THE PERIODIC ISOPERIMETRIC PROBLEM

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ABSTRACT. Given a discrete group G of isometries of $\mathbb{R}^3$, we study the G-isoperimetric problem, which consists of minimizing area (modulo G) among surfaces in $\mathbb{R}^3$ which enclose a G-invariant region with a prescribed volume fraction. If G is a line group, we prove that solutions are either families of round spheres or right cylinders. In the doubly periodic case we prove that for most rank two lattices, solutions must be spheres, cylinders or planes. For the remaining rank two lattices we show, among other results, an isoperimetric inequality in terms of the topology of the isoperimetric surfaces. Finally, we study the case where $G = Pm\overline{3}m$ (the group of symmetries of the integer rank three lattice $\mathbb{Z}^3$) and other crystallographic groups of cubic type. We prove that isoperimetric solutions must be spheres if the prescribed volume fraction is less than $1/6$, and we give an isoperimetric inequality for G-invariant regions that, for instance, implies that the area (modulo $\mathbb{Z}^3$) of a surface dividing the three space in two G-invariant regions with equal volume fractions, is at least 2.19 (the conjectured solution is the classical P Schwarz triply periodic minimal surface whose area is ~ 2.34). Another consequence of this isoperimetric inequality is that $Pm\overline{3}m$-symmetric surfaces (other than families of spheres) cannot be isoperimetric for the lattice group $\mathbb{Z}^3$.

1. INTRODUCTION

The periodic isoperimetric problem is one of the nicest open questions in classical differential geometry: given a discrete group G of isometries of the Euclidean three space, it consists of describing, among G-invariant regions Ω in $\mathbb{R}^3$ with prescribed volume (modulo G), those whose boundary has least area (modulo G). The periodic isoperimetric problem is also the simplest mathematical model to explain certain shapes appearing in a number of nanostructured interface phenomena in materials science, where spherical, cylindrical and lamellar configurations alternate with more sophisticated bicontinuous ones. In these last cases, interfaces are small perturbations of triply periodic constant mean curvature Gyroids ($G = I4_1 32$), Primitive ($G = Pm\overline{3}m$) or Diamond ($G = Fd\overline{3}m$) Schwarz surfaces and other conjectural candidates to solving the G-periodic isoperimetric problem for various crystallographic groups G; see for instance [8], [37], [39] and, in particular, Hyde [20]. Despite its great interest, both in geometry and crystalline surface science, the problem remains mostly unsolved.

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If G is a *point group* (i.e. G is finite), then, as balls solve the (usual) Euclidean isoperimetric problem, the solutions of the G-isoperimetric problem are balls centered at the fixed points of G. If G is a *line group* (i.e. if the orbits of G consist of infinitely many points lying in a solid cylinder), then it can be showed (see Theorem 6) that any G-isoperimetric region is a G-invariant family of either balls or solid cylinders. Thus, in the *nonperiodic* and the *singly periodic* cases the problem is well understood. Outside of these two cases we have only partial results, even when G is a lattice.

If G is neither a point group nor a line group, then the subgroup of translations $\Gamma = \Gamma(G)$ of G is a lattice of rank two or three and G/Γ is a finite group. These options correspond to the *doubly periodic* and *triply periodic* cases. In the first one, G is a (3-dimensional) *plane group* and the quotient space $\mathbb{R}^3/\Gamma$ is the product $T^2 \times \mathbb{R}$ of a flat 2-torus with the real line. If $\mathrm{rank}\,(\Gamma) = 3$, then G is a *space group* and the action of G can be described in terms of the finite group G/Γ acting on the flat 3-torus $T^3 = \mathbb{R}^3/\Gamma$. Hence, in all the cases, the G-isoperimetric problem in $\mathbb{R}^3$, G being a discrete group, is equivalent to the G'-isoperimetric problem on a complete orientable flat three manifold M for a finite group G'. In most parts of this paper it will be more convenient to formulate our problem in this alternative way.

In sections 2 and 3 we review several results on the usual isoperimetric problem on 3-manifolds which can be adapted to the G-invariant context. In particular, we give a complete solution for singly periodic regions.

In section 4 we study the case $M = T(\alpha, \beta) \times \mathbb{R}$, where $T(\alpha, \beta)$ is the flat 2-torus whose lattice is generated by $(1, 0)$ and (α, β), with $\alpha^2 + \beta^2 \geq 1$ and $0 \leq \alpha \leq \frac{1}{2}$ (any flat 2-torus is homothetic to just one of these tori). We prove in Theorem 13 that, given a finite group $G \subset Sym(M)$, the area A and the enclosed volume V of a G-isoperimetric surface of Euler characteristic $\chi < 0$ verify the inequality $9A^3 > 64\pi|\chi|V^2$. If $G = \{1\}$, the natural conjecture says that any isoperimetric surface is either a sphere, a cylinder or a pair of parallel planar tori. We prove this conjecture for $\beta \geq \frac{9\pi}{16} \simeq 1.767$ and we show that spheres are isoperimetric (in their admissible range) for any (α, β), outside a small neighborhood of $(1/2, \sqrt{3}/2)$ (which corresponds to the hexagonal 2-torus). It is worth noting that for the hexagonal 2-torus, the conjecture is very close to failing: we have constructed alternative candidates (which approximate constant mean curvature surfaces of genus two) whose area is bigger than the area of the best among spheres, cylinders and planes enclosing the same volume only by a factor less than 1.0003.

The triply periodic case is the subject of section 5. We focus on the cubic torus $T^3 = \mathbb{R}^3/\mathbb{Z}^3$. In this space we give an isoperimetric inequality for G-invariant regions, G being a group with a fixed point and containing the diagonal rotations of angle $2\pi/3$ (a typical example is the group $G = Pm\overline{3}m$). In particular we prove that any surface dividing T^3 in two G-invariant regions of the same volume has area larger than or equal to 2.19. The conjectured minimizing surface for this problem is the classical Primitive Schwarz minimal surface whose area is approximately 2.34; see Figure 1. We also show that for volumes between 0 and $1/6$, the unique G-isoperimetric surfaces are round spheres. If we forget the symmetry assumption, then the solution, in the equal volumes case, is known to be a pair a planar tori with total area 2; see [6], [15], [34]. Although for other prescribed volumes the problem remains open (it is conjectured that any isoperimetric surface in T^3 is either a
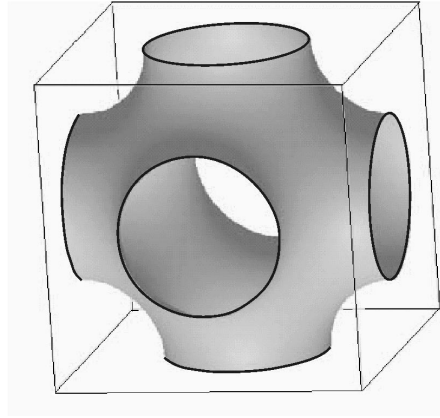
FIGURE 1. P Schwarz minimal surface is conjectured to have least area among surfaces dividing the 3-space into two $Pm\overline{3}m$-invariant regions with equal volume fractions.

sphere or a cylinder or a pair of planar tori), we prove that the only isoperimetric regions in T^3 which are invariant under the group G are round balls. This excludes a very natural family of alternative candidates of Schwarz type and should be useful to solve the isoperimetric problem in the cubic 3-torus.

The authors would like to thank Brian White for providing a proof of Theorem 3.

2. ISOPERIMETRY AND SYMMETRIES IN FLAT THREE MANIFOLDS

Among the global properties of constant mean curvature surfaces in Euclidean geometry, we will need the following ones.

Theorem 1. *Let Σ be a connected properly embedded surface with constant mean curvature in $\mathbb{R}^3$.*

- *(Alexandrov [2]) If Σ is compact, then it must be a round sphere.*
- *(Korevaar, Kusner & Solomon [21]) If Σ lies inside a cylinder, then it must be either a cylinder or a Delaunay unduloid.*
- *(Meeks [27], [21]) If Σ is a topological plane (resp. annulus), then it is a plane (resp. a cylinder or a Delaunay unduloid).*

In particular, we have that a compact surface of genus 1 embedded with constant mean curvature in a complete flat three-manifold is either planar or a flat cylinder or an unduloid. The above theorem is proved by using the Alexandrov moving plane technique. With the same idea we obtain the following standard properties of doubly periodic constant mean curvature surfaces; see for instance [33]. For convenience we state the result in the quotient space, i.e. the product of a flat 2-torus and a real line.

Proposition 2. *If Σ is a compact connected constant mean curvature surface embedded in $T^2 \times \mathbb{R}$ with genus $g \geq 2$, then Σ is symmetric with respect to a horizontal torus $\{z = a\}$ and $\Sigma^+ = \Sigma \cap \{z > a\}$ is a graph over a domain $D \subset T$, where $T - D$ consists of $g - 1$ closed pairwise disjoint topological disks. In particular, two*

surfaces of this type with the same symmetry torus (but not necessarily with the same mean curvature) must intersect.

Given a three-dimensional Riemannian manifold M^3 and a finite group of symmetries $G \subset Sym(M)$, we can consider the isoperimetric problem for G-invariant regions (or simply *G-isoperimetric problem*): among G-invariant regions with prescribed volume in M, find those whose boundary has least area. This extends the usual isoperimetric problem (which corresponds to the case $G = \{1\}$). In this paper we will study the situation where G is a finite group of symmetries of a complete orientable flat 3-manifold M (for details about these groups see for instance [16], [22]).

The fundamental existence and regularity properties (see Almgren [4], Gonzalez, Massari and Tamanini [11] and Morgan [28]) work in our setting and we can state the following result.

Theorem 3. *Let $G \subset Sym(M)$ be a finite group of isometries of a compact 3-manifold M. Then for any $0 < v < V(M)$ there exists a region $\Omega \subset M$ such that*

1. *$g(\Omega) = \Omega$ for any $g \in G$ and $V(\Omega) = v$,*
2. *$\Sigma = \partial\Omega$ minimizes the area among boundaries of all regions satisfying 1.*

Any region Ω satisfying 1 and 2 is compact and has smooth boundary Σ with constant mean curvature.

If M is noncompact, then the assertions above still hold, provided $M/Sym(M)$ is compact.

Proof. We sketch a proof due to Brian White. Given $x \in M$, we denote by G_x the group of orthogonal tranformations in $\mathbb{R}^3$ (viewed as the tangent space of M at x) obtained by linearizing the stabilizer group $\{g \in G \,|\, g(x) = x\}$ of x. The existence of Ω and regularity of Σ outside $X = \{x \in \Sigma : G_x \neq \{1\}\}$ hold by the same arguments used in the usual isoperimetric problem; see [28].

To prove that regularity does hold everywhere consider a point $x \in X$ (note that X consists of a finite union of points, curves and surfaces). Observe that the Monotonicity Theorem and existence of tangent cone holds at x. Indeed, using geodesic normal coordinates at x, we see that the radial deformations one uses to prove monotonicity preserve the right symmetries.

The tangent cone C at x is a cone in $\mathbb{R}^3$ with vertex at the origin that separates the space in two G_x-invariant (not necessarily connected) regions Ω_i, $i = 1, 2$. Moreover C is area-minimizing among surfaces that separate $\mathbb{R}^3$ in two G_x-invariant regions. The cone is a finite union of planar pieces which meet along some half-lines R leaving the origin. If $y \neq 0$ lies in one of these rays, $y \in R$, then after interchanging Ω_1 and Ω_2 if necessary, we have that, for ε small enough, each component W of $B_\varepsilon(y) \cap \Omega_1$ is a wedge with interior angle smaller than π. Clearly, the union of the two half discs in ∂W can be substituted by a surface in W with the same boundary curve and less area. By repeating this construction at all the points of the G_x-orbit of y we obtain a new surface C' which contradicts the least area property of C.

Therefore the tangent cone C is a plane, which implies that G_x is either a cyclic or dihedral group which fixes all the points in a geodesic L passing trough x and is orthogonal to C (if G_x is not of this type, we get that $x \notin \Sigma$). Then Allard's regularity theorem [3] says that the surface Σ is smooth at x. More precisely, we do not know a priori that we can apply Allard regularity at the point x, but we can

apply it in all small balls near x that do not intersect X. This shows that, near x, the surface is smooth (which we already knew) and the tangent planes are nearly perpendicular to L. This means the surface is C^1 (or even $C^{1,\alpha}$) at x and standard removal of singularities theorems in PDE now gives higher regularity around that point. □

A region Ω satisfying items 1 and 2 in the theorem above will be called a *G-isoperimetric region* and its boundary Σ a *G-isoperimetric surface*.

If G is a finite group and Ω is a G-isoperimetric region in a complete orientable flat manifold M, then its boundary Σ must be a closed (not necessarily connected) orientable surface with constant mean curvature H. Consider on Σ the quadratic form coming from the second variation formula for the area,

$$(1) \qquad Q(u, u) = \int_\Sigma \left(|\nabla u|^2 - |\sigma|^2 u^2 \right) dA,$$

where u lies in the Sobolev space $H^1(\Sigma)$ and σ is the second fundamental form of the immersion. As Σ minimizes the area up to second order under symmetry and volume constraints, we have the following useful *stability* property.

Proposition 4 ([5]). $Q(u, u) \geq 0$ *for any G-invariant function $u \in H^1(\Sigma)$ such that $\int_\Sigma u \, dA = 0$.*

The function u corresponds to an infinitesimal G-invariant normal deformation of the surface, and the zero mean value condition means that the deformation preserves the volume infinitesimally. The *Jacobi operator* $L = \Delta + |\sigma|^2$ is related to Q by the formula $Q(u, v) = -\int_\Sigma u L v \, dA$.

Some of the known properties of the standard isoperimetric problem in 3-manifolds can be adapted to the G-invariant setting. In particular we have the following result.

Theorem 5. *Let M be a complete orientable flat 3-manifold, G a finite group of symmetries of M and $\Omega \subset M$ a G-isoperimetric region bounded by a closed surface $\Sigma = \partial\Omega$. Then, either Σ is planar (but not necessarily connected) or Σ/G is connected (in particular, the components of Σ are all congruent).*

If the connected components of Σ have genus $g = 0$, then they are round spheres.

If the connected components of Σ have genus $g = 1$, then they must be flat. In particular, they are tori obtained as a quotient of either a plane or a circular cylinder.

Proof. If Σ/G is not connected, then there are on Σ nonzero locally constant G-invariant functions with zero mean value, and the stability condition in Proposition 4 implies that Σ is totally geodesic.

If the components of Σ are topological spheres, then we can lift them to $\mathbb{R}^3$ and using Theorem 1 we conclude that Σ is a G-invariant family of round spheres.

If the components of Σ are nonflat tori, then Ritoré and Ros [32] find a contradiction for the case $G = \{1\}$: they give a function u with mean value zero and such that $Q(u, u) < 0$. In the symmetric setting the function u has the required extra symmetries, and so the same argument works. □

As a direct extension of Theorem 9 of Ritoré and Ros [32], using the result above we can completely describe the solutions of the symmetric isoperimetric problem

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in the singly periodic case. If G is a line group (i.e. the orbits of G in $\mathbb{R}^3$ consist of infinitely many points lying in a solid cylinder), then there is an infinite cyclic normal subgroup $C \subset G$ (the generator of C is either a translation or a screw motion) such that G/C is a finite group of isometries of the flat manifold $\mathbb{R}^3/C$.

Theorem 6. *If G is a line group, then the solutions of the G-isoperimetric problem are spheres (if the prescribed volume V in $\mathbb{R}^3/G$ is smaller than or equal to a constant $V_0 = V_0(G)$) and cylinders (if $V \geq V_0$).*

Proof. Let $\Sigma \subset \mathbb{R}^3$ be a G-isoperimetric surface. As the planar case is clearly impossible, from Theorem 5 we have that Σ/G must be compact and connected. In particular the connected components of the pullback image of Σ are properly embedded constant mean curvature surfaces in $\mathbb{R}^3$ which lie inside a cylinder. So these components are either spheres, unduloids or cylinders by Theorem 1. From the proof of Theorem 5, we know that unduloids are G-unstable. Finally observe that a unique cylinder around a closed fixed line of G is always better for the isoperimetric problem than several cylinders parallel to that line. Thus G-isoperimetric surfaces are either a G-invariant family of spheres or circular cylinders, and the theorem follows easily. □

If the group G does not contain orientation reversing symmetries, we have the following interesting restrictions.

Theorem 7. *Let M be a complete orientable flat 3-manifold, G a finite group of orientation preserving symmetries of M, and $\Omega \subset M$ a G-isoperimetric region bounded by a closed surface $\Sigma = \partial\Omega$ with constant mean curvature H. Then either Σ is a union of parallel planar 2-tori or Σ/G is a connected Riemann surface of genus $g \leq 4$ satisfying the following restrictions:*

1. *If $g = 4$, then Σ is a minimal surface.*
2. *If $g = 2$ or 3, then $A(\Sigma)H^2 \leq 2\pi|G|$, where $|G|$ denotes the order of G.*

The proof depends on an argument first used by Hersch [17] and Yang and Yau [40]; see [31, 41] and [34]. One can consider a nonconstant meromorphic map $\phi : \Sigma/G \to S^2$ of degree as small as possible. It can be shown that, after composition with a suitable conformal transformation of the 2-sphere, the (G-invariant) coordinate functions of the map $\Sigma \to \Sigma/G \overset{\phi}{\to} S^2(1) \subset \mathbb{R}^3$ are L^2-orthogonal to 1. Then, one concludes the proof of the theorem by using these coordinate functions as test functions in the stability quadratic form (1).

Finally we want to mention some interesting facts (even though we will not use them in this paper). Ross [35] has proved that the Primitive (resp. Diamond) Schwarz minimal surface of genus three is stable for the isoperimetric problem (i.e. its area is a local minimum among surfaces which enclose a fixed volume) in the primitive (resp. face-centered) cubic 3-torus.

If Γ is a lattice generated by two or three orthogonal vectors, then any isoperimetric surface in $M = \mathbb{R}^3/\Gamma$ admits three pairwise orthogonal mirror symmetries; see [18], [30].

Recently, the double bubble problem in flat 2-tori have been solved; see [10]. For some partial results in the 3-dimensional case see [9].

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3. THE ISOPERIMETRIC PROFILE

The goal of this section is to study a certain differential inequality satisfied by the *isoperimetric profile* of a complete orientable flat manifold M with a given finite symmetry group G.

The *isoperimetric profile* $I_G = I_{M,G} : (0, V(M)) \to \mathbb{R}$ is defined as

$$I_G(v) = \inf \{A(\partial\Omega) : \Omega \subset M \text{ is a } G\text{-invariant region with } V(\Omega) = v\}.$$

In the case $G = \{1\}$ we will denote the profile simply by $I = I_M$. If the volume of M is finite, then the complementary region of a G-isoperimetric region is a G-isoperimetric region too, and so we have $I_G(V(M) - v) = I_G(v)$, for any v.

If Ω is a region in M bounded by a compact surface Σ, then we denote by Ω_t and $\Sigma_t = \partial\Omega_t$ the parallel deformation of Σ and Ω, $|t| < \varepsilon$, defined by $\Omega_0 = \Omega$, $\Omega_t = \{p \in M \mid dist_M(p, \Omega) \leq t\}$ for $t > 0$ and $\Omega_t = \{p \in \Omega \mid dist_M(p, \Sigma) \geq |t|\}$ if $t < 0$. Note that the regions Ω_t are G-invariant. Then with the notation $A(t) = A(\Sigma_t)$ and $V(t) = V(\Omega_t)$ we have the well-known formulas

$$(2) \qquad \begin{cases} V(t) = v + tA(\Sigma) + t^2 HA(\Sigma) + \frac{2\pi}{3}\chi(\Sigma)t^3, \\ A(t) = A(\Sigma) + 2tHA(\Sigma) + 2\pi\chi(\Sigma)t^2, \end{cases}$$

where $\chi(\Sigma) = \frac{1}{2\pi}\int K dA$ is the Euler characteristic of Σ. Note that (2) implies $V(t) = V(t_0) + \int_{t_0}^{t} A(s)ds$.

Consider the following family of ordinary differential equations, parametrized by a constant χ:

$$(\star) \qquad\qquad\qquad I^2 I'' + I(I')^2 - 4\pi\chi = 0.$$

Let us look for solutions $I(v)$ of $(\star)$ by making a change of variable $v = v(t)$ with the hypothesis $I(v) = dv/dt$. Such an assumption leads to a much simplified ODE: $d^3v/dt^3 = 4\pi\chi$. Clearly we deduce that positive solutions of $(\star)$ are given by the graph of the curves $\gamma(t) = (v(t), I(v(t)))$, with $I(v(t)) \geq 0$ and

$$(3) \qquad \begin{cases} v(t) = V_0 + A_0 t + H_0 A_0 t^2 + \frac{2\pi\chi}{3}t^3, \\ I(v(t)) = A_0 + 2H_0 A_0 t + 2\pi\chi t^2, \end{cases}$$

where V_0, A_0 and H_0 are real parameters. These expressions coincide with the ones in formula (2). Note that $2H_0 = dI/dv$ at $t = 0$ gives the initial slope of the solution.

In the case $\chi < 0$, it follows from (3) that for V_0, $A_0 > 0$ the function $I(v)$ is a concave function which meets the v-axis orthogonally and has just a single maximum. Moreover I is symmetric with respect to the vertical line through the maximum; see Figure 2.

Recall that given an ODE of the form $F(u'', u', u) = 0$, with $\partial F/\partial u'' > 0$, and a continuous function $w(t)$ defined in an open interval, we say that w satisfies the differential inequality $F(w'', w', w) \leq 0$ (in the weak sense) at $t = t_0$ if there is a solution $u(t)$ of the ODE defined for $|t - t_0| < \varepsilon$, such that $w(t_0) = u(t_0)$ and $w(t) \leq u(t)$. We will say that $v \mapsto J(v)$, $a < v < b$, is a *supersolution* of $(\star)$ if it is continuous and satisfies $J^2 J'' + J(J')^2 - 4\pi\chi \leq 0$ (in the weak sense) at each value of v. Note that if I is a solution of $(\star)$ with constant χ', then I is a supersolution of $(\star)$ for any constant $\chi \geq \chi'$. Several times throughout this paper we will use the following versions of the maximum principle.
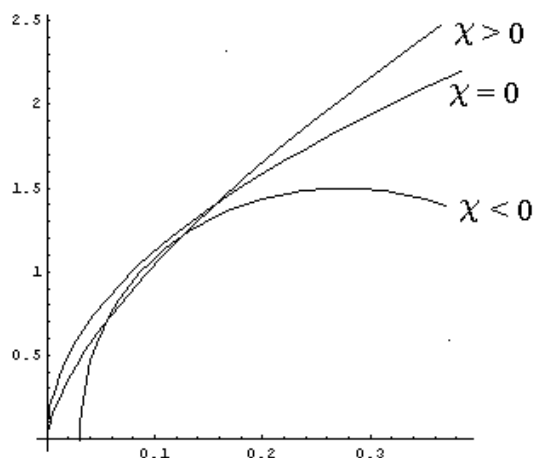
FIGURE 2. Positive solutions of $(\star)$ for different values of the parameter χ.

Lemma 8. *Let $I : [v_1, v_2] \to \mathbb{R}$ be a positive solution of $(\star)$ with constant χ and $J : [v_1, v_2] \to \mathbb{R}$ a positive continuous function which is a supersolution of $(\star)$ in $]v_1, v_2[$ with constant χ'. Assume $\chi' \leq \min\{0, \chi\}$.*

i) If $I - J$ has a nonnegative maximum at the interior of the interval, then $I = J$ in the whole interval. In particular, if $I - J$ is positive and monotonically increasing near v_1, then the same holds in $[v_1, v_2]$.

ii) If I and J coincide at the values v_1 and v_2, then either $I < J$ on open interval (v_1, v_2) or both functions coincide in that interval.

iii) If $J \geq I$ in $[v_1, v_2]$ and $I(v_0) = J(v_0)$, for some v_0 with $v_1 < v_0 < v_2$, then $I = J$ in $[v_1, v_2]$.

iv) If $I(v_1) = J(v_1)$ and $J'_+(v_1) = \limsup_{v \to v_1^+}(J(v) - J(v_1))/(v - v_1)$ is smaller than or equal to $I'(v_1)$, then $I \geq J$. Moreover $I(v_0) = J(v_0)$ at some interior value v_0, implies that $I = J$ in $[v_1, v_0]$.

Proof. To prove i), suppose that $I - J$ attains its maximum at the interior point v and $(I - J)(v) \geq 0$. We can also assume that J is not only a supersolution but a solution of $(\star)$ with constant χ'. Hence, $J(v) \leq I(v)$, $J'(v) = I'(v)$ and $J''(v) \geq I''(v)$. If $\chi > \chi'$, then using $(\star)$ we obtain

$$J''(v) + \frac{(J')^2(v)}{J(v)} = \frac{4\pi\chi'}{J(v)^2} < \frac{4\pi\chi}{I(v)^2} = I''(v) + \frac{(I')^2(v)}{I(v)} \leq J''(v) + \frac{(J')^2(v)}{J(v)}$$

and this contradiction proves the claim. The same argument works in the case $\chi = \chi' < 0$ and $J(v) < I(v)$. If $\chi = \chi'$ and $J(v) = I(v)$, the result follows from the uniqueness of solutions of $(\star)$ with given initial data. Finally, if $\chi = \chi' = 0$ the property can be checked directly (note in this case that I and J are functions of the type $v \mapsto \sqrt{av + b}$).

Items ii) and iii) follow directly from the first one.

To prove iv) choose another solution I_+ of $(\star)$ with constant χ, starting at the same point $(v_1, I(v_1))$ as I and with a slightly greater slope, so that for v close to and greater than v_1, I_+ lies strictly above J. Item i) says that $I_+ - J$ does not

admit an interior positive local maximum and, therefore, it must be monotonically increasing, first near v_1 and then in the whole $(v_1, v_2]$. As I_+ can be take abitrarily close to I, we have that $I \geq J$ in $[v_1, v_2]$ and $(I - J)(v) \leq (I - J)(w)$ for $v < w$. $\quad\square$

Theorem 9. *Let Ω be a G-isoperimetric region with $V(\Omega) = v$ in a complete flat 3-manifold M and denote by H the mean curvature of $\Sigma = \partial\Omega$ (with respect to the inwards pointing unit normal vector field).*

a) The isoperimetric profile I_G is continuous, has left and right derivatives $I'_{G+}(v)$ and $I'_{G-}(v)$, for any $0 < v < V(M)$, and

$$\text{(4)} \qquad\qquad I'_{G+}(v) \leq 2H \leq I'_{G-}(v).$$

Moreover there are G-isoperimetric regions Ω_+ and Ω_- whose mean curvatures are given by $I'_{G+}(v)/2$ and $I'_{G-}(v)/2$, respectively. In particular, I_G is differentiable at v provided that there exists a unique isoperimetric surface enclosing a volume v.

b) For $0 < v < V(M)$, I_G satisfies (in the weak sense) the differential inequality

$$\text{(5)} \qquad\qquad I_G^2 I''_G + I_G(I'_G)^2 - 4\pi\chi(\Sigma) \leq 0,$$

where $\chi(\Sigma)$ is the Euler characteristic of Σ (in particular, I_G is concave). If I_G is twice differentiable in a neighborhood (v_1, v_2) of v, then $I_G^2 I''_G + I_G(I'_G)^2 - 4\pi\chi(\Sigma) = 0$ in (v_1, v_2) if and only if any isoperimetric surface enclosing a volume between v_1 and v_2 is spherical, cylindrical or planar (this holds, in particular, for Σ itself).

c) Let k be the minimum number of points in the orbits of G. Then, for small v, any G-isoperimetric region of volume v is a G-invariant family of k metric balls. In particular

$$\text{(6)} \qquad\qquad I_G(v) = (36k\pi v^2)^{1/3}, \text{ for } v \text{ close to } 0.$$

Proof. The first part of a) is proved as in the nonsymmetric case; see [7]. The existence of the regions $\Omega_\pm$ and the statement c) follow from compactness arguments; see Theorem 2.2 in Morgan and Johnson [29] and Theorem 18 and Proposition 5 in [34].

The differential inequality (5) is a restatement (working only in the three-dimensional case) of a result of Bavard and Pansu [7]. To prove it, we consider the parallel deformation Σ_t of Σ. Using (2) we have that $V'(t) = A(t)$ and writing area in terms of volume we get

$$\frac{dA}{dV} = \frac{A'(t)}{V'(t)} = \frac{A'(t)}{A(t)} \qquad \text{and} \qquad \frac{d^2A}{dV^2} = \frac{A''(t)A(t) - A'(t)^2}{A(t)^3}.$$

Therefore

$$A^2 \frac{d^2A}{dV^2} + A\left(\frac{dA}{dV}\right)^2 = A''(t) = 4\pi\chi(\Sigma).$$

Moreover $I_G(v) = A(v)$ and, by the minimizing property of the profile, $I_G(V) \leq A(V)$ for V near v which proves (5).

The concavity of the profile is clear at the points v where $\chi(\Sigma) \leq 0$. At the points where $\chi(\Sigma) > 0$, Theorem 5 says that isoperimetric surfaces are spheres and the concavity can be checked directly.

To prove the remaining part of b), suppose that

$$\text{(7)} \qquad\qquad I_G^2 I''_G + I_G(I'_G)^2 - 4\pi\chi(\Sigma) = 0 \quad \text{in } (v_1, v_2).$$

Observe that both functions $I_G(V)$ and $A(V)$ are solutions of (7) with $I_G(v) = A(v)$ and $I'_G(v) = A'(v)$. Therefore, I_G and A coincide in a neighborhood of v. Thus the parallel surfaces Σ_t to Σ are isoperimetric surfaces, and so they all have constant mean curvature, which is only possible when Σ is spherical, cylindrical or planar. In particular, $\chi(\Sigma) \geq 0$. Now consider an isoperimetric surface Σ' enclosing a volume $V \in (v_1, v_2)$. Then, we have

$$4\pi\chi(\Sigma') \geq I_G^2(V)I''_G(V) + I_G(V)(I'_G(V))^2 = 4\pi\chi(\Sigma) \geq 0.$$

Therefore the connected components of Σ' have genus 0 or 1 and we conclude by using Theorem 5 that these components are spherical, cylindrical or planar.

Finally, since parallel deformation takes spheres to spheres, equality (7) holds in (v_1, v_2) whenever spheres are isoperimetric in this range. The same holds cylinders and planes, and we have proved b). $\qquad\square$

A closed constant mean curvature G-invariant surface Σ enclosing a domain in M is said to be *G-stable* if the quadratic form (1) verifies $Q(u, u) \geq 0$ for any G-invariant function u with $\int_\Sigma u\, dA = 0$. To finish this section, we will prove that the differential inequality (5) holds for 1-parameter families of G-stable surfaces.

Proposition 10. *Let M be a complete orientable flat 3-manifold, $G \subset Sym(M)$ a finite group and $\{\Sigma_v \mid v_1 < v < v_2\}$ a smooth family of G-stable surfaces in M, such that the volume enclosed by Σ_v is equal to v. Then, the area function $A(v) = A(\Sigma_v)$ satisfies the differential inequality*

$$A^2 A'' + A(A')^2 - 4\pi\chi \leq 0,$$

where χ is the Euler characteristic of Σ_v (which does not depend on v).

Proof. For fixed volume v, denote by $\Sigma = \Sigma_v$, $u : \Sigma_v \to \mathbb{R}$ the infinitesimal normal variation of the family at Σ, and L the Jacobi operator of Σ. Standard variation formulae for the volume, area and mean curvature H give $\int_\Sigma u = -1$, $A'(v) = -2H\int_\Sigma u = 2H$ and $A''(v) = 2H'(v) = Lu$.

Now define $a = -1/A$. As Σ is G-stable and $u - a$ is a G-invariant function with zero mean value, we have

$$0 \leq Q(u - a, u - a) = Q(u, u) - 2Q(u, a) + Q(a, a)$$

$$= -\int_\Sigma uLu + 2a\int_\Sigma Lu - a^2\int_\Sigma |\sigma|^2 = \left(-\int_\Sigma u + 2aA\right)A'' - \frac{1}{A^2}\int_\Sigma (4H^2 - 2K)$$

$$= -A'' - \frac{(A')^2}{A} + \frac{4\pi\chi}{A^2},$$

where K denotes the Gauss curvature of Σ and we have used the Gauss-Bonnet Theorem. $\qquad\square$

4. THE DOUBLY PERIODIC CASE

Any flat 2-torus T^2 is homothetic to a torus $T(\alpha, \beta) = \mathbb{R}^2/\Gamma(\alpha, \beta)$ defined by the lattice $\Gamma(\alpha, \beta)$ spanned by the vectors $(1, 0)$ and (α, β), with $\alpha^2 + \beta^2 \geq 1$, $0 \leq \alpha \leq 1/2$ and $0 < \beta$. The area of $T(\alpha, \beta)$ and the length of its shortest closed geodesic are given by β and 1, respectively. In particular

$$(8) \qquad\qquad A(T(\alpha, \beta)) \geq \frac{\sqrt{3}}{2},$$

with equality for the hexagonal torus, i.e. the quotient of $\mathbb{R}^2$ over the hexagonal lattice generated by two unit vectors forming an angle of $\pi/3$.

The conjectured candidates for isoperimetric surfaces in $T(\alpha,\beta) \times \mathbb{R}$ are spheres, cylinders and pairs of horizontal planes. Namely, for $0 < r < 1/2$ the round sphere of radius r embeds isometrically in $T(\alpha,\beta) \times \mathbb{R}$, and if we write area as function of volume we have $A(V) = (36\pi V^2)^{1/3}$. Analogously, the best cylinders for the isoperimetric problem are those constructed around a closed geodesic of length 1. If the cylinder has radius r, $0 < r < \sqrt{3}/4$, then its area $A(r) = 2\pi r$ and volume $V(r) = \pi r^2$ are related by the equation $A(V) = 2\sqrt{\pi V}$, while the area of a pair of parallel planar tori equals $A(V) = 2\beta$.

We define the *spheres-cylinders-planes profile* (scp profile) of $T(\alpha,\beta) \times \mathbb{R}$ as the function $I_{\mathrm{scp}} : (0,\infty) \to \mathbb{R}^+$ which gives the least area among spheres, cylinders and pairs of parallel planes enclosing a volume V. From the paragraph above we have that (see Figure 3)

$$I_{\mathrm{scp}}(V) = \begin{cases} (36\pi V^2)^{1/3} & \text{if } 0 < V \leq \frac{4\pi}{81} \text{ spherical,} \\ 2\sqrt{\pi V} & \text{if } \frac{4\pi}{81} \leq V \leq \frac{\beta^2}{\pi} \text{ cylindrical and} \\ 2\beta & \text{if } \frac{\beta^2}{\pi} \leq V \text{ planar range.} \end{cases}$$
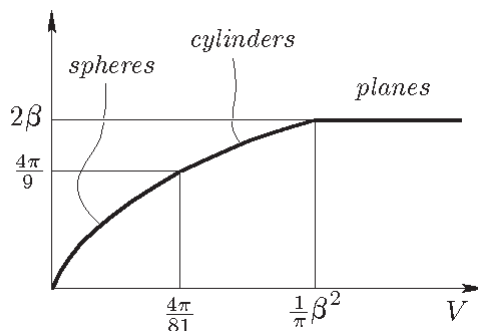

FIGURE 3. The spheres-cylinders-planes profile in $T(\alpha,\beta) \times \mathbb{R}$.

Recall that $2H$ is the slope of I_{scp} at its smooth points. The relationship between area, volume and mean curvature is $H = \frac{A}{3V}$ in the spherical branch, $H = \frac{A}{4V}$ in the cylindrical range and $H = 0$ in the planar range.

If we consider the isoperimetric problem in $T^2 \times \mathbb{R}$ with no extra symmetries, it is an interesting and natural problem to decide for which tori $T(\alpha,\beta)$ the scp profile coincide with isoperimetric profile. Consider the following statement.

Conjecture 1. *The unique solutions of the isoperimetric problem in $T(\alpha,\beta) \times \mathbb{R}$ are spheres, cylinders around closed geodesic and parallel pairs of totally geodesic tori. In particular, the isoperimetric profile of $T(\alpha,\beta) \times \mathbb{R}$ coincides with its* scp *profile.*

Using compactness arguments, Ritoré and Ros [32] proved that the conjecture holds for β large enough. We will now give an explicit upper bound on β.

Let $\Sigma \subset T^2 \times \mathbb{R}$ be a compact, connected, embedded surface with Euler characteristic χ, constant mean curvature H, area A and enclosed volume V. Denote

by $N = (N_1, N_2, N_3)$ the inwards pointing unit normal vector along Σ. We shall consider a family of deformations of Σ in $T^2 \times \mathbb{R}$, obtained by applying the affine transformation $\phi_\lambda : (x_1, x_2, x_3) \mapsto (x_1, x_2, (1+\lambda)x_3)$. Henceforth $O(t^n)$ will denote a function such that $O(t^n)/t^n$ is bounded when $t \to 0$.

Lemma 11. *If Σ is a compact surface with constant mean curvature embedded in $T^2 \times \mathbb{R}$, then the area of the surface $\Sigma_\lambda = \phi_\lambda(\Sigma)$, with $|\lambda| < \varepsilon$, is*

$$A(\Sigma_\lambda) = A + \lambda \int_\Sigma (1 - N_3^2)\, dA + \frac{\lambda^2}{2} \int_\Sigma N_3^2(1 - N_3^2)\, dA + O(\lambda^3),$$

and the volume enclosed by Σ_λ is $(1+\lambda)V$.

Proof. In order to compute the variation of the area, it is enough to consider a piece S of Σ given by the graph of a function $(x, y, u(x, y))$, with $(x, y) \in D$. For that surface we have

$$N_3 = \frac{\pm 1}{\sqrt{1 + |\nabla u|^2}}, \qquad A(\lambda) = A(S_\lambda) = \int_D \sqrt{1 + (1+\lambda)^2 |\nabla u|^2}\, dxdy,$$

and therefore

$$A'(0) = \int_D \frac{|\nabla u|^2}{1 + |\nabla u|^2} \sqrt{1 + |\nabla u|^2}\, dxdy = \int_S (1 - N_3^2)\, dA$$

and

$$A''(0) = \int_D \frac{|\nabla u|^2}{(1 + |\nabla u|^2)^{3/2}}\, dxdy = \int_S N_3^2(1 - N_3^2)\, dA.$$

$\square$

Let G be a finite subgroup of $Sym(T^2 \times \mathbb{R})$. We will assume that G fixes the planar torus $x_3 = 0$. By applying the transformation $(x_1, x_2, x_3) \mapsto (x_1, x_2, (1+\lambda)x_3)$ to the parallel surface Σ_t of Σ we have a two-parameter deformation of G-invariant surfaces $(t, \lambda) \mapsto \Sigma_{t,\lambda}$. This deformation allows us to obtain the following useful information.

Proposition 12. *Let $\Sigma \subset T^2 \times \mathbb{R}$ be a nonplanar G-isoperimetric surface with Euler characteristic χ, mean curvature H, area A and enclosed volume V. Then,*

$$\tag{9} \int_\Sigma N_3^2\, dA = A - 2HV$$

and

$$\tag{10} 4HV - \frac{12H^2V^2}{A} + \frac{4V^2}{A^2}\pi\chi + \int_\Sigma N_3^2(1 - N_3^2)\, dA \geq 0,$$

where $N = (N_1, N_2, N_3)$ is the Gauss map of Σ.

Proof. The area and volume functions of the parallel surfaces $t \mapsto \Sigma_t$ are

$$\tag{11} A(t) = A(\Sigma_t) = A + 2tHA + 2\pi\chi t^2 \,,\ V(t) = V + tA + t^2HA + O(t^3).$$

Therefore, Lemma 11 and the fact that the surfaces Σ and Σ_t have the same normal vector, imply that the area and volume functions of $(t, \lambda) \mapsto \Sigma_{t,\lambda}$ are

$$A(t, \lambda) = A(\Sigma_{t,\lambda}) = A(t) + \lambda \int_\Sigma (1 - N_3^2)\, dA(t) + \frac{1}{2}\lambda^2 \int_\Sigma N_3^2(1 - N_3^2)\, dA(t) + O(\lambda^3)$$

and

$$V(t, \lambda) = (1+\lambda)V(t),$$

where $dA(t) = (1 + 2tH + t^2 K)dA$ stands for the area element of Σ_t. If we take $t = t(\lambda)$, expanding t in a neighborhood of $\lambda = 0$ as

(12) $$t(\lambda) = t_1 \lambda + \frac{t_2}{2}\lambda^2 + O(\lambda^3),$$

a straightforward computation gives that the area function of the composed deformation is

$$A(\lambda) = A(t(\lambda), \lambda) = A + \lambda\left(2AHt_1 + \int_\Sigma (1 - N_3^2)\, dA\right)$$
$$+ \lambda^2\left\{2\pi\chi t_1^2 + AHt_2 + 2Ht_1 \int_\Sigma (1 - N_3^2)\, dA + \frac{1}{2}\int_\Sigma N_3^2(1 - N_3^2)\, dA\right\} + O(\lambda^3).$$

Analogously, the volume function is

(13) $$V(\lambda) = (1+\lambda)V(t(\lambda)) = V + \lambda(V + At_1) + \lambda^2\left\{AHt_1^2 + At_1 + \frac{A}{2}t_2\right\} + O(\lambda^3).$$

From now on, we choose $t(\lambda)$ such that the deformation keeps the volume fixed. By (13), this implies that

$$t_1 = -\frac{V}{A} \quad \text{and} \quad t_2 = \frac{2V}{A} - \frac{2HV^2}{A^2}.$$

Plugging these equalities into the area function it follows that

$$A(\lambda) = A + \lambda\left(-2HV + \int_\Sigma (1 - N_3^2)\, dA\right) + \lambda^2\left\{+2\pi\chi\frac{V^2}{A^2} + 2HV\right.$$
$$\left. -\frac{2H^2V^2}{A} - \frac{2HV}{A}\int_\Sigma (1 - N_3^2)\, dA + \frac{1}{2}\int_\Sigma N_3^2(1 - N_3^2)\, dA\right\} + O(\lambda^3).$$

The constant mean curvature assumption means exactly that the coefficient of λ vanishes, which gives the first formula in the statement. The stability of Σ under G-invariant volume preserving deformations implies that $A''(0) \geq 0$, and the second formula in the proposition follows after substitution of $\int_\Sigma (1 - N_3^2)\, dA$ by $2HV$. $\square$

Theorem 13. *Let G be a finite group of symmetries of $T^2 \times \mathbb{R}$ and Σ a nonplanar G-isoperimetric surface with Euler characteristic $\chi < 0$, mean curvature H, area A and enclosed volume V. Then,*

$$\frac{64\pi}{9}|\chi| < \frac{A^3}{V^2} \quad \text{and}$$
$$\frac{3A}{16V}\left(1 - \sqrt{1 + \chi\frac{64\pi V^2}{9A^3}}\right) < H < \frac{3A}{16V}\left(1 + \sqrt{1 + \chi\frac{64\pi V^2}{9A^3}}\right).$$

Proof. Using the Cauchy-Schwarz inequality and the equality in Proposition 12, we can estimate

$$\int_\Sigma N_3^2(1 - N_3^2)\, dA = \int_\Sigma N_3^2\, dA - \int_\Sigma N_3^4\, dA$$
$$\leq \int_\Sigma N_3^2\, dA - \frac{1}{A}\left(\int_\Sigma N_3^2\, dA\right)^2 = 2HV - \frac{4H^2V^2}{A}.$$

Clearly, the inequality above must be strict, otherwise N_3 would be constant, which contradicts the hypothesis.

Joining the above inequality with the one in Proposition 12 we obtain

$$(14) \qquad 0 < 3H - \frac{8H^2V}{A} + \frac{2V}{A^2}\pi\chi.$$

The right-hand-side of (14) can be considered as a parabola in H with negative main coefficient, thus its discriminant must be positive, i.e. $9A^3 + 64\pi V^2\chi > 0$. Moreover, H must be trapped between the two roots of $3H - \frac{8H^2V}{A} + \frac{2V}{A^2}\pi\chi = 0$. Now the theorem is proved. $\qquad\square$

Corollary 14. *In $T^2 \times \mathbb{R}$, any G-isoperimetric surface with genus greater than one encloses a volume strictly less than $\frac{3}{4\sqrt{\pi}}A(T)^{3/2}$.*

Proof. Use the isoperimetric inequality in Theorem 13 together with $A \le 2A(T)$. $\qquad\square$

As a consequence of Theorem 13, we can solve Conjecture 1 for a large family of spaces of the form $T^2 \times \mathbb{R}$. Let us consider the set

$$\mathcal{A} = \{(\alpha, \beta)|\ T(\alpha, \beta) \times \mathbb{R} \text{ does not satisfy Conjecture 1}\}.$$

Corollary 15. *If $\beta \ge \frac{9\pi}{16} \simeq 1.767$, then $T(\alpha, \beta) \times \mathbb{R}$ satisfies Conjecture 1, i.e. the unique isoperimetric surfaces in $T(\alpha, \beta) \times \mathbb{R}$ are spheres, cylinders and planes.*

Proof. As Conjecture 1 is true for large β (see [32]) to prove the proposition it is enough to see that any pair (α, β) in $\partial\mathcal{A}$ satisfies $\beta < 9\pi/16$. The compactness of the space of isoperimetric surfaces implies that for $(\alpha, \beta) \in \partial\mathcal{A}$ the isoperimetric profile I of $T(\alpha, \beta) \times \mathbb{R}$ coincides with its scp profile, but there exists an isoperimetric surface $\Sigma \subset T(\alpha, \beta) \times \mathbb{R}$ other than spheres, cylinders and planar tori. From Theorem 5 we know that Σ has Euler characteristic $\chi \le -2$. In particular, if A and V denote the area of Σ and the volume it encloses, then we have $A = I(V)$. On the other hand, Theorem 9 implies

$$(15) \qquad I^2 I'' + I(I')^2 - 4\pi\chi \le 0 \quad \text{at} \quad V,$$

which is impossible if V lies at the interior of either the spherical range or the cylindrical or the planar one by item b) of Theorem 9. Thus V lies at one of the transition points of I, that is, either $(V, A) = (4\pi/81, 4\pi/9)$ or $(V, A) = (\beta^2/\pi, 2\beta)$.

In the first case V separates the spherical and the cylindrical branches, and the mean curvature H of Σ must be greater than or equal to the one of the cylinder with the same area (and enclosing the same volume): to see that, observe that, otherwise, for $\varepsilon > 0$ sufficiently small, the equidistant surfaces Σ_t, $0 < t < \varepsilon$, would have less area than the cylinders enclosing the same volume. Hence

$$H^2 A \ge (\frac{A}{4V})^2 A = \frac{9\pi}{4} > 2\pi,$$

which contradicts Theorem 7. Therefore Σ has area $A = 2\beta$ and encloses a volume $V = \beta^2/\pi$. Plugging these values in the first inequality of Theorem 13 we obtain $\beta < 9\pi/16$. $\qquad\square$

Remark 1. From arguments in the proof of the last corollary one deduces that if $(\alpha, \beta) \in \partial\mathcal{A}$ and $\Sigma \subset T(\alpha, \beta) \times \mathbb{R}$ is an isoperimetric surface with Euler characteristic $\chi \le -2$, then the volume enclosed by Σ separates the spherical and cylindrical branches of the profile. After substitution of the values $V = \frac{\beta^2}{\pi}, A = 2\beta$ in the

second inequality of Theorem 13, we obtain the following additional information about the nonstandard isoperimetric surface Σ:

$$\frac{3\pi}{8\beta}\left(1 - \sqrt{1 + \chi\frac{8\beta}{9\pi}}\right) < H < \frac{3\pi}{8\beta}\left(1 + \sqrt{1 + \chi\frac{8\beta}{9\pi}}\right).$$

Proposition 16. *If $\mathcal{A} \neq \emptyset$, then the boundary $\partial\mathcal{A}$ is a graph over its horizontal projection $(\alpha, \beta) \mapsto \alpha$ on $[0, 1/2]$, and $\mathcal{A}$ lies below this graph.*

Proof. Consider $(\alpha, \beta) \in \partial\mathcal{A}$ and suppose that $\beta > \sqrt{1 - \alpha^2}$. As $\mathcal{A}$ is compact, to prove the proposition it is enough to see that, for some $\delta < \beta$, $\{(\alpha, \beta') \mid \delta < \beta' < \beta\}$ is contained in the interior of $\mathcal{A}$.

The isoperimetric profile of $T(\alpha, \beta) \times \mathbb{R}$ coincides with its scp profile, and there exists an isoperimetric surface $\Sigma \subset T(\alpha, \beta) \times \mathbb{R}$ with genus $g \geq 2$. By the proof of Corollary 15 we know that Σ encloses a volume $V = \frac{\beta^2}{\pi}$ and has area $A = 2\beta$. We shall consider a deformation of Σ, composed of the deformation through parallel surfaces $\Sigma \mapsto \Sigma_t \subset T(\alpha, \beta) \times \mathbb{R}$ (t close to zero), followed by the affine deformation (which changes the ambient space!)

$$\varphi_\lambda : T(\alpha, \beta) \times \mathbb{R} \to T(\alpha, (1 + \lambda)\beta) \times \mathbb{R} \mid \varphi_\lambda(x, y, z) = (x, (1 + \lambda)y, z),$$

where λ is close to zero.

The area and volume functions $A(t, \lambda), V(t, \lambda)$ of $(t, \lambda) \mapsto \Sigma_{t,\lambda} = \varphi_\lambda(\Sigma_t)$ are given by expressions similar to those in the proof of Proposition 12 after replacing N_3 with N_2 for obvious reasons. Taking $t = t(\lambda) = t_1\lambda + O(\lambda^2)$, we find

$$A(\lambda) = A(\Sigma_{t(\lambda),\lambda}) = A + \lambda\left(2AHt_1 + \int_\Sigma (1 - N_2^2)\, dA\right) + O(\lambda^2),$$

while

$$V(\lambda) = V(t(\lambda), \lambda) = (1 + \lambda)V(\Omega_{t(\lambda)}) = V + \lambda(V + At_1) + O(\lambda^2).$$

We shall determine the function $t(\lambda)$ by using the condition that the volume function $V(\lambda)$ of the composed deformation coincides with the volume separating the cylindrical zone and the planar zone in the scp profile $I_{\text{scp},\lambda}$ for the new ambient space $T(\alpha, (1 + \lambda)\beta) \times \mathbb{R}$, which is given by $(1 + \lambda)^2 V = (1 + \lambda)^2 \beta^2/\pi$.

This condition is easily seen to be equivalent (at the first order) to the equation $t_1 = V/A$ which gives the area function

$$
\begin{aligned}
A(\lambda) &= A + \lambda\left(2HV + \int_\Sigma (1 - N_2^2)\, dA\right) + O(\lambda^2) \\
&= A + \lambda\left(\int_\Sigma (1 - N_3^2)\, dA + \int_\Sigma (1 - N_2^2)\, dA\right) + O(\lambda^2) \\
&= A + \lambda\int_\Sigma (1 + N_1^2)\, dA + O(\lambda^2),
\end{aligned}
$$

where we have used Proposition 12 in the second equality. Let us compare the area of $\Sigma_{t(\lambda),\lambda}$ and the scp profile (whose value at $(1+\lambda)^2 V$ is just $2(1+\lambda)\beta = (1+\lambda)A$). Order one information is sufficient, and we have

$$
\begin{aligned}
A(\lambda) - I_{\text{scp},\lambda}((1 + \lambda)^2 V) &= \lambda\left(\int_\Sigma (1 + N_1^2)\, dA - A\right) + O(\lambda^2) \\
&= \lambda\int_\Sigma N_1^2\, dA + O(\lambda^2).
\end{aligned}
$$

We point out that the last integral, $\int_\Sigma N_1^2\, dA$, cannot be zero, otherwise N_1 vanishes identically on Σ. Thus Σ must be flat, which contradicts our assumption on the genus of Σ. We conclude by choosing $\lambda < 0$ close to 0 so that $(\alpha, (1+\lambda)\beta)$ lies in the interior of $\mathcal{A}$. $\square$

4.1. **Isoperimetry of spheres.** Our next goal is to prove that except for a very small range in the moduli space of tori, the only isoperimetric surfaces in $T^2 \times \mathbb{R}$ (no symmetries are imposed, i.e. $G = \{1\}$) on the whole spherical range of the scp profile (volumes less than or equal to $4\pi/81$) are spheres. In particular, this property will be true for any rectangular torus. To obtain such a result we will need the following inequality.

Proposition 17. *Let Σ be a compact, connected surface with genus $g \geq 2$ in $T^2 \times \mathbb{R}$ with constant mean curvature H, area A and enclosing a volume V. Then,*

$$AH^2(A - 2HV) + 2\pi\sqrt{A(A - 2HV)} \geq 4\pi A(T^2).$$

Proof. From Proposition 2 we know that Σ is symmetric with respect to a horizontal plane (which can be supposed to be $x_3 = 0$) and that $\Sigma^+ = \Sigma \cap \{x_3 \geq 0\}$ is the graph of a function defined on a domain $D \subset T$, whose boundary ∂D consists of $g - 1$ Jordan curves. As these Jordan curves bound disks in T, the euclidean isoperimetric inequality applied in $T - D$ gives

$$(16) \qquad\qquad L(\partial D)^2 \geq 4\pi(A(T) - A(D)),$$

where as usual, A, L denote area and length of the corresponding objects. The Jacobian of the projection of Σ^+ onto D is given by $-N_3$, N_3 being the third component of the inwards pointing unit normal vector along Σ. Hence $A(D) = -\int_{\Sigma^+} N_3\, dA$. On the other hand, applying the divergence theorem to the vector field $X = \text{tangent part of } e_3$ on Σ^+ (whose divergence is $2HN_3$), we obtain

$$-2HA(D) = 2H\int_{\Sigma^+} N_3\, dA = \int_{\partial\Sigma^+} \langle X, -e_3\rangle\, ds = -L(\partial D),$$

which combined with (16) implies

$$(17) \qquad\qquad H^2 A(D)^2 + \pi A(D) - \pi A(T) \geq 0.$$

From the Schwarz inequality one has

$$(18) \qquad A(D)^2 = \left(\int_{\Sigma^+} N_3\, dA\right)^2 \leq \frac{A}{2}\int_{\Sigma^+} N_3^2\, dA = \frac{A}{4}(A - 2HV),$$

where we have used Proposition 12 in the last equality. Combining (17) and (18) the proposition follows. $\square$

Theorem 18. *Let $\beta_0 = \frac{2\pi}{27}\left(3 - \sqrt{2} + \sqrt{9 - 3\sqrt{2}}\right)$. For any $\beta > \beta_0$, the unique isoperimetric surfaces $\Sigma \subset T(\alpha, \beta) \times \mathbb{R}$ with enclosed volume $0 < V \leq \frac{4\pi}{81}$ are spheres and cylinders (the last ones only for $V = \frac{4\pi}{81}$).*

Remark 2. To give an idea of how restrictive the condition $\beta > \beta_0$ is, note that $\beta_0 \simeq 0.876603$ and the minimum value of β representing tori $T(\alpha, \beta)$ is $\beta = \frac{\sqrt{3}}{2} \simeq 0.866025$. In particular, Theorem 18 applies to all rectangular tori (i.e. $\alpha = 0$ and $\beta \geq 1$).

Proof. Suppose that for a given $T = T(\alpha, \beta)$, there exists an isoperimetric surface in $T \times \mathbb{R}$ with genus $g \geq 2$ enclosing a volume $V \in (0, \frac{4\pi}{81}]$. If β_1 is the largest β among tori satisfying this property (Corollary 15 implies that $\beta_1 \leq 9\pi/16$), then from the maximum principle in Lemma 8 and the compactness of the space of isoperimetric surfaces it follows that, for some α, there must exist an isoperimetric surface $\Sigma \subset T(\alpha, \beta_1) \times \mathbb{R}$ with genus $g \geq 2$, area $A = \frac{4\pi}{9}$ and enclosed volume $V = \frac{4\pi}{81}$. Using Proposition 17 with these area and volume we get

$$(19) \qquad \frac{2\pi}{729} \left(18H^2 - 4H^3 + 27\sqrt{9 - 2H} \right) \geq A(T_1) = \beta_1,$$

where H is the mean curvature of Σ and $T_1 = T(\alpha, \beta_1)$. On the other hand, the inequality $AH^2 \leq 2\pi$ of Theorem 7 ensures that $H \leq \frac{3}{\sqrt{2}}$.

Define $f(h) = \frac{2\pi}{729}(18h^2 - 4h^3 + 27\sqrt{9 - 2h})$, with $h \in [0, \frac{3}{\sqrt{2}}]$. Equation (19) implies that $\beta_1 \leq f(H) \leq \max_{[0, 3/\sqrt{2}]} f$. A direct computation shows that f' has the same sign as the polynomial $P(h) = (36h - 12h^2)^2(9 - 2h) - 27^2$. By the intermediate value theorem it can be checked that P has five real roots, one in each interval $(-1, 0), (0, 1), (\frac{5}{2}, 3), (3, 4), (4, 5)$. In particular, there exists h_0, $0 < h_0 < \frac{3}{\sqrt{2}}$, such that $P(h_0) = 0$, $P < 0$ in $[0, h_0)$ and $P > 0$ in $(h_0, \frac{3}{\sqrt{2}}]$ (in fact, $h_0 \simeq 0.285499$). This implies that f is decreasing in $[0, h_0)$, increasing in $(h_0, \frac{3}{\sqrt{2}}]$ and has a local minimum at h_0. As $f(0) < f(\frac{3}{\sqrt{2}})$, we deduce that $\max_{[0, 3/\sqrt{2}]} f = f(\frac{3}{\sqrt{2}}) = \beta_0$, which finishes the proof. $\qquad \square$

After Remark 1 it is likely that, in order to prove Conjecture 1 for $T(\alpha, \beta) \times \mathbb{R}$, the hardest situation holds for the hexagonal torus $T(\frac{1}{2}, \frac{\sqrt{3}}{2}) \times \mathbb{R}$ and, in this space, for the volume $V = \frac{3}{4\pi}$ which separates the cylindrical and the planar branches. The cylinder (and the pair of planar tori) enclosing this volume have area equal to $\sqrt{3}$. The most serious alternative candidates to solve the isoperimetric problem in that case are constant mean curvature surfaces Σ of genus two which are symmetric with respect to the planar torus $z = 0$ and such that $\Sigma \cap \{z = 0\}$ is a convex curve; see Figure 4 (these surfaces are called *meshes* or *lamellar catenoids* in materials science). The first surfaces of this type were obtained by Lawson [23] by using the conjugate Plateau problem construction; see also Grosse-Brauckmann [12], [13] and Ritoré [30]. By suitably gluing a piece of nodoid and pieces of planes we have constructed a lamellar catenoid shaped surface, with Euler characteristic -2, in $T(\frac{1}{2}, \frac{\sqrt{3}}{2}) \times \mathbb{R}$, enclosing a volume $V = \frac{3}{4\pi}$ and whose area lies between $I_{\text{scp}}(\frac{3}{4\pi}) = \sqrt{3}$ and $1.0003 \times \sqrt{3}$. This surface looks like the ones in Figure 4.

5. Isoperimetric inequalities for triply periodic regions

The purpose of this section is to give an isoperimetric inequality for G-symmetric surfaces in the cubic torus $T = \mathbb{R}^3/\mathbb{Z}^3$, where G is any finite group of isometries of the torus fixing the origin of T and containing the diagonal rotations through the origin. These are rotations of angle $\pm 2\pi/3$ around the axes of directions $(\pm 1, \pm 1, 1)$. We will eventually show that the isoperimetric profile I_G for this problem lies above the spheres-cylinders-planes profile I_{scp}, which in particular implies that G-symmetric surfaces in T other than spheres cannot be isoperimetric in T (for the nonsymmetric problem).
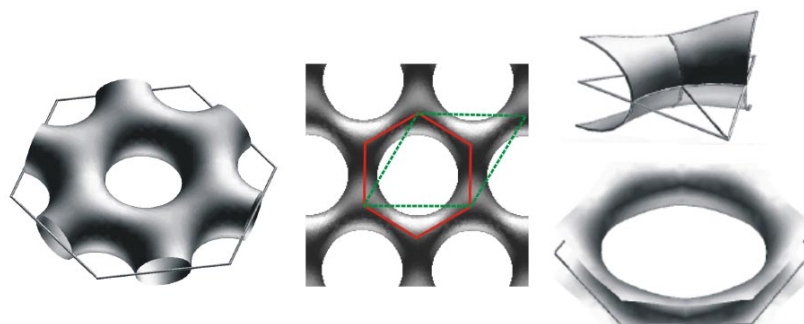
FIGURE 4. Left, center: Genus two surfaces with constant mean curvature in the product of the hexagonal torus and the real line. Right: Fundamental pieces which, after either reflexions or translations, fill up the whole surface. Computer graphics by Karsten Grosse-Brauckmann, [13].

It can be checked that the groups G satisfying the restrictions above are (taking pullback groups in $\mathbb{R}^3$ and using crystallographic notation; see [22], [16]) $P23$, $P432$, $P\overline{4}3m$, $Pm\overline{3}$ and $Pm\overline{3}m$. They leave two points fixed, $(0,0,0)$ and $(\frac{1}{2},\frac{1}{2},\frac{1}{2})$, and the first (resp. the last) one is a subgroup (resp. supergroup) of the others. We conjecture that all these groups have the same isoperimetric surfaces which (when viewed in the torus) consist of either a single sphere (for small volumes) or a genus three surface of Schwarz type (for volumes close to $1/2$). In particular, among surfaces dividing the torus in two equal volume G-invariant regions, the classical Schwarz minimal surface should be the one with smaller area; see Figure 1. We will prove in Theorem 20 that this area must be at least 2.19 (the actual area of Schwarz minimal surface is $\simeq 2.34$). We will also prove that spheres are G-isoperimetric for volume smaller than $1/6 \simeq 0.17$. According to the numerical computations in [1], spheres should be optimal up to volume $\simeq 0.29$.

Proposition 19. *Any solution Σ of the isoperimetric problem for G-symmetric surfaces in the cubic torus T is either a single sphere (centered at a fixed point of G) or has Euler characteristic at most -4.*

Proof. The key property to prove the proposition is that G contains rotations of angle $2\pi/3$ with axes in three independent directions. We can assume that the volume V of the region Ω enclosed by Σ satisfies $0 < V \leq 1/2$. From the concavity of the profile I_G (see Theorem 9) we have that Ω is mean convex, i.e. the mean curvature of Σ (with respect to the inwards pointing normal) is ≥ 0.

It ought to be noted that either Σ is a union of (quotients of) planes or the quotient surface Σ/G is connected (see Theorem 5) and so the components of Σ are all congruent (via symmetries of G) and consist of closed orientable surfaces. In the first case, embeddedness forces the planes to be parallel to a single fixed plane, which is impossible under the action of G.

If these components then have genus zero, by Theorem 1, they must be round spheres of the same radius $r > 0$. As there exists a single sphere centered at a fixed point of G (hence G-invariant) enclosing any volume between 0 and $1/2$, it is clear

that this sphere will do better than any other nonconnected G-invariant family of spheres.

Suppose now that the components of Σ have genus one. These surfaces consist of a union of n (quotients of) cylinders. The orbit under G of a cylinder may yield an embedded disconnected surface, however it turns out to be nonoptimal: to see this, note that such a cylinder is the tubular neighborhood of radius $r > 0$ of a closed geodesic of T of length ℓ, so that the total area of Σ is $A = 2\pi n\ell r$ and the enclosed volume $V = \pi n\ell r^2$. Since we assume that this surface is isoperimetric, it has to do better than the competitor sphere of the same volume. Using the isoperimetric profile for spheres, this condition amounts to

$$36\pi \geq \frac{A^3}{V^2} = \frac{8\pi^3 n^3 \ell^3 r^3}{\pi^2 n^2 \ell^2 r^4} = \frac{8\pi n\ell}{r} \ .$$

However ℓ is certainly greater than or equal to 1, while the property of G mentioned above forces $n \geq 3$. We infer that $r \geq 2/3$, which gives a volume $V \geq 4\pi/3$ which is much more than the limit $1/2$.

Next consider the case where Σ is a connected surface with genus two. We discard this possibility by considering the full lift $\widetilde{\Sigma}$ of Σ to $\mathbb{R}^3$. As the mean convex region enclosed by Σ is a solid donut (see Meeks [26]) it follows that the components of $\widetilde{\Sigma}$ are at most doubly periodic. If they were nonperiodic (resp. singly periodic), then, by Theorem 1, they should be spheres (resp. Delaunay surfaces or cylinders). As both options contradict the genus assumption, we conclude that $\widetilde{\Sigma}$ consists of translated images of a certain doubly periodic constant mean curvature surface in $\mathbb{R}^3$. In particular, each connected component of $\widetilde{\Sigma}$ lies in a slab bounded by two parallel planes to a fixed plane in $\mathbb{R}^3$, which is impossible as $\widetilde{\Sigma}$ is invariant under the pullback group $\widetilde{G}$ of G. Hence the contradiction.

As a conclusion we have shown that either Σ is a (single) sphere or it is a connected surface of genus at least three or it has at least two genus two components, the latter cases giving $\chi(\Sigma) \leq -4$, as we claimed. $\square$

The profile of spheres is given by the function $I_{\text{sph}}(v) = (36\pi v^2)^{1/3}$. We now define a function $J : [0,1] \to \mathbb{R}$ by $J(v) = I_{\text{sph}}(v)$ if $v \leq \frac{1}{6}$, and J is the solution of

$$(20) \qquad\qquad J^2 J'' + J(J')^2 + 16\pi = 0$$

with $J(\frac{1}{6}) = I_{\text{sph}}(\frac{1}{6})$ and $J'(\frac{1}{6}) = I'_{\text{sph}}(\frac{1}{6})$ if $\frac{1}{6} \leq v \leq \frac{5}{6}$, and $J(v) = I_{\text{sph}}(1-v)$ if $\frac{5}{6} \leq v \leq 1$. Note that (20) is the ODE which appears in Theorem 3 corresponding to Euler characteristic -4. The function J is symmetric with respect to $v = 1/2$, and using (3) we deduce that for $\frac{1}{6} \leq v \leq \frac{5}{6}$, the graph of J can be parametrized by

$$v(t) = \frac{1}{6} + \pi^{1/3}t + 2\pi^{2/3}t^2 - \frac{8\pi}{3}t^3 \ , \ a(t) = \pi^{1/3} + 4\pi^{2/3}t - 8\pi t^2 \ .$$

Theorem 20. *The G-isoperimetric profile I_G for a group of symmetries G of the cubic torus $T = \mathbb{R}^3/\mathbb{Z}^3$ fixing a point and containing the diagonal rotations (of angle $2\pi/3$) through this point, is bounded below by the function J, $I_G \geq J$. In particular, any G-symmetric surface dividing T in two equal volumes has area larger than $J(\frac{1}{2}) \geq 2.19$ and any G-isoperimetric surface enclosing a volume less than or equal to $\frac{1}{6}$ is a sphere.*

Proof. As both I_G and J are symmetric with respect to the vertical line through $v = \frac{1}{2}$, we will consider only volumes between 0 and 1/2. As the area of the sphere enclosing a volume $1/2$ is larger than the one of the P minimal surface (which is ~ 2.34), we rule out the case that the entire profile I_G comes from spheres. From Proposition 19, the maximum principle in Lemma 8 and item c) in Theorem 9, we deduce that there exists $v_1 < \frac{1}{2}$ such that $I_G(v) = I_{\mathrm{sph}}(v)$ for $0 \leq v \leq v_1$, and at the point $q = (v_1, I_G(v_1))$ the spheres stop being isoperimetric and the higher topology branch starts. The end point of this branch is $(\frac{1}{2}, I_G(\frac{1}{2}))$.
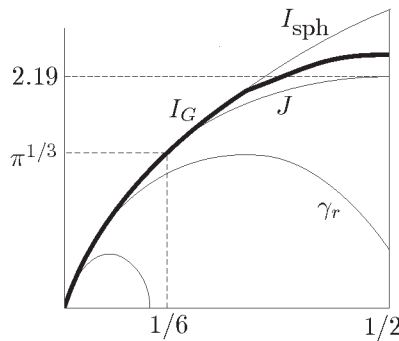

FIGURE 5. G-isoperimetric profile and supersolutions.

Consider a small volume $v = \frac{4}{3}\pi r^3$ in the range where the symmetric profile I_G agrees with the spherical one. Let γ_r be the solution of (20) starting at $p_r = (v, I_{\mathrm{sph}}(v)) = (\frac{4}{3}\pi r^3, 4\pi r^2)$ and tangent to the spherical profile at that point; see Figure 5. According to (3) the graph of γ_r can be parameterized, for $t \geq 0$, as

$$(21) \qquad v_r(t) = \frac{4\pi}{3}r^3 + 4\pi r^2 t + 4\pi r t^2 - \frac{8\pi}{3}t^3 \,, \ \gamma_r(v_r(t)) = 4\pi r^2 + 8\pi r t - 8\pi t^2.$$

Since the spherical profile satisfies $I^2 I'' + I(I')^2 - 8\pi = 0$ and γ_r is a solution of the equation (20), we have from item iv) in Lemma 8 that $\gamma_r(t)$ lies stricly below $I_{\mathrm{sph}}(v(t))$ for $t > 0$.

For r small enough, the curve $\{(v, \gamma_r(v)) | \gamma_r(v) \geq 0\}$ lies below I_G (in fact this curve converges to the origin when r goes to zero) and the highest point of γ_r is $p_{\max}(r) = (4\pi r^3, 6\pi r^2)$. Let

$$r_0 = \sup \left\{ r \,|\, 0 < r < r_1 \text{ and } \gamma_r(v) < I_G(v) \text{ for all } v \leq \frac{1}{2} \text{ such that } \gamma_r(v) \text{ is defined} \right\},$$

where r_1 is the radius of the last isoperimetric sphere (occurring at the branching point q). So, there will be a first contact of γ_{r_0} with I_G (outside the spherical part of the profile I_G). This will occur either at q, or tangentially along the higher topology branch, or at the end point $(\frac{1}{2}, I_G(\frac{1}{2}))$. We include the possibility of γ_{r_0} coinciding with I_G in the third case. The (strict) tangential contact cannot occur again because of the maximal principle, applied with the same constant $\chi = -4$, with I_G being a supersolution of $(\star)$, and item b) in Theorem 9.

Contact at the branching point q is also impossible by the maximum principle in item iv) of Lemma 8 applied to the solution γ_{r_0} and the supersolution I_G. Note that, I_G being trapped between two smooth curves whith are tangent at q, we have

that I'_{G+} and γ'_{r_0} coincide at this point. Moreover, item b) in Theorem 9 prevents γ_{r_0} and I_G coincide in an open interval.

We are left with the third case, where γ_{r_0} hits the isoperimetric profile for the half-volume (the two curves being possibly identical). Obviously the slope of γ_{r_0} at $v = \frac{1}{2}$ is nonnegative, otherwise the curve would have points *above* the profile I_G. Therefore, the maximum of γ_{r_0} is reached for a value of v not less than $1/2$. As the maximum of γ_r is attained at $v = 4\pi r^3$, there exists a special $r_2 \leq r_0$ for which $\gamma'_{r_2}(\frac{1}{2}) = 0$, i.e. γ_{r_2} has its maximum at $v = \frac{1}{2}$. Furthermore γ_{r_2} lies under I_G, as in Figure 5.

Using (21) we can find that

$$r_2 = \frac{1}{2\pi^{1/3}} \simeq 0.341392,$$

so that the initial volume for γ_{r_2} is $\frac{1}{6}$ and, therefore, γ_{r_2} is just the nonspherical branch of J. This proves that $I_G \geq J$. Moreover $J(\frac{1}{6}) = \pi^{1/3} \simeq 1.46459$, while the value at the maximum of J is $J(\frac{1}{2}) = 6\pi r_2^2 = \frac{3}{2}\pi^{1/3} \simeq 2.19689$. $\qquad\square$

In $T = \mathbb{R}^3/\mathbb{Z}^3$, we conjecture that the isoperimetric profile (for the nonsymmetric problem) coincides with the scp profile. From the theorem above we can now conclude a property supporting this conjecture: a natural family of symmetric candidates can be excluded (namely, a one-parameter deformation of the classical Schwarz minimal surface by genus three symmetric surfaces with constant mean curvature; see [1]).

Corollary 21. *The symmetric isoperimetric profile I_G is not less than the spheres-cylinders-planes profile I_{scp}, and is strictly greater when the volume is greater than $\frac{4\pi}{81}$, i.e. in the cylindrical or planar range of I_{scp}. As a consequence, isoperimetric surfaces in the cubic torus (without prescribed extra symmetries) are not G-invariant, except for the spheres.*

Proof. Using the maximum principle in Lemma 8, it suffices to check that I_G lies above I_{scp} at the transition points. At the transition point between spheres and cylinders, $I_G = I_{\mathrm{sph}} = I_{\mathrm{scp}}$ because $v = 4\pi/81 < \frac{1}{6}$. At the other transition point, between cylinders and planar tori, we have from Theorem 20 that $I_G(1/\pi) \geq J(1/\pi) > I_{\mathrm{scp}}(1/\pi) = 2$. The strict inequality can be easily computed numerically, but also proved by a slightly simpler argument, using the monotonicity of J: indeed J reaches height 2 *before* I_{scp}. To see this fact, note that solving $J(v(t)) = 2$ yields

$$t = \frac{\pi^{2/3} - \sqrt{3\pi^{4/3} - 4\pi}}{4\pi},$$

which corresponds to a volume

$$v(t) = \frac{1}{2} - \frac{\sqrt{3\pi^{1/3} - 4}}{12\sqrt{\pi}}(2 + 3\pi^{1/3}) \simeq 0.311,$$

which is strictly less than $1/\pi \simeq 0.318$. $\qquad\square$

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