$, note that we can assume $\langle \widetilde{N}, N_{\mathbf{a},u}\rangle$ smooth and positive everywhere, we can identify $\mathbb{R}^r$ with $V(M_{\mathbf{a},u})$ and $C^{2,\alpha}(\overline{M})$ with $C^{2,\alpha}(\overline{M}_{\mathbf{a},u})$. With these identifications the above arguments give

Lemma 6 *If $M_{\mathbf{a},u} \in \mathcal{M} \cap (\mathcal{A} \times \mathcal{U})$, then the differential of $\overline{H}$ at $(\mathbf{a}, u)$ is given by*

$$2D\overline{H}_{(\mathbf{a},u)} = \overline{L}_{\mathbf{a},u} : \mathcal{B}(M_{\mathbf{a},u}) \longrightarrow C^{\alpha}(\overline{M}_{\mathbf{a},u}),$$

where $\overline{L}_{\mathbf{a},u} = \lambda L_{\mathbf{a},u}$ is the compactified Jacobi operator of $\overline{M}_{\mathbf{a},u}$.

The arguments above also prove the equation (12) that we used in the proof of lemma 2, and that $\overline{L} : \mathcal{B}(M) \longrightarrow C^{\alpha}(\overline{M})$ is bounded as we claimed in section 4.

Lemma 7 *Let $\mathcal{A}, \mathcal{U}$ as in lemma 5. Then, the mean curvature function $H(\mathbf{a}, u)$ of $\psi_{\mathbf{a}} + u\widetilde{N}$, $(\mathbf{a}, u) \in \mathcal{A} \times \mathcal{U}$, is orthogonal to $\langle N_{\mathbf{a},u}, e_1\rangle$, $\langle N_{\mathbf{a},u}, e_2\rangle$, $\det(\psi_{\mathbf{a}} + u\widetilde{N}, N_{\mathbf{a},u}, e_3)$, where $N_{\mathbf{a},u}$ is the Gauss map of $\psi_{\mathbf{a}} + u\widetilde{N}$.*

Proof. Take an element $(\mathbf{a}, u) \in \mathcal{A} \times \mathcal{U}$ and consider the immersion $\psi_{\mathbf{a}} + u\widetilde{N}$. Given $\varepsilon > 0$ small, consider the compact domain $M(\varepsilon)$ in M obtained by removing around each end p_i, $1 \leq i \leq r$, a neighbourhood $\{|w| < \varepsilon\}$, where w denotes the graph coordinate for ψ at p_i. Hence, $\psi_{\mathbf{a}} + u\widetilde{N}$ has r components outside $M(\varepsilon)$, and each one can be written as

$$(\psi_{\mathbf{a}} + u\widetilde{N})(w) = \left(\frac{1}{w}, -a \log |w| + h_1(w)\right), \quad |w| \leq \varepsilon,$$

where $a \in \mathbb{R}$ and $h_1 \in C^{2,\alpha}(\{|w| < \varepsilon\})$. Then

$$2\int_{M(\varepsilon)} H(\mathbf{a}, u) N_{\mathbf{a},u}\, dA_{\mathbf{a},u} = \int_{M(\varepsilon)} \Delta_{\mathbf{a},u}(\psi_{\mathbf{a}} + u\widetilde{N})\, dA_{\mathbf{a},u} = \int_{\partial M(\varepsilon)} \eta_{\mathbf{a},u}\, ds_{\mathbf{a},u},$$

where subindex $\bullet_{\mathbf{a},u}$ denotes that the corresponding object is computed in the induced metric by $\psi_{\mathbf{a}} + u\widetilde{N}$, and $\eta_{\mathbf{a},u}$ is the exterior conormal field to $\psi_{\mathbf{a}} + u\widetilde{N}$ along $\partial M(\varepsilon)$. Analogously,

$$2\int_{M(\varepsilon)} H(\mathbf{a},u)\det(\psi_{\mathbf{a}} + u\widetilde{N}, N_{\mathbf{a},u}, e_3)\ dA_{\mathbf{a},u} = \int_{\partial M(\varepsilon)} \det(\psi_{\mathbf{a}} + u\widetilde{N}, \eta_{\mathbf{a},u}, e_3)\ ds_{\mathbf{a},u}.$$

We can parametrize $\partial M(\varepsilon)$ by r disjoint copies of $w = \varepsilon e^{i\theta}$, $\theta \in [0, 2\pi]$. Put

$$\alpha_\varepsilon(\theta) = (\psi_{\mathbf{a}} + u\widetilde{N})(\varepsilon e^{i\theta}) = \left(\frac{1}{w}, v(w)\right),$$

where $v = -a\log|w| + h_1$. Then, $\eta_{\mathbf{a},u} ds_{\mathbf{a},u} = -\alpha'_\varepsilon \wedge N_{\mathbf{a},u}\ d\theta$. On the other hand, (7) gives

$$N_{\mathbf{a},u} = \frac{1}{\sqrt{1+\varepsilon^2 Q_1}}(\overline{w}^2\nabla_0 v, 1).$$

Derivation of α_ε respect to θ yields $\alpha'_\varepsilon = \left(\frac{-i}{w}, \langle iw, \nabla_0 v\rangle\right)$, hence

$$\alpha'_\varepsilon \wedge N_{\mathbf{a},u} = \frac{1}{\sqrt{1+\varepsilon^2 Q_1}}\left(\langle iw, \nabla_0 v\rangle i\overline{w}^2\nabla_0 v - \frac{1}{w}, \varepsilon^2 \mathrm{Real}\left(\frac{\nabla_0 v}{w}\right)\right).$$

¿From $\nabla_0 v = -\frac{a}{\varepsilon^2}w + \nabla_0 h_1$, we can deduce that

$$\eta_{\mathbf{a},u} ds_{\mathbf{a},u} = \left[\left(\frac{1}{w}, a\right) + \left(\frac{O_1(\varepsilon^2)}{w} + O_2(\varepsilon^2), O_3(\varepsilon)\right)\right] d\theta, \tag{20}$$

where $O_i(\varepsilon^k)$ denotes a function of ε, θ such that $\varepsilon^{-k}O_i(\varepsilon^k)$ is bounded as ε goes to zero, and O_1, O_3 are real valued. From this we can deduce that

$$\lim_{\varepsilon\to 0}\int_{\{|w|=\varepsilon\}} \eta_{\mathbf{a},u}\ ds_{\mathbf{a},u} = \lim_{\varepsilon\to 0}\left[\int_0^{2\pi}\left(\frac{1}{w}, a\right) d\theta + \int_0^{2\pi} O(\varepsilon)\ d\theta\right] = 2\pi a e_3.$$

This proves that $H(\psi_{\mathbf{a}} + u\widetilde{N})$ is orthogonal to $\langle N_{\mathbf{a},u}, e_i\rangle$, $i = 1, 2$. Finally,

$$\int_{\{|w|=\varepsilon\}} \det(\psi_{\mathbf{a}} + u\widetilde{N}, \eta_{\mathbf{a},u}, e_3)\ ds_{\mathbf{a},u} =$$

$$= \int_{\{|w|=\varepsilon\}} \det\left((\frac{1}{w}, 0), \left(\frac{1 + O_1(\varepsilon^2)}{w} + O_2(\varepsilon^2), 0\right), (0,1)\right) d\theta =$$

$$= \int_{\{|w|=\varepsilon\}} \det\left((\frac{1}{w}, 0), (O_2(\varepsilon^2), 0), (0,1)\right) d\theta.$$

As the last expression converges to zero as ε goes to zero, the lemma is proved.

As Weierstrass representation gives that minimal surfaces in $\mathcal{M}$ can be expressed by means of meromorphic data verifying a number of analytic constraints —about order of poles, residues, periods,...— it follows that $\mathcal{M}$ is a real analytic set. A global explicit description of $\mathcal{M}$ from this point of view could be useful, see Pirola [23] for a related situation. A natural question is to find conditions that imply the smoothness of $\mathcal{M}$ around one of its elements. Denote by $\mathcal{M}^* \subset \mathcal{M}$ the subset of non degenerate minimal surfaces.

Theorem 2 *$\mathcal{M}^*$ is an open subset of $\mathcal{M}$. Moreover, $\mathcal{M}^*$ is either empty or a $(r+3)$-dimensional real analytic manifold.*

Proof. Consider $M \in \mathcal{M}^*$, and take neighbourhoods $\mathcal{U}$ of the origin in $C^{2,\alpha}(\overline{M})$ and $\mathcal{A}$ of $\mathbf{c} = \mathbf{log}(M)$ in $\mathbb{R}^r$ such that lemma 5 holds. Consider the map

$$\begin{array}{rcl} F:\ \mathbb{R}^3 \times \mathcal{A} \times \mathcal{U} & \longrightarrow & C^{\alpha}(\overline{M}) \\ (d_1, d_2, d_3, \mathbf{a}, u) & \longmapsto & \overline{H}(\mathbf{a}, u) - d_1\langle N_{\mathbf{a},u}, e_1\rangle - d_2\langle N_{\mathbf{a},u}, e_2\rangle - d_3 \det(\psi_{\mathbf{a}} + u\widetilde{N}, N_{\mathbf{a},u}, e_3) \end{array}$$

where $N_{\mathbf{a},u}$ is the Gauss map of $\psi_{\mathbf{a}}+u\widetilde{N}$. As $\overline{H}$ is regular, it follows that F is also real analytic. If $(\mathbf{a}, u) \in \mathcal{M} \cap (\mathcal{A} \times \mathcal{U})$ represents a minimal surface near M, then using the identifications in lemma 6 we deduce that the differential of F at $(0, \mathbf{a}, u)$, $DF_{(0,\mathbf{a},u)} : \mathbb{R}^3 \times \mathcal{B}(M_{\mathbf{a},u}) \longrightarrow C^{\alpha}(M_{\mathbf{a},u})$ is given by

$$DF_{(0,\mathbf{a},u)}(d_1, d_2, d_3, v) = \frac{1}{2}\overline{L}_{\mathbf{a},u}v - d_1\langle N_{\mathbf{a},u}, e_1\rangle - d_2\langle N_{\mathbf{a},u}, e_2\rangle - d_3 \det(\psi_{\mathbf{a}} + u\widetilde{N}, N_{\mathbf{a},u}, e_3).$$

Using that $\overline{L}_{\mathbf{a},u}(\mathcal{B}(M_{\mathbf{a},u})) = \mathcal{K}_0(M_{\mathbf{a},u})^{\perp}$ it follows that $\mathrm{Kernel}\Big(DF_{(0,\mathbf{a},u)}\Big) = \{0\} \times \mathcal{J}(M_{\mathbf{a},u}) = \mathcal{J}(M_{\mathbf{a},u})$. Evaluating at M, i.e. taking $\mathbf{a} = \mathbf{c}$ and $u = 0$, our assumption in the statement of the theorem gives that $DF_{(0,\mathbf{c},0)}$ is surjective and its kernel is $(r+3)$-dimensional. Using Implicit function theorem we find a neighbourhood $\mathcal{W}$ of $(0, \mathbf{c}, 0)$ in $\mathbb{R}^3 \times \mathcal{A} \times \mathcal{U}$ such that $\mathcal{V} := F^{-1}(0) \cap \mathcal{W}$ is a real analytic $(r+3)$-dimensional submanifold of $\mathcal{W}$. From lemma 7, $\mathcal{V}$ contains only minimal immersions, hence $\mathcal{V} \subset \{0\} \times \mathcal{A} \times \mathcal{U} = \mathcal{A} \times \mathcal{U}$. Moreover, the tangent space of $\mathcal{V}$ at $M_{\mathbf{a},u}$ is $\mathcal{J}(M_{\mathbf{a},u})$ and so, $\mathcal{V} \subset \mathcal{M}^*$. In particular, $\mathcal{M}^*$ is open. To conclude the proof of the theorem it remains to prove that the above analytic structures —which are defined only in a neighbourhood of each point of $\mathcal{M}^*$— are all compatible and so, they define a global real analytic structure on $\mathcal{M}^*$. Even more: by coherence reasons we want to prove that given $M \in \mathcal{M}^*$, the local analytic structure around M constructed above is independent of the particular choices of $\psi_{\mathbf{a}}$ and $\widetilde{N}$. All these facts are a direct consequence of the following argument —recall that an immersion determines uniquely the differentiable structure of the immersed manifold—. Given $\varepsilon > 0$, let $\mathcal{M}^*_\varepsilon$ be the set of surfaces $M \in \mathcal{M}$ such that the procedure above —i.e. the choice of $\psi_{\mathbf{a}}$ and $\widetilde{N}$ and the construction of the analytic neighbourhood $\mathcal{V} = \mathcal{V}_M$— can be applied by using the w-coordinate around each end of M over the disk $\{|w| < \varepsilon\}$. Thus, $\mathcal{M}^*_\varepsilon$ is an open subset of $\mathcal{M}$. Consider the continuous map $j : \mathcal{M}^*_\varepsilon \longrightarrow \mathbb{R} \times C^{2,\alpha}(\{|w| \leq \delta\})$, with $\delta \in]0, \varepsilon[$, given by $j(M) = (a_1, u_1)$, if M is represented in the w-coordinate around the end p_1, $|w| < \varepsilon$, by $w \mapsto \left(\frac{1}{w}, -a_1 \log \rho + u_1\right)$. If $\mathcal{V}_M \subset \mathcal{M}^*_\varepsilon$, then j is clearly analytic on $\mathcal{V}_M$ —j is linear in the arguments $(\mathbf{a}, u)$ because of (19)—. Moreover $v \in \mathcal{J}(M_{\mathbf{a},u})$, $M_{\mathbf{a},u} \in \mathcal{V}_M$, lies in Kernel$(Dj_{M_{\mathbf{a},u}})$ if and only if v vanishes around the first end of $M_{\mathbf{a},u}$. Thus Jacobi equation and the unique continuation principle imply that j is an immersion when restricted to $\mathcal{V}_M$ —the aditional splitness condition for immersions in our setting, see [13], holds because $\mathcal{M}^*$ is finite dimensional—. As given $M_1, M_2 \in \mathcal{M}^*$ we have that their analytic neighbourhoods $\mathcal{V}_{M_1}, \mathcal{V}_{M_2}$ are contained in $\mathcal{M}^*_\varepsilon$ for some $\varepsilon > 0$ small enough, we conclude that both structures are compatible at the points of $\mathcal{V}_{M_1} \cap \mathcal{V}_{M_2}$. The same argument shows that two different analytic structures around $M \in \mathcal{M}^*$ are compatible, and the theorem is proved.

The following result lets us handle easily the smoothness in the manifold $\mathcal{M}^*$:

Lemma 8 *Let $\delta > 0$, $M \in \mathcal{M}^*$ and $\psi_t : M \longrightarrow \mathbb{R}^3$, $|t| < \delta$, be a smooth family of immersions —in the sense that $(t,p) \longmapsto \psi_t(p)$ is smooth— such that ψ_t represents an element $M_t \in \mathcal{M}^*$ with $M_0 = M$. Then, the curve $t \longmapsto M_t \in \mathcal{M}^*$ is smooth and its velocity vector at $t = 0$ is given by $u = \langle \frac{d\psi_t}{dt}\Big|_{t=0}, N\rangle \in \mathcal{J}(M)$, N being the Gauss map of M.*

Proof. We will prove the lemma for $\delta > 0$ small enough. The general result follows in the same way after minor modifications. Clearly $M_t \to M$ smoothly as t goes to zero. Thus, from theorem 2, with the notations above and taking δ small enough, we have that $M_t \in \mathcal{V} \subset \mathcal{A} \times \mathcal{U}$, $|t| < \delta$. Hence M_t is represented by a pair $(\mathbf{a}_t, u_t) \in \mathcal{A} \times \mathcal{U}$. As the components of $\mathbf{a}_t$ can be computed as the integral along some fixed curves in M of an expression depending smoothly on t, it follows that also $\mathbf{a}_t$ depends smoothly on t. The fact that u_t is smooth with respect to t in a bounded domain of M is trivial. It remains to consider the functions u_t at an end p_i of M. In the w-coordinate around p_i, the surfaces M_t are given by $\phi_t(w) = \left(\frac{1}{w}, -a_t \log \rho + h(w) + \frac{u_t(w)}{\langle N, e_3\rangle}\right)$, $0 < |w| \leq \varepsilon$, where M is defined by $w \mapsto \left(\frac{1}{w}, -c \log \rho + h(w)\right)$, $0 < |w| \leq \varepsilon$, and u_t is smooth in $\{|w| \leq \varepsilon\}$. Both a_t and $u_t|_{\{|w|=\varepsilon\}}$ depend smoothly on t. Hence for ε small enough lemma 2 applies, so with its notation, the curve $(a_t, v_t) = E^{-1}\left(a_t, 0, u_t|_{\{|w|=\varepsilon\}}\right)$ depends smoothly on t. As the maximum principle at infinity [14] implies that $u_t(w)$, $|w| \leq \varepsilon$ is uniquely determined by a_t and $u_t|_{\{|w|=\varepsilon\}}$ we conclude that $v_t = u_t$ in $\{|w| \leq \varepsilon\}$, hence we have proved that the curve $t \mapsto M_t$ is smooth. The fact that its velocity vector can be computed in the stated way follows because the normal part of the infinitesimal variation of a deformation is independent of the particular choice of the immersions ψ_t: it depends only on the surfaces M_t themselves. This proves the lemma.

Remark 2

1. *If the family ψ_t is analytic, in the sense that the map $\mathbb{R} \times M \longrightarrow \mathbb{R}^3$, $(t, p) \mapsto \psi_t(p)$ is real analytic, we can conclude that the curve M_t, $|t| < \delta$ is real analytic as follows —note that some care is necessary when you want to check analyticity because neither $\widetilde{N}$ nor $(\mathbf{a}, p) \mapsto \psi_{\mathbf{a}}(p)$ are analytic—:* with the notation of the proof above, a_t and $u_t|_{\{|w|=\varepsilon\}}$ depend analytically on t and thus, the same holds for the curve $t \mapsto \left(a_t,\, u_t|_{\{|w| \leq \varepsilon\}}\right) = E^{-1}\left(a_t, 0,\, u_t|_{\{|w|=\varepsilon\}}\right)$. In particular, $t \mapsto j(M_t) \in \mathbb{R} \times C^{2,\alpha}(\{|w| \leq \varepsilon\})$ is analytic, j being the immersion of $\mathcal{M}^*_\varepsilon$ in the proof of theorem 2, and the remark follows from standard properties of submanifolds.

2. *S. Nayatani [20] has proved that Hoffman-Meeks' surface of three ends and genus γ, $2 \leq \gamma \leq 37$ [10], has dim $\mathcal{K} = 4$. This implies that, up to translations and rotations around the x_3-axis, each Hoffman-Meeks' surface can be deformed by a six-parameter —one-parameter if we do not count dilatations— family of embedded minimal surfaces with genus γ and three ends. This answers partially to a conjecture of Hoffman and Karcher in [9] that states that the space of properly embedded minimal surfaces of finite total curvature, genus $g \geq 2$ and three ends is equal to the one-parameter family of surfaces described in [9].*

6 Related results and applications.

1. We can also obtain information about the structure of the set of minimal surfaces with certain type of symmetries. Let $\widetilde{\mathcal{G}}$ be the subgroup of isometries of $\mathbb{R}^3$ which preserve the upper halfspace and $\mathcal{G}$ the connected component of the identity in $\widetilde{\mathcal{G}}$. $\mathcal{G}$ is a 3-dimensional Lie group generated by the horizontal translations and the rotations around the x_3-axis. Consider a discrete group G of rigid motions preserving the upper halfspace, i.e. $G \subset \widetilde{\mathcal{G}}$. A proper minimal immersion $\psi : M \longrightarrow \mathbb{R}^3$ is said to be G-invariant when for each $\phi \in G$, there exists a conformal diffeomorphism $S_\phi : M \longrightarrow M$ such that $\psi \circ S_\phi = \phi \circ \psi$. If we impose ψ to have finite total curvature and embedded ends with vertical limit normal vectors, no traslation parts are allowed and we will assume that G fixes the origin of $\mathbb{R}^3$. So G is finite and generated by a rotation R_θ around the x_3-axis of angle $\theta = \frac{2\pi}{k}$, $k \in \mathbb{N}$, or by a reflection ϕ in a vertical plane of $\mathbb{R}^3$, or by two reflections ϕ_1, ϕ_2 in distinct vertical planes. Moreover, each end of ψ is preserved by G and S_ϕ extends to $\overline{M}$ as a conformal —holomorphic or antiholomorphic— diffeomorphism, which will be also denoted by S_ϕ. Note that if we write down the immersion ψ around an end p_i in the graph coordinate w as in (4), then $h \circ S_\phi = h$, for each $\phi \in G$. This lets us consider the family $\psi_{\mathbf{a}}$ in section 5 in such a way that the conditions 1,2,3 remain true and moreover $\psi_{\mathbf{a}} \circ S_\phi = \phi \circ \psi_{\mathbf{a}}$, for each $\mathbf{a} \in \mathcal{A}$. Consider a Riemannian metric $d\overline{s}^2$ in the conformal structure of $\overline{M}$, S_ϕ-invariant for all $\phi \in G$, so the conformal factor λ between $d\overline{s}^2$ and the induced metric by ψ is also S_ϕ-invariant, for each $\phi \in G$. Take the tranverse map $\widetilde{N}$ G-invariant, too. Consider the closed subspace $C^{k,\alpha}(\overline{M})^G = \{u \in C^{k,\alpha}(\overline{M}) : u \circ S_\phi = u, \ \forall \phi \in G\}$. Then $\psi_{\mathbf{a}} + u\widetilde{N}$ is a G-invariant immersion for each $(\mathbf{a}, u) \in \mathcal{A} \times \mathcal{U}^G$ for suitable open neighbourhoods $\mathcal{A}$ of $\mathbf{c} = \mathbf{log}(M)$ in $\mathbb{R}^r$ and $\mathcal{U}^G$ of zero in $C^{2,\alpha}(\overline{M})^G$. Hence $\mathbb{R}^r \times C^{2,\alpha}(\overline{M})^G$ is a Banach space and repeating the arguments in lemma 5 we conclude that we can shrink $\mathcal{A}$, $\mathcal{U}^G$ in such a way that the operator $\overline{H} : \mathcal{A} \times \mathcal{U}^G \longrightarrow C^\alpha(\overline{M})^G$ is real analytic. We can also identify the tangent space to $\mathcal{A} \times \mathcal{U}^G$ at $(\mathbf{c}, 0)$ with the space $\mathcal{B}^G = \{v = u_{\mathbf{a}} + u \in \mathcal{B} : \mathbf{a} \in \mathbb{R}^r, u \in C^{2,\alpha}(\overline{M})^G\} \subset \mathcal{B}$, $u_{\mathbf{a}}$ defined as in section 5. Consider the linear operator

$$\overline{L} = 2D\overline{H}_{(\mathbf{c},0)} : \mathcal{B}^G \longrightarrow C^\alpha(\overline{M})^G.$$

We claim that $\overline{L}(\mathcal{B}^G) = \overline{L}(\mathcal{B}) \cap C^\alpha(\overline{M})^G$. Clearly the first space is contained in the second one. To prove the other inclusion, take $v \in \overline{L}(\mathcal{B}) \cap C^\alpha(\overline{M})^G$. There exists a function $u \in \mathcal{B}$ such that $\overline{L}u = v$. Then $u^G = \frac{1}{|G|}\sum_{\phi \in G}(u \circ \phi)$ lies in $\mathcal{B}^G$ —here $|G|$ denotes the number of elements of G—, and $\overline{L}(u^G) = v$ as we claimed. Now consider the Jacobi spaces

$$\mathcal{J}^G = \mathcal{J} \cap \mathcal{B}^G, \quad \mathcal{K}^G = \mathcal{K} \cap \mathcal{B}^G, \quad \mathcal{K}_0^G = \overline{L}(\mathcal{B}^G)^\perp \cap C^\alpha(\overline{M})^G.$$

Hence $\mathcal{K}_0^G \subset \mathcal{K}^G$ and reasoning as in the proof of lemma 4, we obtain $\mathcal{K}_0^G = \mathcal{K}_0 \cap \mathcal{B}^G$ and $\dim \mathcal{J}^G = r + \dim \mathcal{K}_0^G$. On the other hand, given $a \in \mathbb{R}^3$, $\langle N, a\rangle \in \mathcal{B}^G$ if and only if $\phi(a) = a$, for each $\phi \in G$, and $\det(\psi, N, e_3) \in \mathcal{B}^G$ if and only if $\det \phi = 1$, for each $\phi \in G$. Hence we have three possibilities:

1. G is generated by a rotation R_θ around the x_3-axis. In this case, the G-invariant Killing functions for ψ are $\text{Killing}^G = \text{Span}\{\det(\psi, N, e_3), \langle N, e_3\rangle\}$, and defining $\text{Killing}_0^G = \text{Killing}^G \cap \mathcal{K}_0$, it follows that $\text{Killing}_0^G = \text{Span}\{\det(\psi, N, e_3)\}$.

2. $G = \{1, \phi\}$, where ϕ is a reflection in a vertical plane, which can be supposed to be the (x_1, x_3)-plane. Then $\text{Killing}^G = \text{Span}\{\langle N, e_1\rangle, \langle N, e_3\rangle\}$ and $\text{Killing}_0^G = \text{Span}\{\langle N, e_1\rangle\}$.

3. G is generated by two reflections in distinct vertical planes. In this case, $\text{Killing}^G = \text{Span}\{\langle N, e_3\rangle\}$ and $\text{Killing}_0^G = \{0\}$.

Repeating the arguments in the proof of Theorem 2, changing each space by the corresponding G-invariant version, we can prove the following result:

Theorem 3 *Let G be a finite group of rigid motions preserving the upper halfspace. Denote by $\mathcal{M}^G \subset \mathcal{M}$ the set of G-invariant proper minimal immersions in $\mathbb{R}^3$ with finite total curvature, fixed topology and embedded ends with vertical limit normal vectors, endowed with the topology inherited as subset of $\mathcal{M}$. Let $\mathcal{M}^{G,*} = \{M \in \mathcal{M}^G : \mathcal{K}_0^G(M) = \text{Killing}_0^G(M)\}$. Then, $\mathcal{M}^{G,*}$ is an open subset of $\mathcal{M}^G$ which is either empty or admits a natural structure of $(r + \dim \text{Killing}_0^G)$-dimensional real analytic manifold and the tangent space at a point M of $\mathcal{M}^{G,*}$ is given by $\mathcal{J}^G(M)$.*

Remark 3

1. *Note that the condition $\mathcal{K}_0(M) = \text{Killing}_0(M)$ implies $\mathcal{K}_0^G(M) = \text{Killing}_0^G(M)$. In general the converse should not be true.*

2. *Consider the action*

$$\begin{array}{rccc} \Psi : & \mathcal{G} \times \mathcal{M} & \longrightarrow & \mathcal{M} \\ & (\phi, M) & \longmapsto & \phi(M) \end{array} \tag{21}$$

¿From remark 2.1 we have that Ψ is a real analytic action when restricted to $\mathcal{M}^*$. The tangent space of $\mathcal{G}$ at the identity I is generated by the infinitesimal translations along any horizontal vector x, $\dot{T}_x$, and the infinitesimal rotation around the x_3-axis, $\dot{R}$. Lemma 8 gives $(D_1\Psi)_{(I,M)}(\dot{T}_x) = \langle N, x\rangle$ and $(D_1\Psi)_{(I,M)}(\dot{R}) = \det(p, N, e_3)$, N and p being the Gauss map and the position vector on M, respectively. These equalities imply that $(D_1\Psi)_{(I,M)}$ is injective.

Corollary 1 *Suppose that $M \in \mathcal{M}^*$ is invariant by a finite subgroup G of $\widetilde{\mathcal{G}}$. Then any surface M' in the same connected component that M —in the space $\mathcal{M}^*$— is invariant by a subgroup $G' \subset \widetilde{\mathcal{G}}$ conjugate with G.*

Proof. From remark 3 we have that $\mathcal{M}^G \cap \mathcal{M}^*$ is an open subset of $\mathcal{M}^{G,*}$. Given $\varepsilon > 0$, we can consider simultaneously the immersions $j : \mathcal{M}^*_\varepsilon \longrightarrow \mathbb{R} \times C^{2,\alpha}(\{|w| \leq \varepsilon\})$ and its restriction to $\mathcal{M}^G \cap \mathcal{M}^*_\varepsilon$, $j^G : \mathcal{M}^G \cap \mathcal{M}^*_\varepsilon \longrightarrow \mathbb{R} \times C^{2,\alpha}(\{|w| \leq \varepsilon\})$. The fact that j^G is an immersion follows as in the case of j, see the proof of theorem 2. As the image of the first immersion contains the image of the second one, we conclude that $\mathcal{M}^G \cap \mathcal{M}^*$ is a submanifold of $\mathcal{M}^*$. If we look at the differential of the map $\Psi : \mathcal{G} \times \left(\mathcal{M}^G \cap \mathcal{M}^*\right) \longrightarrow \mathcal{M}^*$ given by $(\phi, M) \mapsto \phi(M)$, at the point (I, M), we see, using remark 3.2 that in the three possibilities above, its image is given by $\text{Killing}_0(M) + \mathcal{J}^G(M) = \mathcal{J}(M)$. To obtain the last equality, compute the dimension of the left-hand term taking into account that $\text{Killing}_0(M) \cap \mathcal{J}^G(M) = \mathcal{K}^G_0(M)$. Therefore Ψ is an open map. In particular, $\{\phi(M) : \phi \in \mathcal{G},\ M \in \mathcal{M}^G \cap \mathcal{M}^*\}$ is an open subset of $\mathcal{M}^*$. As it is trivially closed and, by hypothesis non empty, we conclude the proof of the corollary.

Remark 4 *The last phenomenon of "permanence of symmetries under deformations" can be observed on the minimal surfaces $M_{k,x}$, $x \geq 1$ in Hoffman and Karcher [9]: This is a one-parameter family of properly embedded minimal surfaces with genus k and three ends. The surface $M_{k,1}$ is the one obtained by Hoffman and Meeks [10]. These surfaces are all invariant under a subgroup of $\widetilde{\mathcal{G}}$ of order $2(k+1)$. However, the surface $M_{k,1}$ has additional symmetries which do not lie in $\widetilde{\mathcal{G}}$ and which dissapear during the deformation.*

2. The arguments in this paper extend almost without changes to the case in which $M \hookrightarrow \mathbb{R}^3$ is a proper minimal immersion with finite total curvature and embedded non parallel ends. The only difference appears in the space Killing_0 because

1. If M has three ends where the limit normal points to linearly independent directions, then $\text{Killing}_0 = \{0\}$.

2. If all the limit normal directions lie in a plane, say the (x_1, x_2)-plane, then $\text{Killing}_0 = \text{Span}\{\langle N, e_3\rangle\}$.

Note that in both situations, Killing$= \{\langle N, e_1\rangle, \langle N, e_2\rangle, \langle N, e_3\rangle\}$. Hence we can state

Let $\mathcal{M}$ be the space of proper minimal immersions in $\mathbb{R}^3$ with fixed topology, finite total curvature and embedded non parallel ends with prescribed limit normal values, endowed with the uniform topology on compact sets. Given $M \in \mathcal{M}$ such that $\mathcal{K}_0 =$ Killing_0, the space $\mathcal{M}$ is an $(r + \dim\ \text{Killing}_0)$-real analytic manifold in a neighbourhood of M.

As in the above section, the hypothesis of this last statement holds in particular when Killing $= \mathcal{K}$. Montiel and Ros [19] prove that if the branching values of the extended Gauss map $N : \overline{M} \longrightarrow S^2(1)$ are located on an equator of $S^2(1)$, then Killing $= \mathcal{K}$. For instance, the surface M defined by Jorge and Meeks in [11], with genus zero and r Catenoid ends distributed symmetrically in the horizontal plane has nullity three because its Gauss map is given by $z \in \overline{\mathbb{C}} \longmapsto z^{r+1} \in \overline{\mathbb{C}}$. Thus, the above result describes the set of properly immersed genus zero minimal surfaces with finite total curvature and r embedded ends parallel to the ones of M around this surface.

7 The Lagrangian submanifold structure.

Recall that if we consider on $\mathbb{R}^{2r} = \mathbb{R}^r \oplus \mathbb{R}^r$ the canonical symplectic two-form Ω given by

$$\Omega\left((\mathbf{a},\mathbf{b}),(\mathbf{a}',\mathbf{b}')\right) = \langle \mathbf{a},\mathbf{b}'\rangle - \langle \mathbf{a}',\mathbf{b}\rangle,$$

for any two vectors $(\mathbf{a},\mathbf{b}),(\mathbf{a}',\mathbf{b}') \in \mathbb{R}^r \oplus \mathbb{R}^r$, an r-dimensional immersion $f : M^r \longrightarrow \mathbb{R}^{2r}$ is said to be a **Lagrangian immersion** if $f^*\Omega = 0$.

We will continue with the same notation above. Consider the continuous map

$$\begin{array}{rccl} f: & \mathcal{M} & \longrightarrow & \mathbb{R}^{2r} \\ & M & \longmapsto & \left(\mathbf{log}(M), \mathbf{height}(M)\right), \end{array}$$

where $\mathbf{log}, \mathbf{height}$ are defined as in section 1. Let $M \in \mathcal{M}^*$ be a non degenerate surface and $\mathcal{V} \subset \mathcal{A} \times \mathcal{U}$ the analytic neighbourhood of M obtained in the proof of theorem 2. Given $M_{\mathbf{a},u} \in \mathcal{V}$, from (19) we have

$$\left.\begin{array}{l} \mathbf{log}(M_{\mathbf{a},u}) = \mathbf{a}, \\ \mathbf{height}(M_{\mathbf{a},u}) = \left(h(p_1) + \dfrac{u(p_1)}{\langle N(p_1), e_3\rangle}, \ldots, h(p_r) + \dfrac{u(p_r)}{\langle N(p_r), e_3\rangle}\right) \end{array}\right\} \tag{22}$$

As f is the restriction to $\mathcal{V}$ of a real analytic map defined on $\mathcal{A} \times \mathcal{U}$, $f : \mathcal{M}^* \longrightarrow \mathbb{R}^{2r}$ is also real analytic. Moreover, (22) implies that the differential of f at M is given, for any $v = u_{\dot{\mathbf{a}}} + \dot{u} \in \mathcal{J}(M)$ by

$$\begin{array}{l} D\mathbf{log}_M(v) = \dot{\mathbf{a}} = \mathbf{Log}(v), \\ D\mathbf{height}_M(v) = \left(\dot{u}(p_1)\langle N(p_1), e_3\rangle, \ldots, \dot{u}(p_r)\langle N(p_r), e_3\rangle\right) = \mathbf{Height}(v), \end{array}$$

Hence, $Df_M : \mathcal{J}(M) \longrightarrow \mathbb{R}^{2r}$ can be written as $Df_M = (\mathbf{Log}, \mathbf{Height})|_{\mathcal{J}(M)}$, which joint with (15) gives $f^*\Omega = 0$. On the other hand, as Kernel$(Df_M) = \mathcal{K}_0(M)$, we conclude that the rank of Df_M is r for any $M \in \mathcal{M}^*$. From remark 3.2, we get that the map $\Phi : \mathcal{G} \times \mathcal{M}^* \longrightarrow \mathcal{M}^* \times \mathcal{M}^*$ defined as $(\phi, M) \mapsto (M, \phi(M))$ is an immersion. Now we prove the main result of this section.

Theorem 4 *If $\mathcal{M}^*$ is non empty, then the quotient real analytic manifold $\mathcal{M}^*/\mathcal{G}$ is well-defined and the map $f : \mathcal{M}^*/\mathcal{G} \longrightarrow \mathbb{R}^{2r}$ defined by $f([M]) = (\mathbf{log}(M), \mathbf{height}(M))$, $[M]$ being the class of M in $\mathcal{M}^*/\mathcal{G}$, is a real analytic Lagrangian immersion.*

Proof. First we prove that Φ is a proper immersion. If $\{M_n\}_n \subset \mathcal{M}^*$ and $\{\phi_n\}_n \subset \mathcal{G}$ are sequences such that $M_n \to M_\infty \in \mathcal{M}^*$ and $\phi_n(M_n) \to M'_\infty \in \mathcal{M}^*$, then the translational part of ϕ_n is necessarily bounded. To see this, take an horizontal plane whose intersection with M_∞ is compact and transversal and look at the sections of the surfaces M_n and $\phi_n(M_n)$ with this plane. As all these sections are contained in a fixed compact of $\mathbb{R}^3$, we conclude the properness of Φ. Corollary 1 guaranties that the number of preimages by Φ of the points of $\Phi(\mathcal{G} \times \mathcal{M}^*)$ is finite and locally constant —this number coincides, given $M \in \mathcal{M}^*$, with the order of the group $\{\phi \in \mathcal{G} : \phi(M) = M\}$—. Thus $\Phi(\mathcal{G} \times \mathcal{M}^*)$, i.e. the graph of the relation defined on $\mathcal{M}^*$ by the group $\mathcal{G}$, is a closed submanifold of $\mathcal{M}^* \times \mathcal{M}^*$. It is well-known, see [6], that this is equivalent to the existence of an analytic structure on $\mathcal{M}^*/\mathcal{G}$ such that the projection $\mathcal{M}^* \longrightarrow \mathcal{M}^*/\mathcal{G}$ is a submersion. In our case, $\mathcal{M}^*/\mathcal{G}$ is clearly r-dimensional and the constant rank map $f : \mathcal{M}^* \longrightarrow \mathbb{R}^{2r}$ factorizes to the quotient. The properties of f on $\mathcal{M}^*$ imply directly that $f : \mathcal{M}^*/\mathcal{G} \longrightarrow \mathbb{R}^{2r}$ is an analytic Lagrangian immersion and the theorem is proved.

Remark 5 *The tangent space of the quotient manifold at $[M] \in \mathcal{M}^*/\mathcal{G}$ identifies naturally with the quotient $\mathcal{J}(M)/\mathcal{K}_0(M)$.*

Given a Lagrangian immersion $f : M^r \longrightarrow \mathbb{R}^{2r}$, its second fundamental form $\sigma : TM \times TM \longrightarrow TM^\perp$ can be expressed, in a more natural way, by means of a symmetric 3-tensor $C : TM \times TM \times TM \longrightarrow \mathbb{R}$ defined as $C(X, Y, Z) = \Omega(\sigma(X, Y), Z) = \Omega(XY(f), Z(f))$, X, Y and Z being tangent vector fields to M^r. By means of a long computation —which we will not reproduce here because we do not have, at this moment, any application of the next formula—, one can show that the second fundamental form C of the Lagrangian immersion $f : \mathcal{M}^*/\mathcal{G} \longrightarrow \mathbb{R}^{2r}$ is given as follows: If $[u], [v], [w] \in \mathcal{J}(M)/\mathcal{K}_0(M)$, then

$$C_{[M]}\left([u],[v],[w]\right) = \int_M \left\{u\sigma(\nabla v,\nabla w) + v\sigma(\nabla u,\nabla w) + w\sigma(\nabla u,\nabla v)\right\} dA,$$

where σ is the —real valued— second fundamental form of $M \hookrightarrow \mathbb{R}^3$.

Remark 6 *It is a simple fact that the map* $\mathbf{log} : \mathcal{M} \longrightarrow \mathbb{R}^r$ *takes values in the hyperplane* $\mathbb{R}_0^{r-1} = \{(a_1,\ldots,a_r) \in \mathbb{R}^r : a_1 + \ldots + a_r = 0\}$. *If* $M \in \mathcal{M}$ *and* $\dim \mathcal{K}(M) = 4$, *then* $M \in \mathcal{M}^*$ *and* $D\mathbf{log}_M : \mathcal{J}(M) \longrightarrow \mathbb{R}_0^{r-1}$ *is surjective. From this fact we deduce directly that the surfaces in* $\mathcal{M}^*$ *nearby to* M *are parametrized, up to translations and rotations around the vertical axis, by the logarithmic growths of their ends.*

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