Minimal and constant mean curvature surfaces

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1 Introduction

These notes originated from a series of lectures given by Harold Rosenberg at IMPA, found on youtube:
https://www.youtube.com/playlist?list=PLDf7S31yZaYxaM1IsSUI34Dxdz4oTLMM9
These lectures have been complemented with the recent solution by Brendle of the isoperimetric inequality for minimal surfaces (Section 9) and the Alexandrov theorem based on the moving plane technique (Section 21).

There are many texts on minimum surfaces; we recommend the ones by Nitsche [14] and Osserman [15] among the most classics. A more current treatment can be found in the book of Colding and Minicozzi [3], where the focus is via global analysis on Riemannian manifolds.

On the cover image: It is taken from Pinterest. It can be seen at
https://www.pinterest.es/pin/560135272387443211/

Notation.
$L(\gamma)$: length of a curve γ.
$A(\Sigma)$: area of a surface Σ (n-dimensional volume if Σ has dimension n).
$\mathrm{Gr}(f)$: Graph of a function f.
$\mathbb{D}(z_0, r) = \{z \in \mathbb{R}^2 \mid |z - z_0| < r\}$, $z_0 \in \mathbb{R}^2$, $r > 0$.
$\mathbb{D}(r) = \mathbb{D}(0, r)$, $\quad \mathbb{D} = \mathbb{D}(1)$, $\quad \mathbb{S}^1 = \partial\mathbb{D}$.
$\mathbb{B}(p, r) = \{q \in \mathbb{R}^3 \mid |p - q| < r\}$, $p \in \mathbb{R}^3$, $\mathbb{S}^2(p, r) = \partial\mathbb{B}(p, r)$, $r > 0$. $\mathbb{S}^2(r) = \mathbb{S}^2(\vec{0}, r)$, $\mathbb{S}^2 = \mathbb{S}^2(1)$.
$A(p_0, s, t) = \mathbb{B}(p_0, t) - \mathbb{B}(p_0, s)$ with $0 < s < t$.
$B_M(p, r) = \{q \in M \mid d_M(p, q) < r\}$, $p \in M$ in a complete Riemannian manifold, $r > 0$.
$|A_\Sigma|$: norm of the second fundamental form of a hypersurface Σ.

2 Minimal graphs

In his 1762 work *Essai d'une nouvelle méthode pour déterminer les maxima et minima des formules intégrales indefinies* [11], Lagrange considered the following problem: Let $\Omega \subset \mathbb{R}^2$ be a relatively compact open set with smooth boundary, and $\varphi \in C^0(\partial\Omega) \setminus C^0(\partial\Omega, \mathbb{R})$. Consider all possible extensions $f \in C^2(\overline{\Omega})$ of φ to Ω,

$$\mathcal{A}(\varphi) = \{f \in C^2(\Omega) \cap C^0(\overline{\Omega}) \mid f|_{\partial\Omega} = \varphi\},$$

> What can we say about a function $f \in \mathcal{A}(\varphi)$ that minimizes area among all functions in $\mathcal{A}(\varphi)$? (there existence, uniqueness, regularity...)

To answer this question, Lagrange introduced the *Euler-Lagrange technique,* which was the starting point of the Calculus of Variations (a technique that can be applied to other functionals besides area), that we explain next. Suppose that $f \in A(\varphi)$ is a critical point for the above problem.

Take $\xi \in C^2(\overline{\Omega})$ with $\xi|_{\partial\Omega} = 0$. Given $t \in \mathbb{R}$, the function $f_t = f + t\xi$ lies in $\mathcal{A}(\varphi)$, hence

$$\left.\frac{d}{dt}\right|_{t=0} A(\mathrm{Gr}(f_t)) = 0.$$

But $A(\mathrm{Gr}(f_t)) = \int_\Omega \sqrt{1 + \|\nabla f_t\|^2}$ (gradient computed with respect to the standard metric in $\mathbb{R}^2$) and $\|\nabla f_t\|^2 = \|\nabla f + t\nabla\xi\|^2 = \|\nabla f\|^2 + 2t\langle\nabla f, \nabla\xi\rangle + t^2\|\nabla\xi\|^2$, thus

$$\left.\frac{d}{dt}\right|_{t=0} A(\mathrm{Gr}(f_t)) = \int_\Omega \left.\frac{d}{dt}\right|_{t=0} \sqrt{1 + \|\nabla f_t\|^2} = \int_\Omega \frac{\left.\frac{d}{dt}\right|_{t=0} \|\nabla f_t\|^2}{2\sqrt{1+\|\nabla f\|^2}} = \int_\Omega \frac{\langle \nabla f, \nabla\xi\rangle}{\sqrt{1+\|\nabla f\|^2}}$$

$$= \int_\Omega \langle \frac{\nabla f}{\sqrt{1+\|\nabla f\|^2}}, \nabla\xi\rangle \overset{\text{(Stokes)}}{=} -\int_\Omega \xi \;\mathrm{div}\left(\frac{\nabla f}{\sqrt{1+\|\nabla f\|^2}}\right).$$

Since the last expression vanishes for all $\xi \in C^2(\overline{\Omega})$ with $\xi|_{\partial\Omega} = 0$, we conclude that a necessary and sufficient condition for $f \in \mathcal{A}(\varphi)$ to be a critical point of the area functional in $\mathcal{A}(\varphi)$ is that

$$\mathrm{div}\left(\frac{\nabla f}{\sqrt{1+\|\nabla f\|^2}}\right) = 0 \tag{1}$$

The second order PDE (1) is the Euler-Lagrange equation for the area functional area for graphs, and it is quasilinear and elliptic.

Let us look at some natural questions about the above equation. Given $\varphi \in C^2(\partial\Omega)$,

1. If $f \in \mathcal{A}(\varphi)$ satisfies (1), Is the graphical surface $\mathrm{Gr}(f)$ necessarily a global minimum of the area functional among surfaces with the same boundary?

2. Is the minimum area surface among all surfaces with boundary $\mathrm{Gr}(\varphi)$ necessarily a graph over Ω?

If $\Omega = \mathbb{D}$ is the open unit disk and we take φ as the constant zero, the global minimum for area among all surfaces with the same boundary $\mathrm{Gr}(\varphi)$ is given by $f \equiv 0$. Nevertheless, in general the answer to the two above questions is no; for example, if we consider the domain $\Omega = \mathbb{D} \setminus \mathbb{D}(-1, \varepsilon)$ with $\varepsilon > 0$ small, the contour given in Figure 1(b) is the graph of a C^2 function but the surface of least area with that boundary is not a graph over Ω (it is not even contained in $\Omega \times \mathbb{R}$).

However, in the situation of question (1) above, $\mathrm{Gr}(f)$ minimizes area among all surfaces contained in $\Omega \times \mathbb{R}$, as follows from the following calibration argument:

Let $\Omega \subset \mathbb{R}^2$ be a relatively compact open set, $\varphi \in C^0(\partial\Omega)$ and $f \in C^2(\Omega) \cap C^0(\overline{\Omega})$ a solution of the Dirichlet problem

$$\left\{\begin{array}{ll} \mathrm{div}\left(\frac{\nabla f}{\sqrt{1+\|\nabla f\|^2}}\right) = 0 & \text{in } \Omega, \\ f = \varphi & \text{in } \partial\Omega. \end{array}\right. \tag{2}$$

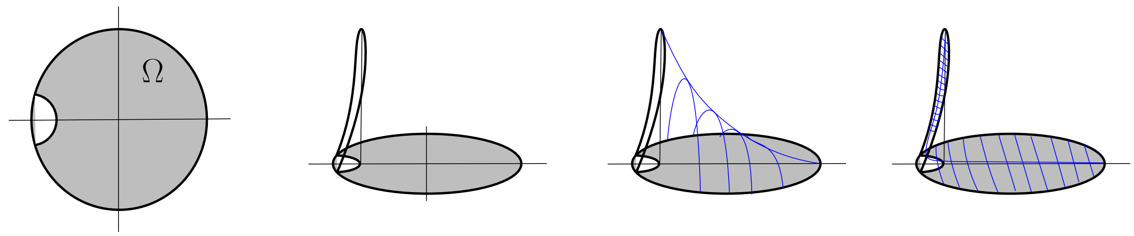


Figure 1: From left to right: (a) the domain $\Omega = \mathbb{D} \setminus \mathbb{D}(-1, \varepsilon)$. (b): Graphical contour over $\partial\Omega$. (c): Graph over Ω with the above contour, not being a minimum for area with that boundary. (d): A non-graphical surface with the same boundary, whose area is strictly smaller than the area of (c).

Let $X(x, y) = (x, y, f(x, y))$, $(x, y) \in \Omega$, be a parameterization of $\mathrm{Gr}(f)$ with Gauss map $N = \frac{1}{W}(-f_x, -f_y, 1)$, where

$$W = \sqrt{1 + \|\nabla f\|^2}. \tag{3}$$

Extend N to $\Omega \times \mathbb{R}$ by $N(x, y, z) = N(x, y)$. Thus, we can view N as a smooth vector field on $\Omega \times \mathbb{R}$. Consider the 2-form over $\Omega \times \mathbb{R}$ given by

$$\omega(U, V) = \det(U, V, N).$$

Geometrically, $\omega_p(u, v)$ measures the volume of the parallelepiped with edges $u, v \in \mathbb{R}^3$ and $N_{X(\pi(p))}$, where $p = (x, y, z) \in \Omega \times \mathbb{R}$ and $\pi(p) = (x, y)$. Hence:

(i) $\omega_p(u, v) \leq 1$ for all $p \in \Omega \times \mathbb{R}$ and $u, v \in \mathbb{R}^3$ orthogonal and unitary.

(ii) $\omega_p(u, v) = 1$ for all $p \in \mathrm{Gr}(f)$ when $\{u, v\}$ is a positive orthonormal basis of $T_p\mathrm{Gr}(f)$.

Analytically, $\omega(\partial_x, \partial_y) = N_3 = \frac{1}{W}$, $\omega(\partial_x, \partial_z) = -N_2 = \frac{f_y}{W}$, $\omega(\partial_y, \partial_z) = N_1 = \frac{-f_x}{W}$, hence

$$\omega = \frac{1}{W}\left(dx \wedge dy + f_y dx \wedge dz - f_x dy \wedge dz\right).$$

and

$$\begin{aligned}
d\omega &= d\left(\tfrac{1}{W} dx \wedge dy\right) + d\left(\tfrac{f_y}{W} dx \wedge dz\right) - d\left(\tfrac{f_x}{W} dy \wedge dz\right) \\
&= \left(\tfrac{1}{W}\right)_z dz \wedge dx \wedge dy + \left(\tfrac{f_y}{W}\right)_y dy \wedge dx \wedge dz - \left(\tfrac{f_x}{W}\right)_x dx \wedge dy \wedge dz \\
&= -\left[\left(\tfrac{f_x}{W}\right)_x + \left(\tfrac{f_y}{W}\right)_y\right] dx \wedge dy \wedge dz \\
&= -\mathrm{div}\left(\tfrac{\nabla f}{W}\right) dx \wedge dy \wedge dz \\
&= 0,
\end{aligned}$$

that is, ω is closed. ω is what's known as a *calibration*, and $\mathrm{Gr}(f)$ is a *calibrated surface for* ω.

Proposition 2.1 *In the above situation, if* $\Sigma \subset \Omega \times \mathbb{R}$ *is a compact, immersed orientable surface with* $\partial\Sigma = \partial \mathrm{Gr}(f)$*, then* $A(\Sigma) \geq A(\mathrm{Gr}(f))$*, and equality holds if and only if* $\Sigma = \mathrm{Gr}(f)$*.*

Proof. Take Σ as in the statement (possibly with non-trivial topology). Let $\Sigma_f = \mathrm{Gr}(f)$. As $\partial\Sigma = \partial\Sigma_f$, we can view $\Sigma \setminus \Sigma_f$ as a 2-chain in $\Omega \times \mathbb{R}$ with boundary $\partial(\Sigma \setminus \Sigma_f) = 0$. As the second homology group $H_2(\Omega \times \mathbb{R}, \mathbb{Z})$ vanishes, there exists a 3-chain Λ in $\Omega \times \mathbb{R}$ with $\partial\Lambda = \Sigma \setminus \Sigma_f$. Using Stokes' theorem,

$$0 = \int_\Lambda d\omega = \int_\Sigma \omega - \int_{\Sigma_f} \omega$$

By (ii), $\omega|_{\Sigma_f}$ coincides with the area element of Σ_f, hence $\int_{\Sigma_f} \omega = A(\Sigma_f)$. Analogously, (i) implies that $\int_\Sigma \omega \leq A(\Sigma)$, whence $A(\Sigma_f) \leq A(\Sigma)$.

If $A(\Sigma_f) = A(\Sigma)$, then $\int_\Sigma \omega = A(\Sigma)$ and (i) implies that $\omega|_\Sigma$ is the area element of Σ. Therefore, $T_p\Sigma$ is parallel to $T_{X(\pi(p))}\Sigma_f$ for every $p \in \Sigma$. From here it is not difficult to check that Σ is a vertical translation of Σ_f, and since both surfaces have the same boundary we conclude that $\Sigma = \Sigma_f$. □

Proposition 2.1 can be generalized to graphs over relatively compact open subsets of $\mathbb{R}^n$, but does not ensure that a graph satisfying (2) must be a least 'area' (n-dimensional volume) hypersurface among hypersurfaces with the same boundary. If we impose convexity to *Omega*, this minimization property is true.

Theorem 2.2 *Let* $\Omega \subset \mathbb{R}^n$ *be a convex, relatively open set, and* $f \in C^2(\Omega) \cap C^0(\overline{\Omega})$ *a solution of (2) for* $\varphi \in C^0(\partial\Omega)$*. Then,* $\mathrm{Gr}(f)$ *minimizes* n*-dimensional volume among all compact hypersurfaces of* $\mathbb{R}^{n+1}$ *with the same boundary as* $\mathrm{Gr}(f)$*.*

Proof. As $\overline{\Omega}$ is convex in $\mathbb{R}^n$, we have that $\overline{\Omega} \times \mathbb{R}$ is also convex. Consider the canonical projection $\Pi\colon \mathbb{R}^{n+1} \to \overline{\Omega} \times \mathbb{R}$ mapping each $p \in \mathbb{R}^{n+1}$ on the unique point $\Pi(p) \in \overline{\Omega} \times \mathbb{R}$ such that

$$\mathrm{dist}(p, \Pi(p)) = \mathrm{dist}(p, \overline{\Omega} \times \mathbb{R}).$$

Pi does not increase distances:

$$\|\Pi(p) - \Pi(q)\| \leq \|p - q\|, \quad \forall p, q \in \mathbb{R}^{n+1}. \tag{4}$$

Take a compact hypersurface $\Sigma \subset \mathbb{R}^{n+1}$ with $\partial\Sigma = \partial\Sigma_f$, where $\Sigma_f = \mathrm{Gr}(f)$. Then, the property that Π does not increase distances implies that

$$A(\Sigma) \geq \mathcal{H}_n(\Pi(\Sigma)) \overset{(*)}{\geq} A(\Sigma_f),$$

where $\mathcal{H}_n$ denotes n-dimensional Hausdorff measure ($\Pi(\Sigma)$ might not be a hypersurface) and in $(*)$ we have used Proposition 2.1 generalized to our current setting after approximation of $\Pi(\Sigma)$ by compact hypersurfaces contained in $\Omega \times \mathbb{R}$. □

How can we generalize the above discussion to a Riemannian manifold?

Theorem 2.3 *Let (M^n, g) be a Riemannian manifold, $\Omega \subset M$ a relatively compact open subset and $\varphi \in C^0(\partial\Omega)$. Then:*

1. *If $f \in C^2(\Omega) \cap C^0(\overline{\Omega})$ satisfies $f|_{\partial\Omega} = \varphi$ and $\mathrm{Gr}(f)$ minimizes 'area' (n-dimensional volume) among all graphs over Ω with boundary $\mathrm{Gr}(\varphi)$ (in $\Omega \times \mathbb{R}$ we consider the product metric $g \times dt^2$), then $\mathrm{div}_M(\frac{\nabla f}{W}) = 0$ in Ω, where $W = \sqrt{1 + \|\nabla_M f\|^2}$.*

2. *If $f \in C^2(\Omega) \cap C^0(\overline{\Omega})$ is a solution of (2) for $\varphi \in C^0(\partial\Omega)$, then $\omega(U_1, \dots, U_n) = dv_g(U_1, \dots, U_n, N)$ is a calibration over $\Omega \times \mathbb{R}$ and $\mathrm{Gr}(f)$ is a calibrated hypersurface for Ω, where dv_g stands for the volume element of $(M \times \mathbb{R}, g \times dt^2)$. Furthermore, $\mathrm{Gr}(f)$ minimizes 'area' among all compact hypersurfaces inside $\Omega \times \mathbb{R}$ with the same boundary as $\mathrm{Gr}(f)$.*

Proof. Exercise. □

Nevertheless, Theorem 2.2 does not extend to arbitrary Riemannian manifolds, because we do not dispose of a projection Π not increasing distances.

A central problem in the conditions of Theorem 2.3 is the following:

When there is a solution $f \in C^2(\Omega) \cap C^0(\overline{\Omega})$ of (2)?

We shall see about this problem later. Now let us go back to graphs over open subsets of the plane. Lagrange essentially only gave an example of graph satisfying (1): the affine functions $f(x, y) = ax + by + c$, whose graphs are affine planes of $\mathbb{R}^3$. In 1786, Meusnier gave two other examples: the *helicoid*

$$f(x, y) = \arctan(y/x) \tag{5}$$

and the catenoid

$$f(x, y) = \arg\cosh \sqrt{x^2 + y^2}$$

(these are non-global graphs). In 1836, Scherk [20, 21] produced more sophisticated examples.

It we develop (1), we will obtain the elliptic quasilinear PDE

$$(1 + f_x^2) f_{yy} - 2 f_x f_y f_{xy} + (1 + f_y^2) f_{xx} = 0, \tag{6}$$

known as the *minimal graph equation.*

Examples and exercises.

1. Show that equation (6) is equivalent to the vanishing of the mean curvature of the graph of f ($\mathrm{Gr}(f)$ is a minimal surface).

2. The helicoid is the image of the embedding $\psi(s, t) = s(\cos t, \sin t, 0) + (0, 0, at)$, $(s, t) \in \mathbb{R}^2$ (a is a non-zero constant). Prove that the helicoid is ruled surface. Use the rotation of angle π around each straight line contained in the helicoid to show that the helicoid is a minimal surface. See Figure 2 left.

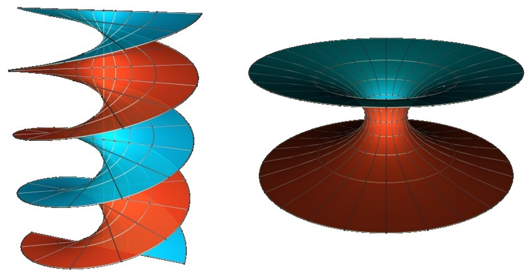


Figure 2: Left: helicoid. Right: catenoid. Figures courtesy of M. Weber.

3. Prove that the surface $C = \{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 - \cosh^2 z = 0\}$ obtained after rotation of the catenary $y = \cosh z$ around the lz-axis is an embedded minimal surface, called *catenoid*. Figure 2 right.

4. (Euler theorem [7]).
Let $f \in C^2([a, b])$ be a positive function and Σ_f the surface of revolution generated by the curve $\{(x, f(x)) \mid x \in [a, b]\}$ after rotation about the x-axis. Demonstrate that if Σ_f has least area among all surfaces of revolution Σ_h where $h \in C^2([a, b])$, $h > 0$, with $h(a) = f(a)$ and $h(b) = f(b)$, then Σ_f is contained in a catenoid, i.e., $f(x) = \lambda \cosh\left(\frac{x-\mu}{\lambda}\right)$, where $\lambda > 0$, $\mu \in \mathbb{R}$.

   Hint: Show that the Euler-Lagrange equation associated to this problem is
$$\sqrt{1 + (f')^2} = \left(\frac{ff'}{\sqrt{1 + (f')^2}}\right)' \quad \text{in } [a, b]. \tag{7}$$
   Show that $f\sqrt{1 + (f')^2} - f'\frac{ff'}{\sqrt{1+(f')^2}}$ is a first integral of (7). Hence there exists $\lambda \in \mathbb{R}$ such that $f\sqrt{1 + (f')^2} - f'\frac{ff'}{\sqrt{1+(f')^2}} = \lambda$. Prove that $\lambda > 0$ and $\frac{ff'}{\lambda^2} = f'f''$. After ruling out the case $f =$ constant (it does not satisfy (7)), work in a neighborhood of a $x_0 \in [a, b]$ where f' has no zeros, invert $y = f(x)$ with $\frac{dx}{dy} = \frac{\lambda^2}{\sqrt{y^2-\lambda^2}}$ and integrate this equation to conclude that $y = y(x) = \lambda \cosh\left(\frac{x-\mu}{\lambda}\right)$.

5. (Doubly periodic Scherk surface) Prove that the surface $S_2 = \{(x, y, z) \in \mathbb{R}^3 \mid \cos x e^z - \cos y = 0\}$ is minimal and invariant under the rank 2 group of translations generated by
$$(x, y, z) \overset{\psi_1}{\mapsto} (x + 2\pi, y, z), \qquad (x, y, z) \overset{\psi_2}{\mapsto} (x, y + 2\pi, z).$$
   Thus, we can view the qotient surface $\Sigma = S_2/(\mathbb{Z}\psi_1 \oplus \mathbb{Z}\psi_2)$ as a minimal surface in the Riemannian manifold $\mathbb{R}^2/(\mathbb{Z}\psi_1 \oplus \mathbb{Z}\psi_2) \equiv (\mathbb{R}/\mathbb{Z})^2 \times \mathbb{R}$ (metric product of a two-dimensional

torus and the real line). Σ has genus zero and four ends, and S_2 has infinite genus and just one end.

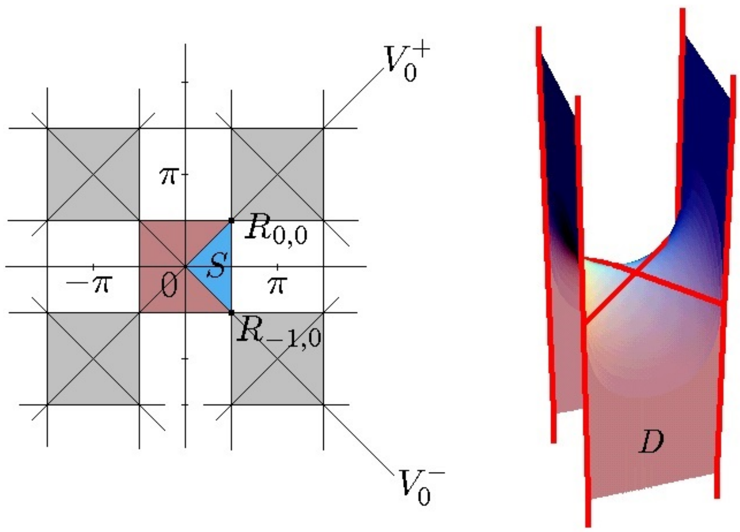


Figure 3: The doubly periodic Scherk surface is a minimal graph over the interior of the shadowed open squares. Besides containing vertical lines passing through each vertex of the squares, the surface contains horizontal lines at height zero, namely the diagonals of the above squares. The graph takes boundary values $\pm\infty$ on opposite edges of each square. If we divide the square in four triangles as in the figure (in blue one of these triangles), then the boundary values are $0, 0, \pm\infty$.

We might ask how Scherk discovered this surface: it is natural to look for solutions of (6) of the form $f(x, y) = g(x) + h(y)$. If we denote by $\cdot = \partial/\partial_x$ and ∂/∂_y, then (6) writes $[1 + h'(y)^2]\ddot{g}(x) + [1 + \dot{g}(x)^2]h'(y) = 0$, or equivalently,

$$\frac{\ddot{g}(x)}{1 + \dot{g}(x)^2} = -\frac{h''(y)}{1 + h'(y)^2}.$$

Hence the two members of the last equation equal the same number $a \in \mathbb{R}$. Integrating,

$$g(x) = c_2 - \frac{\log\left(\cos\left(ax + c_1\right)\right)}{a}, \quad h(y) = \frac{\log\left(\cos\left(ay - c_3\right)\right)}{a} + c_4,$$

where $c_1, c_2, c_3, c_4 \in \mathbb{R}$. Thus,

$$g(x) + h(y) = \log\left(\frac{\cos\left(ay - c_4\right)}{\cos\left(ax + c_1\right)}\right) + \left(c_2 + c_3\right),$$

which is the expression $z(x,y) = \log\left(\frac{\cos y}{\cos x}\right)$ up to translations and changes of scale.

6. (Singly periodic Scherk surface, Figure 4 left) Prove that the surface $S_1 = \{(x,y,z) \in \mathbb{R}^3 \mid \sin z = \sinh x \sinh y\}$ is minimal and invariant under the infinite cyclic group of translations
$$(x,y,z) \mapsto (x,y,z+2k\pi), \quad k \in \mathbb{Z}.$$

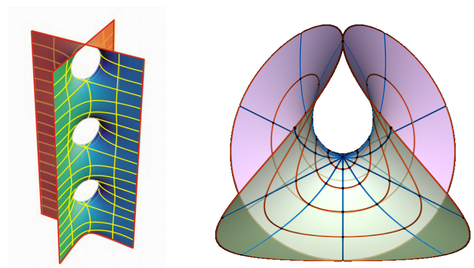


Figure 4: Left: singly periodic Scherk surface. Right: Enneper surface. Figures courtesy of M. Weber.

7. (Enneper surface, Figure 4 right) Prove that the map
$$\psi : \mathbb{R}^2 \to \mathbb{R}^3 \mid \psi(u,v) = \left(u - \frac{u^3}{3} + uv^2, v - \frac{v^3}{3} + vu^2, u^2 - v^2\right).$$
is a minimal immersion of the plane into $\mathbb{R}^3$ (not an embedding).

3 The Plateau problem

Let us consider the doubly periodic Scherk surface, which is generated from a minimal graph $\mathrm{Gr}(f)$ over the interior of a square with boundary values $+\infty, -\infty, +\infty, -\infty$ on consecutive edges. We can imagine this graph as the limit as $n \to \infty$ of minimal graphs $\mathrm{Gr}(f_n)$ on the closed square with boundary values $n, -n, n, -n$ ($n \in \mathbb{N}$):

Does the minimal graph f_n exist? Note that the boundary values are no longer given as in (2), because they are not expressed as the graph of a function over the boundary of the square. How can we find a critical point of the area functional with this kind of boundary values?

One way to solve this problem is to consider the boundary values as a closed polygonal curve Γ in $\mathbb{R}^3$ with eight consecutive horizontal and vertical edges, as in Figure 3, and find a compact minimal disk with this boundary as a solution to the *Plateau problem* with boundary Γ.

The Plateau problem admits many formulations, all sharing the same general principle:

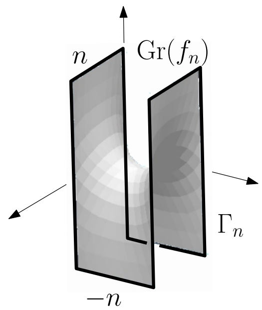


Figure 5: Minimal graph over a square.

Given a compact $(k-1)$-dimensional submanifold (not necessarily connected) Γ of a Riemannian manifold (M^n, g), can one find a submanifold $\Sigma^k \subset M$ with $\partial\Sigma = \Gamma$ of least 'area' (k-dimensional volume) among all submanifolds of M with that boundary?

In order to clarify the above question, it is necessary to specify what is meant by boundary (topological boundary, boundary of a manifold-with-boundary) and by 'area' (k-dimensional Hausdorff measure, k-dimensional volume for a Riemannian manifold of the same dimension, etc). Clearly, one must assume that Γ spans a k-dimensional submanifold of M in some sense (for example, $[\Gamma] = 0$ in the homology group $H^{k-1}(M)$).

We will formulate the Plateau problem FOR DISCS in more precise conditions:

Plateau problem for discs in a Riemannian manifold:

Given a rectifiable Jordan curve Γ[1] *in a Riemannian manifold (M^3, g). Is there a smooth map $X\colon \overline{\mathbb{D}} = \{x^2 + y^2 \leq 1\} \to M$ such that $X|_{\mathbb{S}^1} = \psi$ and $A(X(\overline{\mathbb{D}})) \leq A(\widetilde{X}(\overline{\mathbb{D}}))$ for all $\widetilde{X} \in C^\infty(\overline{\mathbb{D}}, M)$ with $\widetilde{X}|_{\mathbb{S}^1} = \psi$?*

Remark 3.1

1. Although we have talked about the area of $X(\overline{\mathbb{D}})$ or $\widetilde{X}(\overline{\mathbb{D}})$, we are not imposing that $X, \widetilde{X}$ are immersions (it may be more correct to write 2-dimensional Hausdorff measure, but we will not do that for the sake of simplicity).

2. In order to ensure the there existence of some $\widetilde{X} \in C^\infty(\overline{\mathbb{D}}, M)$ with $\widetilde{X}|_{\mathbb{S}^1} = \psi$, we must impose Γ to be homotopically trivial in M^3. This is not a restriction when $M = \mathbb{R}^3$.

3. To try to solve the Plateau problem, it is reasonable to impose that (M, g) be complete. For example, in $\mathbb{R}^2 \setminus \{(0,0)\}$ we cannot find a least length arc joining two points $p, -p \in \mathbb{R}^2 \setminus \{(0,0)\}$, as the arc of minimum length passes through the origin. The same idea leads us to conclude that there is no disk in $\mathbb{R}^3 \setminus \{(0,0,0)\}$ that minimizes area with boundary a circumference centered at the origin.

[1] This means that there exists an Lipschitz embedding $\psi\colon \mathbb{S}^1 \to \Gamma$.

The most general there existence result for the Plateau problem is the following:

Theorem 3.2 (Douglas-Radó-Morrey) *Given a rectifiable Jordan curve Γ homotopically zero in a complete Riemannian manifold (M^3, g), there exists $X \in C^\infty(\overline{\mathbb{D}}, M)$ such that $X(\mathbb{S}^1) = \Gamma$ and $A(X(\overline{\mathbb{D}})) \leq A(\widetilde{X}(\overline{\mathbb{D}}))$ for all $\widetilde{X} \in C^\infty(\overline{\mathbb{D}}, M)$ con $\widetilde{X}(\mathbb{S}^1) = \Gamma$.*

Remark 3.3

1. Douglas [5, 6] and Radó [17] independently proved Theorem 3.2 in its version $M = \mathbb{R}^3$, between 1929 and 1933. Douglas [5, 6] won the first Fields medal (shared with Ahlfors) for this result. The most general version in (M^3, g) is due to Morrey.
2. We will prove the version in $\mathbb{R}^n$ of Theorem 3.2 in Section 8.
3. Osserman proved that the Douglas-Radó-Morrey solution X to the Plateau problem is an immersion at every point of $\mathbb{D}$ (for boundary points this is unknown). X does not have to be an embedding, since we can simply take as G any knot in $\mathbb{R}^3$.
4. In $\mathbb{R}^4$, the Douglas-Radó-Morrey solution X to the Plateau problem does not have to be an immersion in $\mathbb{D}$: Consider the holomorphic map $X(z) = (z^2, z^3)$ from $\mathbb{C}$ to $\mathbb{C}^2 \equiv \mathbb{R}^4$. Then, $X|_{\overline{\mathbb{D}}}$ is the solution to the Plateau problem with boundary $\Gamma = X(\mathbb{S}^1)$ (this follows from the fact that every holomorphic curve in a Kaehler manifold minimizes area by a calibration argument), but X fails to be an immersion at $z = 0$.

Coming back to the example with which we started this section, given $n \in \mathbb{N}$, there exists a Douglas-Radó solution $\Sigma_n = X_n(\overline{\mathbb{D}})$ to the Plateau problem with boundary the polygonal curve Γ_n given by Figure 3. In addition, an argument based on the Radó's theorem that we will see later implies that such solution is embedded (in fact, it is a graph over its projection on the base square) and is unique. This uniqueness ensures that Σ_n inherits each symmetry of Γ_n. (in particular, we conclude that $\vec{0} \in \Sigma_n$ reasoning by composition of the rotation of angle π with respect to the x-axis with the rotation by π with respect to the z-axis).

The next step will be checking that we can take limits in Σ_n as $n \to \infty$ (and thus recover the doubly periodic Scherk surface). To do that, we will use that Σ_n is graph for each n, together with compactness results that we will study in Section 15. Let us analyze this limit process in more detail. Consider the same problem where the square obtained as a projection of Γ_n is replaced by a straight rectangle $R(a, b)$ of base a and height b with $a \gg b > 0$. Intuitively, the solution of the Plateau problem with edge Γ_n has a very low central point (next to height $-n$, see Figure 40 right), and if $n \to \infty$ then Σ_n does not converge to a graph over $R(a, b)$ but to two vertical straight rectangles over two of the edges of the $R(a, b)$ (this phenomenon will be explained by the Jenkins-Serrin's method in Section 20; In fact, the Jenkins-Serrin Theorem implies that if $a \neq b$, there is no minimal graph over $R(a, b)$ with boundary values $+\infty, -\infty, +\infty, -\infty$ on consecutive sides of $R(a, b)$).

We will finish this section with a result on the extrinsic area growth of minimal graphs in $\mathbb{R}^3$.

Theorem 3.4 *Let $\Omega \subset \mathbb{R}^2$ be an open set and $f \in C^2(\Omega)$ a solution of (1). Given $q \in \Omega$ and $r > 0$ such that $\overline{\mathbb{D}}(q,r) \subset \Omega$, let $p := (q, f(q)) \in \mathrm{Gr}(f)$. Then:*

$$A(\Sigma \cap \mathbb{B}(p,r)) \leq 2\pi r^2.$$

Proof. Consider the Jordan curve Γ of class C^2 given by the graph of $f|_{\partial \mathbb{D}(p,r)}$. By Theorem 2.2, the graph of $f|_{\overline{\mathbb{D}}(p,r)}$ minimizes area among all compact surfaces in $\mathbb{R}^3$ with boundary Γ. As $\Gamma \subset \mathbb{S}^2(p,r)$ is a Jordan curve, $\mathbb{S}^2(p,r) \setminus \Gamma$ has two connected components Σ_1, Σ_2. The closure of Σ_i is a compact surface with boundary Γ, for $i = 1, 2$. Thus,

$$A(\Sigma \cap \mathbb{B}(p,r)) \leq \min\{A(\Sigma_1), A(\Sigma_2)\} \leq \frac{1}{2} A(\mathbb{S}^2(p,r)) = 2\pi r^2. \qquad \square$$

4 Submanifolds. First variation of area formula

Let Σ a k-dimensional submanifold of a Riemannian manifold (M^n, g). Let us denote by $\nabla, \overline{\nabla}$ the Levi-Civita connections of Σ and M, respectively. The second fundamental form of Σ is the symmetric bilinear form

$$\sigma(X,Y) = \left(\overline{\nabla}_X Y\right)^{\perp},$$

where X, Y are tangent vector fields to Σ. The *Gauss equation* relates the sectional curvatures of Σ and M with the second fundamental form:

$$K_\Sigma(X,Y)\|X \wedge Y\|^2 = K_M(X,Y)\|X \wedge Y\|^2 + \langle \sigma(X,X), \sigma(Y,Y)\rangle - \|\sigma(X,Y)\|^2,$$

where $K_\Sigma(X,Y)$ (resp. $K_M(X,Y)$ is the sectional curvature of the plane spanned by X, Y in $T\Sigma$ (resp. in TM), provided that X, Y are linearly independent, and $\|X \wedge Y\|^2 = \|X\|^2\|Y\|^2 - \langle X, Y\rangle^2$ (so the Gauss equation makes sense even when X, Y are linearly dependent). In the particular case $k = 2$, K_Σ is the Gauss curvature of Σ and X, Y are orthogonal and unitary, the Gauss equation writes

$$K_\Sigma = K_M(X,Y) + \det \sigma.$$

The *mean curvature vector* of Σ is the normal vector field on Σ given by

$$k\vec{H} = \sum_{i=1}^{k} \sigma(E_i, E_i), \tag{8}$$

where $E_q, \ldots, E_k$ is a local orthonormal basis of $T\Sigma$ (the above sum does not depend on the choice of orthonormal basis). The submanifold Σ is said to be *minimal* if $\vec{H}$ vanishes identically.

The norm of the second fundamental form is the function

$$|\sigma| = \sqrt{\sum_{i,j=1}^{k} \|\sigma(E_i, E_j)\|^2} \in C^\infty(\Sigma).$$

If $N_1, \dots, N_{n-k}$ is a local orthonormal basis of the normal bundle of Σ, then

$$\sigma(X,Y) = \sum_{h=1}^{n-k} \langle \sigma(X,Y), N_h \rangle N_h = \sum_{h=1}^{n-k} \langle \left(\overline{\nabla}_X Y\right)^{\perp}, N_h \rangle N_h = \sum_{h=1}^{n-k} \langle \overline{\nabla}_X Y, N_h \rangle N_h$$

$$= \sum_{h=1}^{n-k} \langle Y, -\overline{\nabla}_X N_h \rangle N_h.$$

The endomorphism $A_h \colon T\Sigma \to T\Sigma$, $A_h(X) = -\overline{\nabla}_X N_h$ is called the *Weingarten endomorphism,* or *shape operator* of Σ associated to the unit normal N_h.

For a hypersurface, there is only one unit normal vector, up to sign. We will denote by A_Σ the shape operator in this case. Thus,

$$|\sigma| = |A_\Sigma|.$$

As A_Σ is self-adjoint with respect to the induced metric on Σ, A_Σ diagonalizes in an orthonormal basis. Thus, given $p \in \Sigma$ there exists an orthonormal basis $e_1, \dots e_{n-1}$ of $T_p\Sigma$ (principal directions) such that $A_\Sigma e_i = k_i e_i$ for some $k_i \in \mathbb{R}$ (principal curvatures), $i = 1, \dots, n-1$. Thus,

$$(n-1)\vec{H}(p) \overset{(8)}{=} \sum_{i=1}^{n-1} \sigma_p(e_i, e_i) = \sum_{i=1}^{n-1} k_i N_p.$$

Also for hypersurfaces, we define the *mean curvature function* as $H = \frac{1}{n-1} \sum_{i=1}^{n-1} k_i$. ($\Sigma$ is minimal if and only if $H = 0$). Decomposing the second fundamental form is its components proportional to I_n and traceless σ_0, we have

$$\sigma = H \cdot I_{n-1} + \sigma_0.$$

Since I_{n-1}, σ_0 are orthogonal in the space of symmetric matrices (pointwise), we have

$$|A_\Sigma|^2 = |\sigma|^2 = (n-1)H^2 + |\sigma_0|^2 \geq (n-1)H^2, \tag{9}$$

with equality only at umbilical points.

Given a (not necessarily tangent) vector field X along a k-dimensional submanifold of a Riemannian manifold (M^n, g), we define the *divergence* of X by generalization of the classical notion of divergence for tangent vector fields to Σ:

$$\operatorname{div}_\Sigma(X) = \sum_{i=1}^{k} \langle (\overline{\nabla}_{e_i} X), e_i \rangle, \tag{10}$$

where $e_1, \dots, e_k$ is any local orthonormal basis of $T\Sigma$.

Lemma 4.1 *Let Σ be a k-dimensional submanifold of a Riemannian manifold (M^n, g).*

1. *If ξ is a normal field along Σ, then $\mathrm{div}_\Sigma(\xi) = -k\langle \vec{H}, \xi\rangle$.*

2. *If X is a (not necessarily tangent or normal) vector field along Σ, then $\mathrm{div}_\Sigma(X) = \mathrm{div}_\Sigma(X^T) - k\langle \vec{H}, X^\perp\rangle$.*

Proof. Let $e_1, \dots e_k$ be a local orthonormal basis of $T\Sigma$.

$$\mathrm{div}_\Sigma(\xi) \overset{\text{(def.)}}{=} \sum_{i=1}^k \langle (\overline{\nabla}_{e_i}\xi)^T, e_i\rangle = \sum_{i=1}^k \langle -A_\xi e_i, e_i\rangle = -\sum_{i=1}^k \langle \sigma(e_i, e_i), \xi\rangle = -k\langle \vec{H}, \xi\rangle,$$

and we have 1. As for 2, $\mathrm{div}_\Sigma(X) \overset{\text{(def.)}}{=} \mathrm{div}_\Sigma(X^T) + \mathrm{div}_\Sigma(X^\perp)$ and we conclude by using 1. $\square$

Lemma 4.2 *A submanifold Σ^k of $\mathbb{R}^n$ is minimal if and only if its coordinate functions are harmonic on Σ.*

Proof. Let $\psi\colon \Sigma^k \looparrowright \mathbb{R}^n$ be an isometric immersion. Taking $X = a \in \mathbb{S}^{n-1}(1)$ in (10) we have $\mathrm{div}_\Sigma(a) = 0$, hence

$$\begin{aligned}\Delta_\Sigma\langle \psi, a\rangle = \mathrm{div}_\Sigma(\nabla_\Sigma\langle\psi, a\rangle) &\overset{\text{Lemma 4.1(2)}}{=} \mathrm{div}_\Sigma(\overline{\nabla}\langle\psi, a\rangle) + k\langle \vec{H}, (\overline{\nabla}\langle\psi, a\rangle)^\perp\rangle \\ &= \mathrm{div}_\Sigma(a) + k\langle \vec{H}, a^\perp\rangle = k\langle \vec{H}, a\rangle.\end{aligned}$$

$\square$

Proposition 4.3 (First variation of area formula) *Let Σ^k be a submanifold of a Riemannian manifold (M^n, g), and let X be a compactly supported smooth vector field on Σ (not necessarily tangent). If $F\colon \Sigma \times (-\varepsilon, \varepsilon) \to M$ is a variation of Σ (i.e., $F(p, 0) = p$ $\forall p \in \Sigma$) with variational field X ($\frac{\partial F}{\partial t}(p, 0) = X(p)$, $\forall p \in \Sigma$), then*

$$\left.\frac{d}{dt}\right|_{t=0} A(F_t) = -k\int_\Sigma \langle X, \vec{H}\rangle + \int_{\partial\Sigma}\langle X, \eta\rangle, \tag{11}$$

where $F_t\colon \Sigma \looparrowright M$, $F_t(p) = F(t, p)$ (this is an immersion for $|t|$ sufficiently small) and η is the unit conormal vector pointing outwards Σ along $\partial\Sigma$.

Remark 4.4 When $\vec{H} \neq 0$, $\vec{H}$ gives the direction of maximal decrease of area of Σ (for example, in the round sphere this happens in the direction that points to the center of the sphere).

Proof. Take local coordinates $(x_1, \ldots, x_k)$ in Σ. Thus, $\{\frac{\partial F}{\partial x_i}(p,t)\}_{i=1}^k$ is a basis of $T_{F_t(p)}F_t(\Sigma)$ at every (p,t). Call $g_{ij} = \langle \frac{\partial F}{\partial x_i}, \frac{\partial F}{\partial x_j} \rangle$ at each (p,t), to the coefficients of the induced metric. The area element of $F_t(\Sigma)$ is

$$dA_{F_t(\Sigma)} = G(t)\, dx_1 \wedge \ldots \wedge dx_k \equiv G(t)\, d\mathbf{x},$$

where $G(t) = \det(g_{i,j}(\cdot, t))$, hence $\sqrt{G(0)}\, d\mathbf{x} = dA_\Sigma$ and

$$A(F_t) = \int_\Sigma dA_{F_t(\Sigma)} = \int_\Sigma \sqrt{G(t)}\, d\mathbf{x} = \int_\Sigma \sqrt{G(t)}\, \sqrt{\det(g^{ij}(0))}\, dA_\Sigma,$$

where $(g^{ij}(t))_{i,j} = (g_{ij}(t))_{i,j}^{-1}$. Taking derivatives at $t = 0$,

$$\begin{aligned}
\frac{d}{dt}\Big|_{t=0} [A(F_t)] &= \int_\Sigma \frac{d}{dt}\Big|_{t=0} \left(\sqrt{G(t)}\right) \sqrt{\det(g^{ij}(0))}\, dA_\Sigma, \\
&= \int_\Sigma \frac{1}{2\sqrt{G(0)}} \frac{d}{dt}\Big|_{t=0} (G(t))\, \sqrt{\det(g^{ij}(0))}\, dA_\Sigma, \\
&\overset{(*)}{=} \frac{1}{2} \int_\Sigma \operatorname{trace}\left[(g'_{ij}(0))_{i,j} (g^{ij}(0))_{i,j}\right]\, dA_\Sigma, \\
&= \frac{1}{2} \int_\Sigma \sum_{i,j=1}^k g'_{ij}(0) g^{ij}(0)\, dA_\Sigma,
\end{aligned}$$

where in $(\star)$ we have used that if $A(t)$ is a smooth curve of regular matrices, then $(\det A)' = \det A \cdot \operatorname{trace}(A' A^{-1})$. As

$$g'_{ij}(0) = \frac{d}{dt}\Big|_{t=0} \langle \frac{\partial F}{\partial x_i}, \frac{\partial F}{\partial x_j} \rangle = \langle \nabla^M_{\frac{\partial F}{\partial t}|_{t=0}} \frac{\partial F}{\partial x_i}, \frac{\partial F}{\partial x_j} \rangle + \langle \frac{\partial F}{\partial x_i}, \nabla^M_{\frac{\partial F}{\partial t}|_{t=0}} \frac{\partial F}{\partial x_j} \rangle,$$

then

$$\sum_{i,j=1}^k g'_{ij}(0) g^{ij}(0) = 2 \sum_{i,j=1}^k \langle \nabla^M_{\frac{\partial F}{\partial t}|_{t=0}} \frac{\partial F}{\partial x_i}, \frac{\partial F}{\partial x_j} \rangle g^{ij}(0). \tag{12}$$

Decompose X in its tangent and normal parts to Σ. As the derivative in the left-hand-side of (11) is linear, we can prove (11) in two separate cases, namely $X =$ tangent and $X =$ normal to Σ. In the first case, the Gauss formula allows us to write the last expression as

$$2 \sum_{i,j=1}^k \langle \nabla^\Sigma_{\frac{\partial F}{\partial t}|_{t=0}} \frac{\partial F}{\partial x_i}, \frac{\partial F}{\partial x_j} \rangle g^{ij}(0).$$

As X is tangent to Σ, we can take $X = \frac{\partial F}{\partial t}\big|_{t=0}$ as one of the vector fields in the canonical basis associated to the local coordinates $(x_1, \ldots, x_k)$, thus $[\frac{\partial F}{\partial t}\big|_{t=0}, \frac{\partial F}{\partial x_i}] = 0$ and we re-write the last expression as

$$2\sum_{i,j=1}^{k}\langle\nabla^{\Sigma}_{\frac{\partial F}{\partial x_i}}X,\tfrac{\partial F}{\partial x_j}\rangle g^{ij}(0)=2\,\mathrm{div}_{\Sigma}(X).$$

Therefore, in this case $X=$ tangent to Σ we have

$$\frac{d}{dt}\bigg|_{t=0}[A(F_t)] \quad = \quad \int_{\Sigma}\mathrm{div}_{\Sigma}(X)\,dA_{\Sigma}\overset{(**)}{=}\textstyle\int_{\partial\Sigma}\langle X,\eta\rangle,$$

where in $(**)$ we have assumed that Σ is orientable in order to apply Stokes' theorem (this is not strictly necessary, but we will assume it for the sake of simplicity), and we have proved (11) in the case $X=$ tangent to Σ.

Now suppose that $X=$ normal to Σ. Coming back to (12), we can view $(x_1,\dots,x_k,t)$ as local coordinates in $\Sigma\times(-\varepsilon,\varepsilon)$, and thus $[\frac{\partial F}{\partial t}\big|_{t=0},\frac{\partial F}{\partial x_i}]=0$ hence we can exchange order in the derivative variables (now in M, before in Σ) and so,

$$\sum_{i,j=1}^{k}g'_{ij}(0)g^{ij}(0)=2\sum_{i,j=1}^{k}\langle\nabla^{M}_{\frac{\partial F}{\partial x_i}}X,\tfrac{\partial F}{\partial x_j}\rangle g^{ij}(0)=\textstyle\sum_{i=1}^{k}\langle\nabla^{M}_{\frac{\partial F}{\partial x_i}}X,\frac{\partial F}{\partial x_i}\rangle=\mathrm{div}_{\Sigma}(X),$$

where we have taken the local coordinates in Σ in such a way that $g_{i,j}(0)=\delta_{ij}$ (the last divergence does not require that X be tangent to Σ, see equation (10). Using Lemma 4.1, $\mathrm{div}_{\Sigma}(X)=-k\langle\vec{H},X\rangle$ and we deduce (11). □

Definition 4.5 A submanifold Σ^k of a Riemannian manifold (M^n,g) is called *minimal* if $\vec{H}=0$, i.e., Σ is a critical point of the area functional for compactly supported variations.

Corollary 4.6 (Weak formulation of minimality) *A submanifold Σ^k of a Riemannian manifold (M^n,g) is minimal if and only if for each compactly supported smooth vector field X on Σ such that $X|_{\partial\Sigma}=0$, it holds* $\displaystyle\int_{\Sigma}\mathrm{div}_{\Sigma}(X)=0$.

Proof. We again suppose Σ is orientable in order to apply Stokes' theorem. By Lemma 4.1, if X is a vector field as in the statement of this corollary,

$$\int_{\Sigma}\mathrm{div}_{\Sigma}(X)=\int_{\Sigma}\mathrm{div}_{\Sigma}(X^T)-k\int_{\Sigma}\langle\vec{H},X^{\perp}\rangle=-k\int_{\Sigma}\langle\vec{H},X^{\perp}\rangle,$$

and the characterization of minimality follows from the L^2-density of the linear space of compactly supported normal vector fields that vanish on $\partial\Sigma$. □

Corollary 4.7 (Convex hull property) *If a compact submanifold Σ^k of $\mathbb{R}^n$ is minimal, then Σ lies in the convex hull of its boundary (in particular, $\partial\Sigma \neq \varnothing$):*

$$\Sigma \subset \mathcal{E}(\partial\Sigma) := \bigcap_{H\in A} H,$$

where $A = \{H$ half-space of $\mathbb{R}^n$ with $\partial\Sigma \subset H\}$.

Proof. Let $H = \{x \in \mathbb{R}^n \mid \langle x, a\rangle \leq c\}$ be a half-space of $\mathbb{R}^n$ that contains $\partial\Sigma$ ($a \in \mathbb{S}^{n-1}(1)$, $c \in \mathbb{R}$). As Σ is minimal, the function $f\colon \Sigma \to \mathbb{R}$, $f(p) = x - \langle p, a\rangle$ is harmonic on Σ by Lemma 4.2), and $f|_{\partial\Sigma} \geq 0$ because $\partial\Sigma \subset H$. Using the maximum principle for harmonic functions, we have $f \geq 0$ in Σ. □

Compactness cannot be dropped from the hypothesis of the last corollary (the exterior of a disk in a plane of $\mathbb{R}^3$ is a counterexample).

5 Weierstrass representation

Suppose $\Sigma \looparrowright \mathbb{R}^n$ is a minimal surface. By Lemma 4.2, the coordinate functions $x_1, \ldots, x_n$ of Σ are harmonic on Σ. Given $p \in \Sigma$, we choose a conformal parameterization (isothermal) $\psi = \psi(z)\colon U \to \psi(U) \subset \Sigma$, where U is an open subset of $\mathbb{C}$. Thus, $x_j \circ \psi\colon U \to \mathbb{R}$ is a harmonic function on U. Calling $z = u + iv$, the Cauchy-Riemann equations imply that the function

$$\phi_j = \frac{\partial(x_j \circ \psi)}{\partial u} - i\frac{\partial(x_j \circ \psi)}{\partial v} \tag{13}$$

is holomorphic on U, and we can express the immersion up to a translation as

$$(x_j \circ \psi)(z) = \mathrm{Re}\left(\int_{z_0}^{z} \phi_j(z)\, dz\right) + \mathrm{cte}, \tag{14}$$

where in order the above integral to be well-defined, we assume that U is simply connected. Moreover,

$$\begin{aligned}\sum_{j=1}^{n} \phi_j^2 &= \sum_{j=1}^{n}\left[\left(\frac{\partial(x_j \circ \psi)}{\partial u}\right)^2 - \left(\frac{\partial(x_j \circ \psi)}{\partial v}\right)^2 - 2i\frac{\partial(x_j \circ \psi)}{\partial u}\frac{\partial(x_j \circ \psi)}{\partial v}\right] \\ &= \left\|\frac{\partial\psi}{\partial u}\right\|^2 - \left\|\frac{\partial\psi}{\partial v}\right\|^2 - 2i\left\langle\frac{\partial\psi}{\partial u}, \frac{\partial\psi}{\partial u}\right\rangle = 0,\end{aligned}$$

where the last equality holds because ψ is conformal. Furthermore,

$$\sum_{j=1}^{n}|\phi_j|^2 = \sum_{j=1}^{n}\left[\left(\frac{\partial x_j \circ \psi)}{\partial u}\right)^2 + \left(\frac{\partial x_j \circ \psi)}{\partial v}\right)^2\right] = \left\|\frac{\partial\psi}{\partial u}\right\|^2 + \left\|\frac{\partial\psi}{\partial v}\right\|^2 = 2\left\|\frac{\partial\psi}{\partial u}\right\|^2 > 0.$$

The equivalence in Lemma 4.2 allows us to state the converse:

Theorem 5.1 *Let U be a simply connected domain in $\mathbb{C}$ and $\phi_1, \ldots, \phi_n \colon U \to \mathbb{C}$ holomorphic functions such that $\sum_{j=1}^n \phi_j^2 = 0$ and $\sum_{j=1}^n |\phi_j|^2 > 0$ in U. Then, the map $X \colon U \to \mathbb{R}^n$ given by*

$$X(z) = \left(\mathrm{Re} \int_{z_0}^{z} \phi_j(z)\, dz \right)_{i=1}^{n} \tag{15}$$

is a minimal immersion from U into $\mathbb{R}^n$ (z_0 is any point in U).

Remark 5.2 The hypothesis $\pi_1(U) = 0$ in the theorem ensures that the integral in the definition of X does not depend on the path that joins z_0 with z. We can eliminate the hypothesis $\pi_1(U) = 0$ by directly imposing that the integral does not depend on the path (the *problem of periods* is solved), so it is possible to generalize the theorem by replacing U by an arbitrary Riemann surface, and each holomorphic function ϕ_j by a holomorphic differential $\phi_j(z)\, dz$ globally defined on the Riemann surface.

Let $\Sigma \subset \mathbb{R}^3$ be an orientable minimal surface. If $(U, z = u + iv)$ is a local holomorphic coordinate on Σ, then we have the holomorphic functions ϕ_1, ϕ_2, ϕ_3 on U given by (13), that satisfy $\sum_{j=1}^3 \phi_j^2 = 0$. The induced metric on M is $ds^2 = \lambda^2 |dz|^2$, where $\lambda = \left\| \frac{\partial X}{\partial u} \right\| = \left\| \frac{\partial X}{\partial v} \right\|$ and X is given by (15). It is not hard to check that $\phi_j\, dz$ is a globally defined holomorphic 1-form on Σ (it does not depend on the local holomorphic coordinate z).

We define

$$f = \phi_1 - i\phi_2, \quad g = \frac{\phi_3}{f} \quad \text{in } U. \tag{16}$$

Remark 5.3 If $f \equiv 0$, we must explain the above definition of g. $f \equiv 0$ is equivalent to $\phi_1 \equiv i\phi_2$, i.e., $\phi_1^2 + \phi_2^2 \equiv 0$ or $\phi_3^2 \equiv 0$. This is equivalent to $\phi_3 \equiv 0$ and by (15), to the property that Σ is contained in a horizontal plane.

For the moment we will assume that Σ is not contained in a horizontal plane, hence $f \not\equiv 0$. In this case, f and g are respectively a holomorphic and a meromorphic function in U ($f\, dz$ is a holomorphic 1- form in Σ and $g \colon \Sigma \to \overline{\mathbb{C}}$ is a globally defined meromorphic function).

We can solve for ϕ_1, ϕ_2, ϕ_3 in terms of f, g:

Lemma 5.4 *In the above situation,*

$$\phi_1 = \frac{1}{2}(1 - g^2) f, \quad \phi_2 = \frac{i}{2}(1 + g^2) f, \quad \phi_3 = fg.$$

Proof. $(\phi_1 - i\phi_2)(\phi_1 + i\phi_2) = \phi_1^2 + \phi_2^2 = -\phi_3^2 = -f^2 g^2$, hence $\phi_1 + i\phi_2 = -\frac{f^2 g^2}{\phi_1 - i\phi_2} = -\frac{f^1 g^2}{f} = -fg^2$. Thus, $\phi_1 = \frac{1}{2}[(\phi_1 + i\phi_2) + (\phi_1 - i\phi_2)] = \frac{1}{2}(-fg^2 + f)$, and we have the first formula. The remainder is obvious. $\square$

Theorem 5.1 and Lemma 5.4 imply that any minimal surface $\Sigma \subset \mathbb{R}^3$ is determined either by the holomorphic 1-form $(\phi_1, \phi_2, \phi_3)\, dz$ or by the pair $(g, f\, dz)$. The first one is called the *Weierstrass*

1-form, and the second one is the *Weierstrass pair* of Σ. Every geometric object on Σ can be described in therms of the Weierstrass representation. For instance, the induced metric ds^2 and the Gauss curvature K are given by

$$ds^2 = \frac{1}{4}(1+|g|^2)^2|f|^2|dz|^2, \qquad K = -\left(\frac{4g'}{(1+|g|^2)^2|f|}\right)^2. \tag{17}$$

From the first formula above one deduces that g has a pole of order $k \in \mathbb{N}$ at $p \in \Sigma$ if and only if f has a zero of order $2k$ at p (we are assuming that Σ is not a horizontal plane). And from the second formula we have that the zeros of K (umbilical points of Σ) are isolated if Σ is not a piece of a plane.

The meromorphic function g has the following geometric interpretation: Consider the Gauss map $N \circ X = \frac{X_u \times X_v}{\|X_u \times X_v\|}$. Then, the expression of N in terms of the Weierstrass pair is:

$$N \circ X = \left(\frac{2\mathrm{Re}(g)}{1+|g|^2}, \frac{2\mathrm{Im}(g)}{1+|g|^2}, \frac{|g|^2-1}{1+|g|^2}\right), \tag{18}$$

In other words, g is the composition of N with the stereographic projection from the North pole of $\mathbb{S}^2$. In particular, N is a anti-conformal map. Another consequence of (18) is that if Σ is contained in a horizontal plane, then g (defined by (18), recall that (16) does not make sense in this case) is constant 0 or ∞.

6 Monotonicity formula. Local density of a minimal hypersurface

Recall the following result:

Proposition 6.1 (Co-area formula) *Let (M^n, g) be a Riemannian manifold and $h \colon M \to \mathbb{R}$ a Lipschitz function such that $\forall t \in \mathbb{R}$, $h^{-1}(-\infty, t]$ is compact. Given an integrable function $f \colon M \to \mathbb{R}$, we have*

$$\int_{\{h \leq t\}} f\|\nabla h\| = \int_{-\infty}^{t} \left(\int_{\{h=\tau\}} f \, dV_{g_\tau}\right) d\tau, \tag{19}$$

where dV_{g_τ} is the volume element of the submanifold $\{h=\tau\}$ with respect to the induced metric (h is smooth almost everywhere in M since it is Lipschitz, and its regular values are dense in $\mathbb{R}$ by Sard's theorem).

If additionally ∇h only vanishes in a measure zero set of M, then

$$\int_{\{h \leq t\}} f = \int_{-\infty}^{t} \left(\int_{\{h=\tau\}} \frac{f}{\|\nabla h\|} dV_{g_\tau}\right) d\tau. \tag{20}$$

In the particular case that $\|\nabla h\| \equiv 1$ a.e. M (for example, if $h = d_M(\cdot, p_0)$ is the distance function to a point $p_0 \in M$), taking $f \equiv 1$ we have

$$\operatorname{Vol}(\{h \leq t\}) = \int_{-\infty}^{t} A(\{h = \tau\})\, d\tau. \tag{21}$$

Lemma 6.2 *Sea $\Sigma^k \looparrowright \mathbb{R}^n$ an immersed minimal submanifold and $p_0 \in \mathbb{R}^n$. Then:*

1. $\Delta_\Sigma(\|p - p_0\|^2) = 2k$.

From now on, we will assume that Σ is orientable and proper in $\mathbb{R}^n \setminus \{p_0\}$. Let $A(p_0, \varepsilon, r) = \mathbb{B}(p_0, r) \setminus \mathbb{B}(p_0, \varepsilon)$, $0 < \varepsilon < r$. Then:

2. $k \cdot \operatorname{Vol}[\Sigma \cap A(p_0, \varepsilon, r)] = \displaystyle\int_{\partial[\Sigma \cap A(p_0,\varepsilon,r)]} \|(p - p_0)^T\|.$

3. $\dfrac{d}{ds}\operatorname{Vol}[\Sigma \cap A(p_0, \varepsilon, s)] = s \displaystyle\int_{\Sigma \cap \partial\mathbb{B}(p_0,s)} \frac{1}{\|(p - p_0)^T\|}.$

4. $\dfrac{d}{ds}\left(\dfrac{\operatorname{Vol}[\Sigma \cap A(p_0, \varepsilon, s)]}{s^k}\right) = \dfrac{1}{s^{k+1}} \displaystyle\int_{\partial[\Sigma \cap \mathbb{B}(p_0,s)]} \frac{\|(p - p_0)^\perp\|^2}{\|(p - p_0)^T\|} + \frac{1}{s^{k+1}} \int_{\partial[\Sigma \cap \mathbb{B}(p_0,\varepsilon)]} \|(p - p_0)^T\|.$

Proof.

$$\begin{aligned}
\Delta_\Sigma(\|p - p_0\|^2) &= \sum_{i=1}^{n} \Delta_\Sigma\left((x_i - x_i(p_0))^2\right) \\
&= 2\sum_{i=1}^{n} (x_i - x_i(p_0))\Delta_\Sigma(x_i - x_i(p_0)) + 2\sum_{i=1}^{n} \|\nabla_\Sigma(x_i - x_i(p_0))\|^2 \\
&\overset{(*)}{=} 2\sum_{i=1}^{n} \|\nabla_\Sigma(x_i - x_i(p_0))\|^2 = 2\sum_{i=1}^{n} \|e_i^T\|^2,
\end{aligned}$$

where in $(*)$ we have used Lemma 4.2 and $\{e_1, \ldots, e_n\}$ is the usual basis of $\mathbb{R}^n$. Let $\{\xi_{k+1}, \ldots, \xi_n\}$ be a local orthonormal basis of $(T\Sigma)^\perp$. Adding up in $i = 1, \ldots, n$ the equality $1 = \|e_i\|^2 = \|e_i^T\|^2 + \sum_{j=k+1}^{n} \langle e_i, \xi_j\rangle^2$, we have

$$\begin{aligned}
n &= \sum_{i=1}^{n} \|e_i^T\|^2 + \sum_{i=1}^{n} \sum_{j=k+1}^{n} \langle e_i, \xi_j\rangle^2 = \sum_{i=1}^{n} \|e_i^T\|^2 + \sum_{j=k+1}^{n} \left(\sum_{i=1}^{n} \langle e_i, \xi_j\rangle^2\right) \\
&= \sum_{i=1}^{n} \|e_i^T\|^2 + \sum_{j=k+1}^{n} \|\xi_j\|^2 = \sum_{i=1}^{n} \|e_i^T\|^2 + (n - k),
\end{aligned}$$

from where $\sum_{i=1}^{n} \|e_i^T\|^2 = k$ and we deduce item 1.

Suppose from now on that Σ is proper in $\mathbb{R}^n \setminus \{p_0\}$ and take $0 < \varepsilon < r$. By item 1,

$$2k \cdot \mathrm{Vol}[\Sigma \cap A(p_0, \varepsilon, r)] = \int_{\Sigma \cap A(p_0,\varepsilon,r)} \Delta_\Sigma(\|p - p_0\|^2)) = \int_{\partial_{r,\varepsilon}} \langle \nabla_\Sigma(\|p-p_0\|^2), \eta\rangle,$$

where $\partial_{r,\varepsilon} = \partial[\Sigma \cap A(p_0, \varepsilon, r)] = [\Sigma \cap \partial\mathbb{B}(p_0, r)] \setminus [\Sigma \cap \partial\mathbb{B}(p_0, \varepsilon)]$ (we have taking into account the orientations) and η is the outward pointing unit conormal to $\Sigma \cap A(p_0, \varepsilon, r)$ along $\partial_{r,\varepsilon}$.

Since

$$\langle \nabla_\Sigma(\|p-p_0\|^2), \eta\rangle = \langle \overline{\nabla}(\|p-p_0\|^2), \eta\rangle = 2\langle p - p_0, \eta\rangle,$$

we have

$$k \cdot \mathrm{Vol}[\Sigma \cap A(p_0, \varepsilon, r)] = \int_{\partial_{r,\varepsilon}} \langle p - p_0, \eta\rangle = \int_{\partial_{r,\varepsilon}} \langle (p-p_0)^T, \eta\rangle. \tag{22}$$

Given $p \in \partial_{r,\varepsilon}$, $T_p\Sigma$ is the orthogonal direct sum of $\langle \eta\rangle$ and $T_p\partial[\Sigma \cap A(p_0, \varepsilon, r)]$ (we will suppose that the spheres of radii ε, r centered at p_0 are transversal to Σ; this is true a.e. in the radius by Sard's theorem). As $T_p\partial[\Sigma \cap A(p_0, \varepsilon, r)] \subset T_p\partial A(p_0, \varepsilon, r) = \langle p - p_0\rangle^\perp$, we deduce that given $v \in T_p\partial[\Sigma \cap A(p_0, \varepsilon, r)]$, we have $\langle (p-p_0)^T, v\rangle = \langle p - p_0, v\rangle = 0$, hence $(p-p_0)^T \perp T_p\partial[\Sigma \cap A(p_0, \varepsilon, r)]$ and thus, $(p - p_0)^T$ is parallel to η, i.e.,

$$\langle (p-p_0)^T, \eta\rangle = \|(p-p_0)^T\| \quad \text{in } \partial_{r,\varepsilon}. \tag{23}$$

From (22) and (23) we directly deduce item 2.

To prove item 3 we will use the co-area formula with $h = \|p - p_0\||_\Sigma$ and conclude

$$\mathrm{Vol}[\Sigma \cap A(p_0, \varepsilon, s)] = \int_\varepsilon^s \left(\int_{\Sigma \cap \partial\mathbb{B}(p_0,\tau)} \frac{1}{\|\nabla_\Sigma(\|p-p_0\|)\|} \right) d\tau = \int_\varepsilon^s \left(\int_{\Sigma \cap \partial\mathbb{B}(p_0,\tau)} \frac{\|p-p_0\|}{\|(p-p_0)^T\|} \right) d\tau.$$

Taking derivatives in s,

$$\frac{d}{ds}\mathrm{Vol}[\Sigma \cap A(p_0, \varepsilon, s)] = \int_{\Sigma \cap \partial\mathbb{B}(p_0,s)} \frac{\|p-p_0\|}{\|(p-p_0)^T\|},$$

and we have item 3.

As for item 4,

$$\begin{aligned}
\frac{d}{ds}\left(\frac{\mathrm{Vol}[\Sigma \cap A(p_0, \varepsilon, s)]}{s^k} \right) &= \frac{1}{s^k}\frac{d}{ds}\left(\mathrm{Vol}[\Sigma \cap A(p_0, \varepsilon, s)] \right) - \frac{k}{s^{k+1}}\mathrm{Vol}[\Sigma \cap A(p_0, \varepsilon, s)] \\
&\overset{\text{(items 1,3)}}{=} \frac{1}{s^{k-1}} \int_{\Sigma \cap \partial\mathbb{B}(p_0,s)} \frac{1}{\|(p-p_0)^T\|} - \frac{1}{s^{k+1}} \int_{\partial[\Sigma \cap A(p_0,\varepsilon,s)]} \|(p-p_0)^T\|
\end{aligned}$$

$$= \frac{1}{s^{k-1}} \int_{\Sigma \cap \partial\mathbb{B}(p_0,s)} \frac{1}{\|(p-p_0)^T\|} - \frac{1}{s^{k+1}} \int_{\partial[\Sigma \cap \mathbb{B}(p_0,s)]} \|(p-p_0)^T\| + \frac{1}{s^{k+1}} \int_{\partial[\Sigma \cap \mathbb{B}(p_0,\varepsilon)]} \|(p-p_0)^T\|. \tag{24}$$

Decomposing $p - p_0$ in its tangent and normal parts to Σ and taking norms, $\|p - p_0\|^2 = \|(p - p_0)^T\|^2 + \|(p - p_0)^\perp\|^2$ hence

$$\|(p - p_0)^T\| = \frac{\|p - p_0\|^2}{\|(p - p_0)^T\|} - \frac{\|(p - p_0)^\perp\|^2}{\|(p - p_0)^T\|}. \tag{25}$$

From (24) and (25) we have

$$\begin{aligned}
\frac{d}{ds}\left(\frac{\mathrm{Vol}[\Sigma \cap A(p_0, \varepsilon, s)]}{s^k}\right) &= \frac{1}{s^{k-1}} \int_{\Sigma \cap \partial \mathbb{B}(p_0,s)} \frac{1}{\|(p - p_0)^T\|} - \frac{1}{s^{k+1}} \int_{\partial[\Sigma \cap \mathbb{B}(p_0,s)]} \frac{\|p - p_0\|^2}{\|(p - p_0)^T\|} \\
&+ \frac{1}{s^{k+1}} \int_{\partial[\Sigma \cap \mathbb{B}(p_0,s)]} \frac{\|(p - p_0)^\perp\|^2}{\|(p - p_0)^T\|} + \frac{1}{s^{k+1}} \int_{\partial[\Sigma \cap \mathbb{B}(p_0,\varepsilon)]} \|(p - p_0)^T\| \\
&= \frac{1}{s^{k+1}} \int_{\partial[\Sigma \cap \mathbb{B}(p_0,s)]} \frac{\|(p - p_0)^\perp\|^2}{\|(p - p_0)^T\|} + \frac{1}{s^{k+1}} \int_{\partial[\Sigma \cap \mathbb{B}(p_0,\varepsilon)]} \|(p - p_0)^T\|.
\end{aligned}$$

and item 4 is proved. □

Theorem 6.3 *Let $p_0 \in \mathbb{R}^n$, $\Sigma^k \looparrowright \mathbb{R}^n \setminus \{p_0\}$ a properly immersed minimal submanifold and $0 < s < t$. Then:*

$$\frac{\mathrm{Vol}(\Sigma \cap A(p_0, \varepsilon, t))}{t^k} - \frac{\mathrm{Vol}(\Sigma \cap A(p_0, \varepsilon, s))}{s^k} =$$

$$\int_{\Sigma \cap A(p_0,s,t)} \frac{\|(p - p_0)^\perp\|^2}{\|p - p_0\|^{k+2}} + \frac{1}{k}\left(\frac{1}{s^k} - \frac{1}{t^k}\right) \int_{\partial[\Sigma \cap \mathbb{B}(p_0,\varepsilon)]} \|(p - p_0)^T\| \geq 0. \tag{26}$$

In particular, the function

$$s > \varepsilon \mapsto \frac{\mathrm{Vol}(\Sigma \cap A(p_0, \varepsilon, s))}{\omega_k s^k}$$

is not decreasing in s, where $\omega_k = \mathrm{Vol}(\mathbb{B}^k(\vec{0}, 1))$ (unit ball in $\mathbb{R}^k$).

Proof.

$$\frac{\mathrm{Vol}(\Sigma \cap A(p_0, \varepsilon, t))}{t^k} - \frac{\mathrm{Vol}(\Sigma \cap A(p_0, \varepsilon, s))}{s^k} = \int_s^t \frac{d}{d\tau}\left(\frac{\mathrm{Vol}[\Sigma \cap A(p_0, \varepsilon, \tau)]}{\tau^k}\right) d\tau$$

$$\overset{(*)}{=} \int_s^t \frac{1}{\tau^{k+1}}\left(\int_{\partial[\Sigma \cap \mathbb{B}(p_0,\tau)]} \frac{\|(p - p_0)^\perp\|^2}{\|(p - p_0)^T\|}\right) d\tau + \int_s^t \frac{1}{\tau^{k+1}}\left(\int_{\partial[\Sigma \cap \mathbb{B}(p_0,\varepsilon)]} \|(p - p_0)^T\|\right) d\tau,$$

where in $(*)$ we have used item 4 of Lemma 6.2. We will re-write the first integral using that if $h = \|p - p_0\|$, then $\nabla_\Sigma h = \frac{(p-p_0)^T}{\|p-p_0\|}$; observe that the integral $\int_{\partial[\Sigma \cap \mathbb{B}(p_0,\varepsilon)]} \|(p - p_0)^T\|$ is independent of τ, hence the above displayed expression is

$$\int_s^t \frac{1}{\tau^{k+1}}\left(\int_{\partial[\Sigma \cap \mathbb{B}(p_0,\tau)]} \frac{1}{\|\nabla_\Sigma h\|} \frac{\|(p - p_0)^\perp\|^2}{\|p - p_0\|}\right) d\tau + \left(\int_s^t \frac{d\tau}{\tau^{k+1}}\right) \int_{\partial[\Sigma \cap \mathbb{B}(p_0,\varepsilon)]} \|(p - p_0)^T\|$$

$$= \int_s^t \left(\int_{\partial[\Sigma\cap\mathbb{B}(p_0,\tau)]} \frac{1}{\|\nabla_\Sigma h\|} \frac{\|(p-p_0)^\perp\|^2}{\|p-p_0\|^{k+2}} \right) d\tau + \frac{1}{k}\left(\frac{1}{s^k} - \frac{1}{t^k}\right) \int_{\partial[\Sigma\cap\mathbb{B}(p_0,\varepsilon)]} \|(p-p_0)^T\|$$

$$\overset{(**)}{=} \int_{\Sigma\cap A(p_0,s,t)} \frac{\|(p-p_0)^\perp\|^2}{\|p-p_0\|^{k+2}} + \frac{1}{k}\left(\frac{1}{s^k} - \frac{1}{t^k}\right) \int_{\partial[\Sigma\cap\mathbb{B}(p_0,\varepsilon)]} \|(p-p_0)^T\|,$$

where in (**) we have used the co-area formula. □

In the particular case that Σ is proper in a neighborhood of p_0 with the notation of Theorem 6.3, we can take $\varepsilon \to 0$ and obtain the classical version of the monotonicity formula (observe that the second integral in the right-hand-side of (26) tends to zero).

Corollary 6.4 *Let $\Sigma^k \looparrowright \mathbb{R}^n$ be a properly immersed minimal submanifold, $p_0 \in \mathbb{R}^n$ and $0 < s < t$. Then:*

$$\frac{\mathrm{Vol}(\Sigma\cap\mathbb{B}(p_0,t))}{t^k} - \frac{\mathrm{Vol}(\Sigma\cap\mathbb{B}(p_0,s))}{s^k} = \int_{\Sigma\cap A(p_0,s,t)} \frac{\|(p-p_0)^\perp\|^2}{\|p-p_0\|^{k+2}} \geq 0.$$

In particular, the function

$$s > 0 \mapsto \frac{\mathrm{Vol}(\Sigma\cap\mathbb{B}(p_0,s))}{\omega_k s^k}$$

is not decreasing in s. Furthermore, if this function is constant, then Σ is a k-plane passing through p_0.

Proof. We only have to study what happens if $s > 0 \mapsto \frac{\mathrm{Vol}(\Sigma\cap\mathbb{B}(p_0,s))}{\omega_k s^k}$ is constant. In this case, $(p-p_0)^\perp$ is identically zero in $A(p_0,s,t)$ for each $0 < s < t$, hence $p - p_0 \in T_p\Sigma$ for every $p \in \Sigma$. This tells us that Σ is invariant under every homothety centered at p_0, so Σ is a cone over $\Sigma \cap \partial\mathbb{B}(p_0,1)$. The only way that this can happen being Σ differentiable at p_0 is that Σ is a k-plane passing through p_0. □

There are four possibilities for a point $p_0 \in \mathbb{R}^n$ in terms of Corollary 6.4:

1. If p_0 is not an accumulation point of Σ, then as Σ is proper in $\mathbb{R}^n \setminus \{p_0\}$ we have $\mathrm{Vol}(\Sigma\cap\mathbb{B}(p_0,s)) = 0$ for $s > 0$ sufficiently small, hence $\frac{\mathrm{Vol}(\Sigma\cap\mathbb{B}(p_0,s))}{\omega_k s^k} = 0$ for these values of s.
2. If Σ extends across p_0 as an embedded surface, then Σ can be locally written as the graph of a function of class C^∞ defined in a disk of small radius in $T_{p_0}\Sigma$, and thus, $\lim_{s\to0^+} \frac{\mathrm{Vol}(\Sigma\cap\mathbb{B}(p_0,s))}{\omega_k s^k} = 1$.
3. If Σ extends across p_0 as an immersed surface with p_0 being a point of self-intersection with $m \geq 2$ sheets ($m \in \mathbb{N}$), we can apply to each sheet the previous case, hence $\lim_{s\to0^+} \frac{\mathrm{Vol}(\Sigma\cap\mathbb{B}(p_0,s))}{\omega_k s^k} = m$.

4. If p_0 is an accumulation point of Σ but we do not know if Σ extends across p_0 (in this case p_0 is a singularity of Σ), then there exists the limit (called the *density of* Σ *at* p_0*:*

$$\Theta(p_0) = \Theta(\Sigma, p_0) = \lim_{s \to 0^+} \frac{\mathrm{Vol}(\Sigma \cap \mathbb{B}(p_0, s))}{\omega_k s^k} \geq 0. \tag{27}$$

Remark 6.5 Cases 2 and 3 above also can be seen as particular cases of 4, and in those cases the density of Σ at p_0 is 1 and m, respectively.

There is an intrinsic version of the monotonicity formula, due to Yau [23, §7]. Although it is valid for Cartan-Hadamard manifolds of every dimension, we will only see here the case of the ambient space being $\mathbb{R}^3$.

Proposition 6.6 *Let* Σ *be a complete, immersed minimal surface in* $\mathbb{R}^3$ *and* $p_0 \in \Sigma$*. Then, the function*

$$s > 0 \mapsto \frac{\mathrm{Area}(B_\Sigma(p_0, s))}{\pi s^2}$$

is non-decreasing in s*, where* $B_\Sigma(p_0, s)$ *denotes the intrinsic ball in* Σ *centered at* p_0 *with radius* s*. Furthermore, if this last function is constant, then* Σ *is a plane.*

Proof. Consider the function $p \in \Sigma \mapsto \|p - p_0\|^2$. By item 1 of Lemma 6.2, $\Delta_\Sigma(\|p - p_0\|^2) = 4$. Using the divergence theorem in $B_\Sigma(p_0, s)$ (we can assume that the boundary of this ball is smooth by Sard's theorem),

$$4\ \mathrm{Area}(B_\Sigma(p_0, s)) = \int_{B_\Sigma(p_0,s)} \Delta_\Sigma(\|p - p_0\|^2) = \int_{\partial B_\Sigma(p_0,s)} \langle \nabla_\Sigma(\|p - p_0\|^2), \eta \rangle,$$

where η is the outward pointing unit conormal vector to $B_\Sigma(p_0, s)$ along its boundary. Moreover,

$$\begin{array}{rcl} \displaystyle\int_{\partial B_\Sigma(p_0,s)} \langle \nabla_\Sigma(\|p - p_0\|^2), \eta \rangle & = & \displaystyle 2 \int_{\partial B_\Sigma(p_0,s)} \|p - p_0\| \langle \nabla_\Sigma(\|p - p_0\|), \eta \rangle \\ & \stackrel{(A)}{\leq} & \displaystyle 2s \int_{\partial B_\Sigma(p_0,s)} \|\nabla_\Sigma(\|p - p_0\|)\| \\ & \stackrel{(B)}{\leq} & \displaystyle 2s \int_{\partial B_\Sigma(p_0,s)} \|\overline{\nabla}(\|p - p_0\|)\| \\ & = & 2s\ \mathrm{Length}(\partial B_\Sigma(p_0, s)), \end{array}$$

where in (A) we have used that $B_\Sigma(p_0, s) \subset \mathbb{B}(p_0, s)$. Thus,

$$2\ \mathrm{Area}(B_\Sigma(p_0, s)) \leq s\ \mathrm{Length}(\partial B_\Sigma(p_0, s)). \tag{28}$$

Hence,

$$\begin{aligned}\frac{d}{ds}\left(\frac{\text{Area}[B_\Sigma(p_0,s)]}{s^2}\right) &= \frac{1}{s^3}\left[s\frac{d}{ds}\left(\text{Area}[B_\Sigma(p_0,s)]\right) - 2\text{Area}[B_\Sigma(p_0,s)]\right] \\ &= \frac{1}{s^3}\left[s\ \text{Length}(\partial B_\Sigma(p_0,s)) - 2\text{Area}[B_\Sigma(p_0,s)]\right],\end{aligned}$$

which is non-negative by (28). This proves that $s \mapsto \frac{1}{s^2}\text{Area}[B_\Sigma(p_0,s)]$ is non-decreasing. If this function is constant, the above computation leads us to equality in (A) and (B) a.e. in s. Equality in (B) implies that Σ contains each straight line passing through p_0, thus Σ is a cone with vertex at p_0. As Σ is smooth, it must be plane. $\square$

7 A Theorem by Ekholm, White and Wienholtz

Definition 7.1 If $\Gamma \subset \mathbb{R}^3$ is a Jordan curve of class C^2 with arclength parameter s and curvature $\kappa(s)$, its *total curvature* is the number

$$C(\Gamma) = \int_\Gamma \kappa(s)\, ds.$$

By Fenchel's Theorem, $C(\Gamma) \geq 2\pi$ with equality if and only if Γ is a planar convex curve. If moreover Γ is a knot[2], then $C(\Gamma) > 4\pi$ by Fary-Milnor's Theorem.

Theorem 7.2 (Ekholm, White, Wienholtz, 2002) *Let $\Gamma \subset \mathbb{R}^3$ be a Jordan curve of class C^2 with total curvature $C(\Gamma) \leq 4\pi$. Let $M \looparrowright \mathbb{R}^2$ be a compact, immersed minimal surface with boundary Γ. Then, M is embedded.*

Remark 7.3 The bound 4π in Theorem 7.2 is sharp: there exist Jordan curves Γ_ε with $C(\Gamma_\varepsilon) = 4\pi + \varepsilon$ and compact, non-embedded minimal surfaces $M_\varepsilon \looparrowright \mathbb{R}^2$ with $\partial M_\varepsilon = \Gamma_\varepsilon$ for every $\varepsilon > 0$ arbitrarily small: consider a circle in $\{z = 0\}$ traveled two times (hence with total curvature 4π) and perturb this immersed closed curve Γ_0 by a Jordan curve Γ_ε with $C(\Gamma_\varepsilon) = 4\pi + \varepsilon$, in such a way that Γ_ε is a knot, as in Figure 6, arbitrarily close to Γ_0. The solution of the Plateau problem with boundary Γ_ε (which exists by the results in Section 8) is a non-embedded surface because Γ_ε is a knot.

Proof. [of Theorem 7.2]
By simplicity, we will give a proof assuming that the total curvature of Γ is strictly less than 4π.

Take $p \in M \setminus \Gamma$. We can assume that $\Gamma \cap \mathbb{B}(p,1) = \varnothing$ after a homothety, which does not change the total curvature of Γ. Define

$$C = \text{Cone}_p(\Gamma) = \{p + t(x-p) \mid x \in \Gamma, t \in [0,1]\}, \quad E = \text{Cone}_p^{\text{ext}}(\Gamma) = \{p + t(x-p) \mid x \in \Gamma, t \geq 1\}.$$

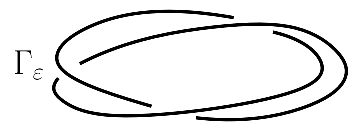


Figure 6: The bound 4π in Theorem 7.2 is sharp.

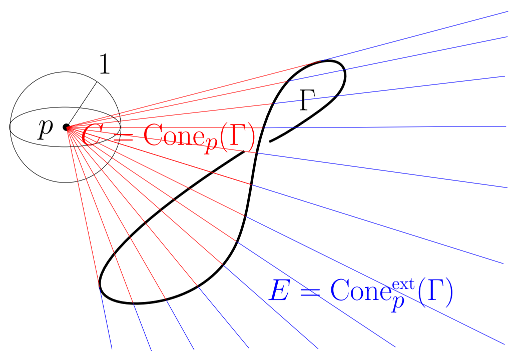


Figure 7: Cone and exterior cone with respect to a point.

Consider the topological surface without boundary $\widetilde{M} = M \cup E$. We will prove the theorem assuming the next proposition holds, and after this we will prove the proposition.

Proposition 7.4 (Generalized monotonicity formula) *The function*

$$s > 0 \mapsto \frac{A(\widetilde{M} \cap \mathbb{B}(p,r))}{\pi r^2}$$

is non-decreasing in r.

The proof of Theorem 7.2 has three steps:

1. If $p \in M \setminus \Gamma$, then $\Theta(M,p) \leq \Theta(C,p)$ (with the notation of (27)).
2. $\Theta(C,p) \leq \frac{1}{2\pi}\int_\Gamma \kappa(s)\,ds$.
3. If $p \in M$ is a self-intersection point of M, then $2 \leq \Theta(M,p) \leq \Theta(C),p) \leq \frac{1}{2\pi}\int_\Gamma \kappa(s)\,ds < \frac{4\pi}{2\pi} = 2$, which is a contradiction.

It remains to prove steps 1 and 2 above, which will be done in two lemmas and in Proposition 7.4.

[2] There is no continuous injective map $\psi\colon \overline{\mathbb{D}}(0,1) \to \mathbb{R}^3$ such that $\psi|_{\mathbb{S}^1} : \Gamma \to \Gamma$ is a homeomorphism.

Lemma 7.5 *If $p \in M \setminus \Gamma$, then $\Theta(M,p) \leq \Theta(C,p)$.*

Proof. Let $\beta \subset \mathbb{S}^2(1)$ be the curve obtained by intersecting C with $\mathbb{S}^2(1)$. Let $r \in (0,1)$ and $h(q) = \|q-p\|$, $q \in \mathbb{R}^3 \setminus \{p\}$. By the co-area formula,

$$A[C \cap \mathbb{B}(p,r)] = \int_0^r \left(\int_{\{h=t\}} \frac{ds_t}{\|\nabla_C h\|} \right) dt.$$

Since $\nabla_C h = \overline{\nabla} h = \frac{q-p}{\|q-p\|}$ is unitary, we have

$$A[C \cap \mathbb{B}(p,r)] = \int_0^r L(\{h=t\})\, dt = \int_0^r L(t\beta)\, dt = \int_0^r tL(\beta)\, dt = L(\beta) \int_0^r t\, dt = L(\beta)\frac{r^2}{2}.$$

Thus,

$$\Theta(C,p) = \lim_{r \to 0^+} \frac{A[C \cap \mathbb{B}(p,r)]}{\pi r^2} = \lim_{r \to 0^+} \frac{L(\beta)\frac{r^2}{2}}{\pi r^2} = \frac{L(\beta)}{2\pi}. \tag{29}$$

As $E \cap \mathbb{B}(p,1) = \varnothing$, we have $M \cap \mathbb{B}(p,r) = \widetilde{M} \cap \mathbb{B}(p,r)$ for every $r \in (0,1)$, hence $A[M \cap \mathbb{B}(p,r)] = A[\widetilde{M} \cap \mathbb{B}(p,r)]$ for every $r \in (0,1)$. This implies

$$\Theta(M,p) = \lim_{r \to 0^+} \frac{A[\widetilde{M} \cap \mathbb{B}(p,r)]}{\pi r^2} \overset{\text{(Proposition 7.4)}}{\leq} \lim_{R \to \infty} \frac{A[\widetilde{M} \cap \mathbb{B}(p,R)]}{\pi R^2}. \tag{30}$$

To compute the limit in the right-hand-side of (30), we can replace a compact part of $\widetilde{M}$ and the limit remains unchanged. For example, let us consider the exterior cone over β:

$$\widetilde{M}_1 = \text{Cone}_p^{\text{ext}}(\beta) = \{p + t(x-p) \mid x \in \beta, t \geq 1\}.$$

Then,

$$\lim_{R \to \infty} \frac{A[\widetilde{M} \cap \mathbb{B}(p,R)]}{\pi R^2} = \lim_{R \to \infty} \frac{A[\widetilde{M}_1 \cap \mathbb{B}(p,R)]}{\pi R^2}.$$

Again by the co-area formula, if $R > 1$ then

$$A[\widetilde{M}_1 \cap \mathbb{B}(p,R)] = \int_1^R \left(\int_{\{h=t\}} \frac{ds_t}{|\nabla_{\widetilde{M}_1} h|} \right) dt = \int_1^R L(\{h=t\})\, dt$$

$$= \int_1^R L(t\beta)\, dt = \int_1^r tL(\beta)\, dt = L(\beta) \int_1^r t\, dt = L(\beta)\frac{R^2-1}{2}.$$

Hence,

$$\lim_{R \to \infty} \frac{A[\widetilde{M}_1 \cap \mathbb{B}(p,R)]}{\pi R^2} = \lim_{R \to \infty} \frac{L(\beta)\frac{R^2-1}{2}}{\pi R^2} = \frac{L(\beta)}{2\pi}. \tag{31}$$

And the lemma follows from (30), (31) and (29). $\square$

Lemma 7.6 $\Theta(C,p) \le \dfrac{1}{2\pi}\displaystyle\int_\Gamma \kappa(s)\,ds.$

Proof. By (29), it suffices to show that $L(\beta) \le \int_\Gamma \kappa(s)\,ds$. Let Σ be the portion of $\widetilde{M}_1$ bounded by β and Γ. By Gauss-Bonnet (observe that Σ is flat),

$$\int_\Gamma \kappa_g^\Sigma + \int_\beta \kappa_g^\Sigma = 2\pi\chi(\Sigma) = 0, \tag{32}$$

where κ_g^Σ denotes the geodesic curvature of the corresponding curve in Σ. If u is the arclength parameter of β and $\cdot = \frac{d}{du}$, the Gauss equation for Σ tells us that

$$\ddot\beta = \nabla^\Sigma_{\dot\beta}\dot\beta + \sigma^\Sigma(\dot\beta,\dot\beta) = \kappa_g^\Sigma\beta + \kappa_n^\Sigma N_\beta^\Sigma, \tag{33}$$

where κ_n^Σ is the normal curvature of β in Σ and N^Σ is the Gauss map of Σ.

The Gauss equation for $\mathbb{S}^2(p,1)$ allows us to write

$$\ddot\beta = \nabla^{\mathbb{S}^2(p,1)}_{\dot\beta}\dot\beta + \sigma^{\mathbb{S}^2(p,1)}(\dot\beta,\dot\beta) = \kappa_g^{\mathbb{S}^2(p,1)}N_\beta^\Sigma - \|\dot\beta\|^2\beta = \kappa_g^{\mathbb{S}^2(p,1)}N_\beta^\Sigma - \beta. \tag{34}$$

Comparing (33) and (34) we obtain $\kappa_g^\Sigma = -1$ (y $\kappa_n^\Sigma = \kappa_g^{\mathbb{S}^2(1)}$), hence

$$L(\beta) = \int_\beta du = -\int_\beta \kappa_g^\Sigma\,du \overset{(32)}{=} \int_\Gamma \kappa_g^\Sigma\,ds \le \int_\Gamma |\kappa_g^\Sigma|ds \le \int_\Gamma \|\Gamma''\|\,ds = \int_\beta \kappa(s)\,ds,$$

and the lemma is proved. □

We now prove Proposition 7.4. The argument is analogous to that of the original proof of the monotonicity formula for minimal surfaces. For this reason, we will give a different proof, valid in the case that $\widetilde{M}$ minimizes area, i.e., each compact piece of $\widetilde{M}$ is a minimum for the area among all compact surfaces with the same boundary (although the proposition remains true without this additional hypothesis).

Take $r>0$. As the cone C_r with vertex at p and base curve $\widetilde{M}\cap\mathbb{S}^2(p,r)$ is a compact surface with the same boundary as $\widetilde{M}\cap\mathbb{B}(p,r)$ and we are assuming that $\widetilde{M}$ is area-minimizing, it holds

$$A[\widetilde{M}\cap\mathbb{B}(p,r)] \le A(C_r) = \int_0^r \left(\int_{\{h=t\}} \frac{ds_t}{\|\nabla_{C_r}h\|}\right)dt = \int_0^r \left(\int_{\{h=t\}} ds_t\right)dt = \int_0^r L[C_r\cap\{h=t\}]\,dt,$$

where $h(q) = \|q-p\|$. But $C_r \cap \{h=t\}$ is the image of $\widetilde{M}\cap\mathbb{S}^2(p,r)$ under the homothety of factor t/r centered at p, hence the last integral equals

$$\int_0^r \frac{t}{r}L[\widetilde{M}\cap\mathbb{S}^2(p,r)]\,dt = L[\widetilde{M}\cap\mathbb{S}^2(p,r)]\int_0^r \frac{t}{r}\,dt = \frac{r}{2}L[\widetilde{M}\cap\mathbb{S}^2(p,r)].$$

By the co-area formula, we have

$$\frac{d}{dr}\left(A[\widetilde{M}\cap\mathbb{B}(p,r)]\right)=\int_{\{h=r\}}\frac{ds_r}{\|\nabla_{\widetilde{M}}h\|}\overset{(*)}{\geq}L[\widetilde{M}\cap\mathbb{S}(p,r)],$$

where in $(*)$ we have used that $\|\nabla_{\widetilde{M}}h\|\leq 1$ ($\widetilde{M}$ contains M). Thus,

$$A[\widetilde{M}\cap\mathbb{B}(p,r)]\leq\frac{r}{2}L[\widetilde{M}\cap\mathbb{S}^2(p,r)]\leq\frac{r}{2}\frac{d}{dr}\left(A[\widetilde{M}\cap\mathbb{B}(p,r)]\right),$$

which implies $\dfrac{d}{dr}\left(\dfrac{A[\widetilde{M}\cap\mathbb{B}(p,r)]}{r^2}\right)\geq 0.$ □

8 The Douglas-Radó solution to the Plateau problem for disks

Consider the *Plateau problem for disks* (proposed by J.A.F. Plateau[3] in 1873):

> Let $\Gamma\subset\mathbb{R}^n$ be a rectifiable Jordan curve (there exists a Lipschitz embedding $\psi\colon\mathbb{S}^1\to\Gamma$). ¿Is there a Lipschitz map $\phi\colon\overline{\mathbb{D}}=\{x^2+y^2\leq 1\}\to\mathbb{R}^n$ such that $\phi|_{\mathbb{S}^1}$ is a parameterization of Γ and $A(\phi(\overline{\mathbb{D}}))\leq A(\psi(\overline{\mathbb{D}}))$ for every Lipschitz map $\psi\colon\overline{\mathbb{D}}\to\mathbb{R}^n$ such that $\psi|_{\mathbb{S}^1}$ parameterizes Γ?

Some comments on the previous problem:

1. A rectifiable Jordan curve G can span more than one minimal surface, even with different topology (Figure 8).

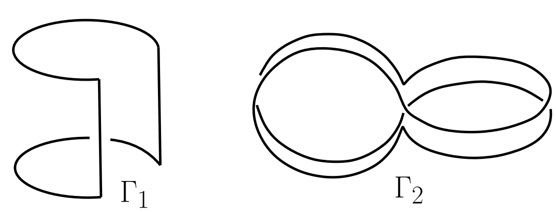


Figure 8: Γ_1 bounds at least two minimum disks, one with less area than the other. Γ_2 bounds at least two minimum disks, both with the same area. Also, Γ_2 bounds one surface of genus 1, of smaller area than the previous ones.

Figure 9: Γ is a rectifiable Jordan curve (finiteness of its length is guaranteed by the summability of a series) that bounds infinitely many minimal surfaces, that can be constructed of arbitrary finite genus, or even with infinite genus (monster surface).

2. A rectifiable Jordan curve Γ can bound infinitely many minimal surfaces, with infinitely many different topological types, and can even bound a minimal surface of infinite genus (Figure 9).

3. A rectifiable Jordan curve Γ could be the boundary of a non-orientable minimal surface. If Γ is a knot, Γ cannot be boundary of an embedded surface, but it can bound an non-embedded minimal surface (and even non-orientable, see Figure 10).

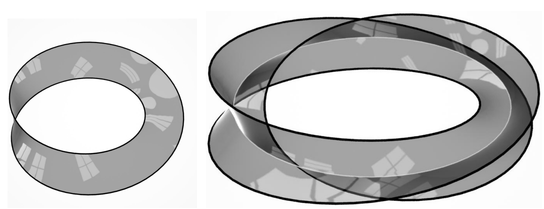


Figure 10: Left: Γ bounds a minimal Möbius band. Right: Γ is a knot and bounds a non-embedded and non-orientable minimal surface.

4. The rectifiability hypothesis on Γ is necessary: there exists a non-rectifiable Jordan curve $\Gamma \subset \mathbb{R}^3$ (equivalently, of infinite length) which is not the boundary of any disk with finite area.

5. The minimum area among all surfaces with boundary a rectifiable Jordan curve Γ may not be a surface (it is always a rectifiable current of dimension 2): let us take Γ as a perturbation of a circle traveled 3 times (Figure 11). If we intersect Γ with a transversal disk D we will

[3]Indeed, Plateau was looking for minimal disks whose boundary is a prescribed Jordan curve, not necessarily area-minimizing.

produce 3 points A, B, C, and when rotating D around Γ, these points spin around each other like the permutation $\begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix}$, so that when we turn D once along Γ, the points rotate an angle $2\pi/3$ in D. To imagine the minimizer of the area with boundary Γ, observe that given 3 points not aligned in $\mathbb{R}^2$, the configuration C of curves linking the three points that minimizes the length is formed by 3 line segments with a common vertex where the segments form (2 to 2) an angle of $2\pi/3$. C is an example of non-smooth solution of a length minimization problem. The 'solution' M of the Plateau problem for the curve Γ is a 2-dimensional current that cuts each disc D in a figure like the C on the plane.

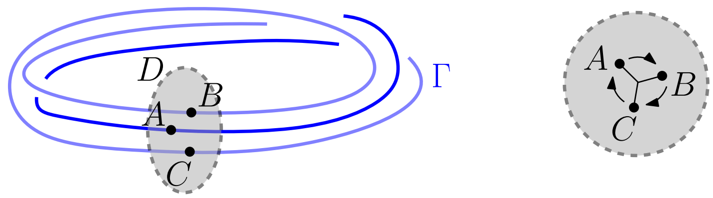


Figure 11: The minimum for the area with boundary a Jordan curve Γ can be a 2-dimensional current that is not a surface.

Theorem 8.1 (Douglas-Radó) *If $\Gamma \subset \mathbb{R}^n$ is a rectifiable Jordan curve, there exists a Lipschitz map $\phi \colon \overline{\mathbb{D}} = \{x^2 + y^2 \leq 1\} \to \mathbb{R}^n$ such that $\phi|_{\mathbb{S}^1}$ is a parameterization of Γ and $A(\phi(\overline{\mathbb{D}})) \leq A(\psi(\overline{\mathbb{D}}))$ for every Lipschitz map $\psi \colon \overline{\mathbb{D}} \to \mathbb{R}^n$ such that $\psi|_{\mathbb{S}^1}$ parameterizes Γ.*

Remark 8.2 1. Osserman, Gulliver and Alt proved that $\phi|_{\mathrm{Int}(\mathbb{D})}$ is an immersion.

2. It can happen that the minimum of the area among ALL surfaces (without restricting the topology to disks) with boundary Γ exists but it is not a disk (Figures 11 and 12).

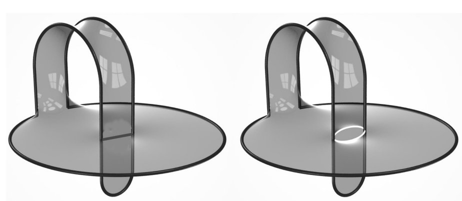


Figure 12: Left: Minimal disk bounded by Γ. Right: The least area surface bounded by Γ is not a disk.

Before starting the proof of Theorem 8.1, let us make some considerations. Let $\Gamma \subset \mathbb{R}^n$ be a rectifiable Jordan curve. Define the set

$$X_\Gamma = \{\psi \colon \mathbb{D}^2 \to \mathbb{R}^n \mid \psi \text{ is Lipschitz and } \psi|_{\mathbb{S}^1} \text{ parameterizes monotonically } \Gamma\},$$

where by parameterizing monotonically Γ we mean that when moving the parameter θ in $\mathbb{S}^1$ monotonically, $\psi(\theta)$ travels along Γ in a non-decreasing way; that is, $\psi(\theta)$ cannot turn back but could remain constant at some interval of $\mathbb{S}^1$. Rigorously, we are imposing that $\psi|_{\mathbb{S}^1} \colon \mathbb{S}^1 \to \Gamma$ is onto and that $(\psi|_{\mathbb{S}^1})^{-1}(C)$ is connected for each connected subset C of Γ. In particular, $\psi|_{\mathbb{S}^1} \colon \mathbb{S}^1 \to \Gamma$ does not need to be a homeomorphism (it is not necessarily injective). This generalization of the notion of monotonicity is reasonable because we will take limits on a sequence of maps in X_Γ whose areas decrease to the infimum of the area in X_Γ, and the injectivity condition on the boundary of the elements in the sequence could be lost when passing to the limit.

The family X_G is not empty: let us take $p \in \Gamma$ and consider the cone over Γ with vertex p, which has finite area by the co-area formula. It is possible to parameterize this cone by a map in X_Γ.

Consider the area functional

$$A \colon X_\Gamma \to [0, \infty), \quad A(\psi) = \int_{\mathbb{D}} \mathrm{Jac}(\psi)\, dxdy, \tag{35}$$

where $\mathrm{Jac}(\psi) = \sqrt{\|\psi_x\|^2 \|\psi_y\|^2 - \langle \psi_x, \psi_y \rangle^2}$.

Since A is bounded from below in X_Γ, there exists

$$a(\Gamma) := \inf_{\psi \in X_\Gamma} A(\psi) \in (0, \infty). \tag{36}$$

Now we can restate the theorem as:

Theorem 8.3 (Douglas-Radó) *If $\Gamma \subset \mathbb{R}^n$ is a rectifiable Jordan curve, then there exists $\phi \in X_\Gamma$ such that $A(\phi) = a(\Gamma)$.*

The first idea that comes to us to prove Theorem 8.1 is to consider a sequence $\{\psi_k\}_k \subset X_\Gamma$ with $A(\psi_k) \searrow a(\Gamma)$ (in the sequel, a *minimizing sequence*) and wonder if we can find a convergent subsequence of $\{\psi_k\}_k$ to a limit $\phi \in X_\Gamma$ with $A(\phi) = a(\Gamma)$. This idea does not work, for two reasons:

1. (Geometric reason). Let us take $\Gamma = \mathbb{S}^1 \subset \{z = 0\}$ in $\mathbb{R}^3$, so that $a(\Gamma) = \pi$, attained in X_Γ only by the closed disk $\overline{\mathbb{D}} \subset \{z = 0\}$. It is possible to construct immersions $\psi_k \in X_\Gamma$ with $A(\psi_k) = \pi + \frac{1}{k}$, whose image is arbitrarily close to $\overline{\mathbb{D}}$ except in k arbitrarily thin vertical 'tentacles' over k points in $\mathbb{D}$ (see Figure 13). In addition, it is possible to construct the ψ_k in such a way that $\{\psi_k(\mathbb{D})\}_k$ is dense in $\mathbb{R}^3$ (taking the tentacles in an appropriate way, not necessarily vertical). Hence we will not be able to find any subsequence of $\{\psi_k\}_k$ that converges to the desired limit.

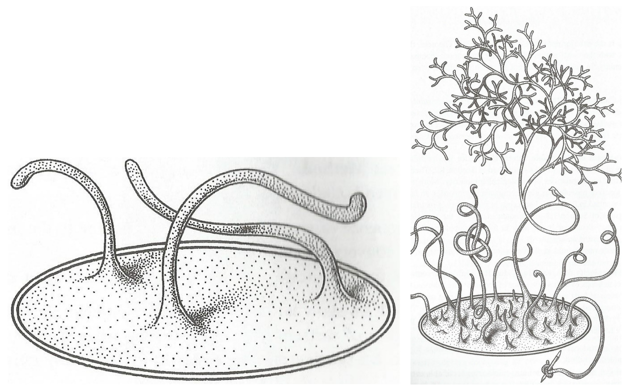


Figure 13: Figures taken from [13].

2. (Non-geometric reason). Let us take a sequence of diffeomorphisms $f_k : \overline{\mathbb{D}} \to \overline{\mathbb{D}}$. If $\{\psi_k\}_k \subset X_\Gamma$ is a minimizing sequence, then $\{\psi_k \circ f_k\}_k \subset X_\Gamma$ is also a minimizing sequence (the area is invariant under reparameterizations). Since the group of diffeomorphisms of $\overline{\mathbb{D}}$ is non-compact, it is not expected to be able to take a convergent subsequence of $\{\psi_k \circ f_k\}_k$.

The second issue above comes from the invariance of the area under reparameterizations. This is one of the reasons why we will replace area by *energy,* in the same way it is done with length and energy when studying geodesics in a Riemannian manifold.

Definition 8.4 Given $X \in X_\Gamma$, define

$$\text{Energy}(\psi) = E(\psi) = \int_{\mathbb{D}} |d\psi|^2 = \int_{\mathbb{D}} \left(\|\psi_x\|^2 + \|\psi_y\|^2\right).$$

Lemma 8.5 $2A(\psi) \leq E(\psi)$, $\forall \psi \in X_\Gamma$. *Moreover, equality holds if and only if* ψ *is conformal a.e. in* $\mathbb{D}$.

Proof. $2\|\psi_x \times \psi_y\| = 2\|\psi_x\|\|\psi_y\||\sin\sphericalangle(\psi_x, \psi_y)| \leq 2\|\psi_x\|\|\psi_y\| = -\left(\|\psi_x\| - \|\psi_y\|\right)^2 + \left(\|\psi_x\|^2 + \|\psi_y\|^2\right) \leq \|\psi_x\|^2 + \|\psi_y\|^2$. Equality in the lemma follows from analyzing when the above inequalities become equalities. □

Define

$$b(\Gamma) := \inf_{\psi \in X_\Gamma} E(\psi) \in (0, \infty).$$

Proposition 8.6 *In the above situation,* $2a(\Gamma) = b(\Gamma)$.

Proof. Taking infimum in X_Γ, Lemma 8.5 gives that $2a(\Gamma) \leq b(\Gamma)$.

Reciprocally, take $\{\psi_k\}_k \subset X_\Gamma$ such that $A(\psi_k) \searrow a(\Gamma)$. We can assume that ψ_k is C^2, by a density argument. Assume the following property holds:

(P) Given $k \in \mathbb{N}$, there exists a diffeomorphism f_k from $\mathbb{D}$ to itself such that $E(\psi_k \circ f_k) \leq 2A(\psi_k) + \frac{1}{k}$.

Then, $b(\Gamma) \leq E(\psi_k \circ f_k) \leq 2A(\psi_k) + \frac{1}{k}$, hence taking $k \to \infty$ we have $b(\Gamma) \leq 2a(\Gamma)$.

Next we prove property (P). Given $r > 0$, define

$$\psi_{k,r} \colon \mathbb{D} \to \mathbb{R}^{n+2}, \quad \psi_{k,r}(x,y) = (\psi_k(x,y), rx, ry)\,.$$

$\psi_{k,r}$ is injective, of class C^2, and $A(\psi_{k,r})$, $E(\psi_{k,r})$ depend continuously on r. In particular, there exists $\varepsilon > 0$ such that $\forall r \in (0,\varepsilon)$,

$$|A(\psi_{k,r}) - A(\psi_k)| < \frac{1}{k}. \tag{37}$$

As $\psi_{k,r}(\mathbb{D})$ is an embedded disk in $\mathbb{R}^{n+2}$, the induced metric by $\psi_{k,r}$ in $\mathbb{D}$ is Riemannian, hence it admits isothermal coordinates. Thus, there exists $f_k \colon \mathbb{D} \to \mathbb{D}$ such that $\overline{\psi_{k,r}} := \psi_{k,r} \circ k_n$ is conformal. Therefore, Lemma 8.5 implies

$$E(\overline{\psi_{k,r}}) = 2A(\overline{\psi_{k,r}}) \overset{(*)}{=} 2A(\psi_{k,r}), \tag{38}$$

where in $(*)$ we have used the invariance of area under reparameterizations. From (37) and (38) we have that $\forall r \in (0,\varepsilon)$,

$$\lim_{k\to\infty} E(\overline{\psi_{k,r}}) = 2 \lim_{k\to\infty} A(\psi_{k,r}) = 2 \lim_{k\to\infty} A(\psi_k) = 2a(\Gamma). \tag{39}$$

On the other hand, $\overline{\psi_{k,r}}(x,y) = (\psi_{k,r}(f_k(x,y)), r\, f_k(x,y))$, hence by definition of energy,

$$E(\psi_k \circ f_k) \leq E(\overline{\psi_{k,r}}), \tag{40}$$

thus

$$\overline{\lim} E(\psi_k \circ f_k) \leq \lim_{k\to\infty} E(\overline{\psi_{k,r}}) \overset{(39)}{=} 2a(\Gamma).$$

As $\psi_k \circ f_k \in X_\Gamma$ for each k, the above inequality implies $b(\Gamma) \leq \overline{\lim} E(\psi_k \circ f_k) \leq 2a(\Gamma)$. $\square$

Remark 8.7 Note that from the above proof we deduce that if $\{\psi_k\}_n \subset X_\Gamma$ satisfies $A(\Gamma_k) \searrow a(\Gamma)$, then $\overline{\lim} E(\psi_k \circ f_k) = b(\Gamma)$.

We can argue similarly with the lim inf: from (40) we have

$$\underline{\lim} E(\psi_k \circ f_k) \leq \lim_{k\to\infty} E(\overline{\psi_{k,r}}) \overset{(39)}{=} 2a(\Gamma) = b(\Gamma),$$

and since $\psi_k \circ f_k \in X_\Gamma$ for each k, then $b(\Gamma) \leq \underline{\lim} E(\psi_n \circ f_n) \leq b(\Gamma)$. Thus, there exists the limit of $E(\psi_k \circ f_k)$ and equals $b(\Gamma) = 2a(\Gamma)$.

We want to prove Theorem 8.3. Let us take a rectifiable Jordan curve $\Gamma \subset \mathbb{R}^n$ and a minimizing sequence $\{\psi_k\}_k \subset X_\Gamma$ with $A(\psi_n) \searrow a(\Gamma)$. By the proof of Proposition 8.6, we can exchange every ψ_k by the composition of ψ_k with a diffeomorphism of $\overline{\mathbb{D}}$ (denoted in the same way) so that $E(\psi_k)$ converges to $2a(\Gamma)$ (and $A(\psi_k)$ still converges to $a(\Gamma)$, because the area is invariant by reparameterizations).

For $k \in \mathbb{N}$ fixed, $\psi_k|_{\mathbb{S}^1}$ is a Lipschitz map taking values in $\mathbb{R}^n$. Applying to each component of $\psi_k|_{\mathbb{S}^1}$ the usual minimization of the Rayleigh quotient for the laplacian (we are using that the first eigenvalue of the laplacian in $\mathbb{D}$ is strictly positive), we conclude that there exists a harmonic extension $f_k \in H^1(\mathbb{D}, \mathbb{R}^n) \cap C^0(\overline{\mathbb{D}}, \mathbb{R}^n)$ of $\psi_k|_{\mathbb{S}^1}$, i.e.,

$$\begin{cases} \Delta f_k = 0 & \text{in } \mathbb{D}, \\ f_k = \psi_k & \text{in } \mathbb{S}^1, \end{cases}$$

such that f_k minimizes energy among all functions in $H^1(\mathbb{D}, \mathbb{R}^n) \cap C^0(\overline{\mathbb{D}}, \mathbb{R}^n)$ with boundary values $\psi_k|_{\mathbb{S}^1}$. As $f_k \in X_\Gamma$, we have

$$2a(\Gamma) \leq 2A(f_k) \overset{\text{(Lemma 8.5)}}{\leq} E(f_k) \leq E(\psi_k).$$

Taking limits we deduce that $E(f_k)$ converges to $2a(\Gamma)$ and $A(f_k)$ converges to $a(\Gamma)$.

The next step consists of finding a convergent subsequence of $\{f_k\}_k$. To do this, we will normalize the above construction in the following way: Fix points $a, b, c \in \mathbb{S}^1$ and $A, B, C \in\in \Gamma$ (different).

Lemma 8.8 *For each $k \in \mathbb{N}$, there exists a Möbius transformation $g_k \colon \overline{\mathbb{D}} \to \overline{\mathbb{D}}$ such that*

$$(f_k \circ g_k)(a) = A, \quad (f_k \circ g_k)(b) = B, \quad (f_k \circ g_k)(c) = C.$$

Proof. It is well-known that if $z_1, z_2, z_3 \in \mathbb{S}^1$ are distinct points and $w_1, w_2, w_3 \in \mathbb{S}^1$ are also distinct, there exists a unique Möbius transformation $g \colon \overline{\mathbb{D}} \to \overline{\mathbb{D}}$ such that $g(z_j) = w_j$, $j = 1, 2, 3$. For $k \in \mathbb{N}$ fixed, as $f_k|_{\mathbb{S}^1} : \mathbb{S}^1 \to \Gamma$ is onto, there exist $w_1, w_2, w_3 \in \mathbb{S}^1$ such that

$$f_k(w_1) = A, \quad f_k(w_2) = B, \quad f_k(w_3) = C.$$

Moreover, we can take w_1, w_2, w_3 distinct because A, B, C are different and $f_k|_{\mathbb{S}^1}$ is monotone in the sense of the definition of X_Γ. Thus, there exists a a unique Möbius transformation $g \colon \overline{\mathbb{D}} \to \overline{\mathbb{D}}$ such that $g(z_j) = w_j$, $j = 1, 2, 3$, and the lemma is proved. □

Since g_k is conformal, Lemma 8.5 implies that $E(f_k) = E(f_k \circ g_k)$. Hence $E(f_k \circ g_k)$ converges to $2a(\Gamma)$ and $A(f_k \circ g_k)$ converges to $a(\Gamma)$.

Lemma 8.9 *In the above situation, $\{(f_k \circ g_k)|_{\mathbb{S}^1}\}_k$ is equicontinuous: given $\varepsilon > 0$, there exists $\delta > 0$ such that if $z_1, z_2 \in \mathbb{S}^1$ satisfy $|z_1 - z_2| < \delta$, then $|(f_k \circ g_k)(z_1) - (f_k \circ g_k)(z_2)| < \varepsilon$.*

Proof. Before proving Lemma 8.9 we will need the following result, that shows how the energy controls the local dilatation of a map.

Lemma 8.10 (Courant-Lebesgue) *Given $E_0 > 0$, a Lipschitz map $f\colon \overline{\mathbb{D}} \to \mathbb{R}^n$ with $E(f) \leq E_0$, $p \in \overline{\mathbb{D}}$ and $R \in (0,1)$, there exists $r \in [R, \sqrt{R}]$ such that if $L(r)$ denotes the length of $f(C(p,r)\cap\overline{\mathbb{D}})$, then*

$$L(r)^2 \leq \frac{4\pi E(f)}{\log(1/R)} \leq \frac{4\pi E_0}{\log(1/R)}.$$

Proof. Since the function

$$\rho \in [R, \sqrt{R}] \mapsto L(\rho) = \text{Length}[f(C(p,\rho)\cap\overline{\mathbb{D}})]$$

is continuous, it achieves a minimum in the compact set $[R, \sqrt{R}]$. Thus, there exists $r \in [R, \sqrt{R}]$ such that $L(r) \leq L(\rho)$ for all $\rho \in [R, \sqrt{R}]$.

Calling s to the arclength parameter of $f(C(p,\rho)\cap\overline{\mathbb{D}})$ and using Schwarz' inequality, we have:

$$L(r)^2 \leq L(\rho)^2 = \left(\int_{C(p,\rho)\cap\overline{\mathbb{D}}} \left\|\tfrac{\partial f}{\partial s}\right\| ds\right)^2$$

$$\leq \text{Length}[C(p,\rho)\cap\overline{\mathbb{D}}] \int_{C(p,\rho)\cap\overline{\mathbb{D}}} \left\|\tfrac{\partial f}{\partial s}\right\|^2 ds \leq 2\pi\rho \int_{C(p,\rho)\cap\overline{\mathbb{D}}} \left\|\tfrac{\partial f}{\partial s}\right\|^2 ds.$$

Dividing by ρ and integrating from R to $\sqrt{R}$,

$$\int_R^{\sqrt{R}} \frac{L(r)^2}{\rho} d\rho \leq 2\pi \int_R^{\sqrt{R}} \left(\int_{C(p,\rho)\cap\overline{\mathbb{D}}} \left\|\tfrac{\partial f}{\partial s}\right\|^2 ds\right) d\rho. \tag{41}$$

The integral of the left-hand-side of (41) is $L(r)^2 \log(1/\sqrt{R}) = \frac{L(r)^2}{2}\log(1/R)$, hence

$$L(r)^2 \leq \frac{4\pi}{\log(1/R)} \int_R^{\sqrt{R}} \left(\int_{C(p,\rho)\cap\overline{\mathbb{D}}} \left\|\tfrac{\partial f}{\partial s}\right\|^2 ds\right) d\rho \leq \frac{4\pi}{\log(1/R)} E(f).$$

□

We now come back to the proof of Lemma 8.9. As Γ is a Jordan curve in $\mathbb{R}^n$, the chord-arc ratio is bounded in Γ. More precisely, given $\varepsilon > 0$, there exists $\lambda = \lambda(\varepsilon)$ only depending on Γ such that for any $P, Q \in \Gamma$ with $0 < |P-Q| < \lambda$, P, Q separate Γ into two arcs $\Gamma_1(P,Q), \Gamma_2(P,Q)$ such that the diameter in $\mathbb{R}^n$ of $\Gamma_1(P,Q)$ is less than ε.

Now take

$$\varepsilon \in (0, \min\{|A-B|, |A-C|, |B-C|\}), \tag{42}$$

which produces a $\lambda = \lambda(\varepsilon) > 0$ with the above chord-arc property. In particular, $\Gamma_1(P,Q)$ contains at most one of the points A, B, C (if for instance $\Gamma_1(P,Q)$ contains A, B, then $\operatorname{diam}(\Gamma_1(P,Q) \geq |A-B| > \varepsilon$, which is a contradiction).

Take $\delta_0 \in (0,1)$ such that

$$2\sqrt{\delta_0} < \min\{|a-b|, |a-c|, |b-c|\}. \tag{43}$$

Let $E_0 > 0$ be an upper bound of $E(f_k \circ g_k)$ for each $k \in \mathbb{N}$ (recall that $E(f_k \circ g_k)$ converges to $2a(\Gamma)$). Since the function

$$R > 0 \mapsto \left(\frac{4\pi E_0}{\log(1/R)}\right)^{1/2}$$

tends to 0^+ as $R \searrow 0$, there exists $R = R(\varepsilon) \in (0, \delta_0)$ such that

$$\left(\frac{4\pi E_0}{\log(1/R)}\right)^{1/2} < \lambda(\varepsilon). \tag{44}$$

Now take $z_0 \in \partial D$. Applying Lemma 8.10 to $f_k \circ g_k$ with $p = z_0$ and with the number R we have just obtained, we deduce that there exists $r \in [R, \sqrt{R}]$ satisfying the inequality of the Courant-Lebesgue Lemma. Thus, if z, z' are the intersection points of $\partial\mathbb{B}(z_0, r)$ with $\mathbb{S}^1$, then

$$\|(f_k \circ g_k)(z) - (f_k \circ g_k)(z')\| \leq L(r) \overset{\text{(Lemma 8.10)}}{\leq} \left(\frac{4\pi E_0}{\log(1/R)}\right)^{1/2} \overset{(44)}{<} \lambda(\varepsilon).$$

Therefore, the above chord-arc property of Γ ensures that

$$\operatorname{diam} \Gamma_1\left((f_k \circ g_k)(z), (f_k \circ g_k)(z')\right) < \varepsilon, \tag{45}$$

and thus, $\Gamma_1\left((f_k \circ g_k)(z), (f_k \circ g_k)(z')\right)$ only contains one of the points A, B, C.

Consider the arc $(f_k \circ g_k)\left(\overline{\mathbb{B}}(z_0, r) \cap \mathbb{S}^1\right) \subset \Gamma$, that satisfies the following properties:

(P1) $(f_k \circ g_k)\left(\overline{\mathbb{B}}(z_0, r) \cap \mathbb{S}^1\right)$ has the same extrema as $\Gamma_1\left((f_k \circ g_k)(z), (f_k \circ g_k)(z')\right)$ (by construction).

(P2) $(f_k \circ g_k)\left(\overline{\mathbb{B}}(z_0, r) \cap \mathbb{S}^1\right)$ only contains one of the points A, B, C (this follows from $0 < r \leq \sqrt{R} < \sqrt{\delta_0} < 2\sqrt{\delta_0}$ and from (43)).

As $\Gamma_1\left((f_k \circ g_k)(z), (f_k \circ g_k)(z')\right)$ also satisfies properties (P1) and (P2) above, we deduce that

$$(f_k \circ g_k)\left(\overline{\mathbb{B}}(z_0, r) \cap \mathbb{S}^1\right) = \Gamma_1\left((f_k \circ g_k)(z), (f_k \circ g_k)(z')\right). \tag{46}$$

Finally, given $w, w' \in \mathbb{B}(z_0, r) \cap \mathbb{S}^1)$, from (46) we have

$$(f_k \circ g_k)(w), (f_k \circ g_k)(w') \in \Gamma_1\left((f_k \circ g_k)(z), (f_k \circ g_k)(z')\right),$$

hence (45) implies $\|(f_k \circ g_k)(w) - (f_k \circ g_k)(w')\| < \varepsilon$. This proves the equicontinuity of $\{(f_k \circ g_k)|_{\mathbb{S}^1}\}_k$ (the dependence on $z_0 \in \mathbb{S}^1$ can be ruled out taking a finite number of these points on $\mathbb{S}^1$ by compactness and using that r does not depend on z_0). Now Lemma 8.9 is proved. $\square$

Lemma 8.11 *There exists $f \in X_\Gamma$ and a subsequence of $\{f_k \circ g_k\}_k$ that converges on compact subsets of $\mathbb{D}$ to f.*

Proof. Note that given $k, h \in \mathbb{N}$, $(f_k \circ g_k) - (f_h \circ g_h)$ is harmonic in $\mathbb{D}$ and continuous in $\overline{\mathbb{D}}$, hence

$$\|(f_k \circ g_k) - (f_h \circ g_h)\|_{L^\infty(\mathbb{D})} = \|(f_k \circ g_k) - (f_h \circ g_h)\|_{L^\infty(\mathbb{S}^1)} \tag{47}$$

(maximum norm). By Lemma 8.9 and Arzelá-Ascoli theorem, $\{(f_k \circ g_k)|_{\mathbb{S}^1}\}_k$ is relatively compact in $C^0(\mathbb{S}^1, \Gamma)$. Thus, after passing to a subsequence (denoted in the same way), we have that $(f_k \circ g_k)|_{\mathbb{S}^1}$ converges to a continuous function from $\mathbb{S}^1$ to Γ in maximum norm, and this limit is Lipschitz in $\mathbb{S}^1$ (because there is a common bound for the Lipschitz constants of $(f_k \circ g_k)|_{\mathbb{S}^1}$). Thus, for k, h large enough in this subsequence, the right-hand-side of (47) can be made arbitrarily small. Therefore, given $K \subset \mathbb{D}$ compact, $\|(f_k \circ g_k) - (f_h \circ g_h)\|_{L^\infty(K)}$ can be made arbitrarily small if k, h are large, i.e. $\{(f_k \circ g_k)|_K\}_k$ is a Cauchy sequence with the maximum norm. Thus, $\{(f_k \circ g_k)|_K\}_k$ converges to a harmonic function $f\colon K \to \mathbb{R}^n$. Moving the compact set K in an increasing sequence whose union is $\mathbb{D}$ and using a diagonal argument, we produce a harmonic map $f\colon \mathbb{D} \to \mathbb{R}^n$ that is the limit on compact subsets of $\mathbb{D}$ of $(f_k \circ g_k)|_{\mathbb{S}^1}$ as $k \to \infty$. In addition, f extends continuously to $\mathbb{S}^1$ as the limit of the restrictions of $(f_k \circ g_k)|_{\mathbb{S}^1}$ as $k \to \infty$. From here it is not difficult to verify that $f \in X_\Gamma$ and the Lemma is proved. $\square$

Now we can finish the proof of the Theorem by Douglas and Radó: If $f \in X_\Gamma$ is the map given by Lemma 8.11, then given a compact set $K \subset \mathbb{D}$,

$$A(f|_K) = \lim_{k\to\infty} A((f_k \circ g_k)|_K) \le \lim_{k\to\infty} A(f_k \circ g_k) = a(\Gamma),$$

hence the dominated convergence theorem implies that $A(f) = a(\Gamma)$, which finishes the proof of the theorem. $\square$

Let us see some properties of the Douglas-Radó solution to the Plateau problem for disks. Until the end of this section, $\Gamma \subset \mathbb{R}^n$ will denote a rectifiable Jordan curve and $f\colon \overline{\mathbb{D}} \to \mathbb{R}^n$ a Lipschitz map, harmonic in $\mathbb{D}$, with $E(f) = 2A(f) = 2a(\Gamma)$. In particular, f is conformal in D by Lemma 8.5. In addition, $f|_{\mathbb{S}^1}\colon \mathbb{S}^1 \to \Gamma$ is a monotone non-decreasing parameterization of Γ.

Lemma 8.12 *Let $A = \{w \in \mathbb{D} \mid (\mathrm{Jac} f)(w) = 0\}$ (points in A are called ramification points of f). Then, A has no accumulation points in $\mathbb{D}$.*

Proof. Take $w = x + iy \in A$. As df_w is not injective and $\|f_x\| = \|f_y\|$, $\langle f_x, f_y\rangle = 0$, we have $df_w = 0$. If we write $f = (f^1, \ldots, f^n)$ in its components, then $\|f_x(w)\|^2 = 0$ writes as

$$[f_x^1(w)]^2 + \ldots + [f_x^n(w)]^2 = 0.$$

(and we can replace x by y in this last equation). On the other hand, as f is harmonic in $\mathbb{D}$ we have $\Delta f^k = 0$ for each $k = 1, \ldots, n$, hence by the Cauchy-Riemann equations

$$f_x^k - i f_y^k \text{ is holomorphic in } \mathbb{D},\ \forall k = 1, \ldots, n.$$

A point w lies in A if and only if $f_x^k - i f_y^k$ vanishes at w. Since $f_x^k - i f_y^k$ is holomorphic and not constantly zero (if identically zero, f would be constant in $\mathbb{D}$, a contradiction), the identity principle for holomorphic functions implies that A has no accumulation points in $\mathbb{D}$. □

Lemma 8.13 *$f|_{\mathbb{S}^1} \colon \mathbb{S}^1 \to \Gamma$ is a homeomorphism.*

Proof. Arguing by contradiction, suppose that there exists a non-trivial interval $\beta \subset \mathbb{S}^1$ such that $f(\beta)$ is a point of Γ. Take a circular domain $E \subset \mathbb{D}$, topologically a disk, whose boundary is the union of a circle arc contained in β and another one contained in $\mathbb{D}$. Let φ be a Möbius transformation mapping $\mathbb{D}$ onto the open upper half-plane $\mathbb{C}^+ \subset \mathbb{C}$. The image of E will be a half-disk contained in $\mathbb{C}^+$, with a connected portion of its boundary contained in $\partial\mathbb{C}^+$ (see Figure 14).

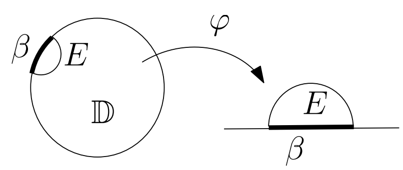


Figure 14: φ is a Möbius transformation.

After composing f with φ (we will not change the notation) and translating the image in $\mathbb{R}^n$, we can suppose that the restriction of f to the half-disk E maps the segment $\beta \subset \partial\mathbb{C}^+$ into $\vec{0} \in \mathbb{R}^n$. This implies that each component f^k of f is identically zero on β. As f^k is harmonic, we can extend f^k to the lower half-disk $E^* = \{\overline{w} \mid w \in E\}$ by $f^k(x, y) = f^k(x, -y)$, $\forall (x, y) \in E^*$. Thus we have a real-valued function f^k defined in the disk $E \cup E^* \cup (\beta \cap \partial E)$. f^k will be the real part of a holomorphic function $h^k \colon E \cup E^* \cup (\beta \cap \partial E) \to \mathbb{C}$.

As $\nabla f^k = 0$ on $\beta \cap \partial E$, we conclude that the derivative of h^k vanishes in $\beta \cap \partial E$. By the identity principle for holomorphic functions, we have that h^k is constant in $E \cup E^* \cup (\beta \cap \partial E)$, and thus f^k is constant in E. Doing this for $k = 1, \ldots, n$ we deduce that f is constant in E and thus, f is constant in $\mathbb{D}$ (by the identity principle for harmonic functions), which is a contradiction. □

Lemma 8.14 *If Γ contains an analytic arc, then $A = \{w \in \mathbb{D} \mid (\mathrm{Jac} f)(w) = 0\}$ has no accumulation points in the preimage by $f|_{\mathbb{S}^1} \colon \mathbb{S}^1 \to \Gamma$ of this arc.*

Proof. Suppose that A has an accumulation point in an interval $\beta \subset \mathbb{S}^1$ such that $f(\beta) = \Gamma_0$ is an analytic arc of Γ. After composing f with an analytic diffeomorphism between open sets of $\mathbb{R}^3$, we can assume that $f(\beta)$ is a segment in $\mathbb{R}^n$ contained in the axis $\{x_1 = \ldots = x_{n-1} = 0\}$. By the argument in the proof of Lemma 8.13 applied to the components $f^1, \ldots, f^{n-1}$ of f, we conclude that each of these components extends to a disk $E \cup E^* \cup (\beta \cap \partial E)$ (we are using the notation in that proof). Fix $k \in \{1, \ldots, n-1\}$. As $\nabla f^k = 0$ in $\beta \cap \partial E$, we can repeat the argument in the last paragraph of the proof of Lemma 8.13 to deduce that $f^1, \ldots, f^{n-1}$ are constant in $\mathbb{D}$. Thus, $f(\Gamma)$ is contained in a straight line of $\mathbb{R}^n$ in the direction of the x_n-axis. This contradicts Lemma 8.13. $\square$

Remark 8.15 Lemma 8.12 implies that the number of branch points of f in each compact of $\mathbb{D}$ is finite, and Lemma 8.13 implies that if Γ is an analytic Jordan curve, then the number branch points of f is finite. A lot more can be said about this, although we will not prove the following properties:

1. Even if Γ is just C^∞, the number of branch points of f is finite. This is a consequence of a Gauss-Bonnet type formula, which relates the sum of the orders of the branch points of f (this concept has not been defined, but is related to the order of vanishing of the holomorphic map whose components are obtained as in the proof of Lemma 8.12) with the total curvature of the branched immersion f and the total curvature of G. This formula can be found in page 213 of the book *Handbook of Differential Geometry,* Vol 1, Elsevier North Holland (1999).

2. The branch points of a Douglas-Radó solution to the Plateau problem are of two types: *true branch points*, around which $f(\mathbb{D})$ fails to be an immersed surface, and *false branch points*, around which $f(\mathbb{D})$ is an immersed surface but f fails to be an immersion. In 1970, Osserman proved that any Douglas-Radó solution for a rectifiable Jordan curve $\Gamma \subset \mathbb{R}^3$ is free of interior true branch points. Later, Gulliver proved there are also no interior false branching points, and therefore, every Douglas-Radó solution to the Plateau problem in $\mathbb{R}^3$ produces a minimal immersion in the interior. If Γ is analytic or its total curvature is strictly less than 4π, boundary branch points can also be discarded (Gulliver and Leslie, 1973).

3. There are examples of analytic Jordan curves $\Gamma \subset \mathbb{R}^4$ whose Douglas-Radó solution always has interior branch points: the application $z \in \overline{D} \overset{f}{\mapsto} (z^2, z^3)$ is the least area surface whose boundary is the analytic Jordan curve obtained as the image of $z \in \mathbb{S}^1 \mapsto (z^2, z^3)$. f has a true branch point at $z = 0$.

4. As for uniqueness of the Douglas-Radó solution of the Plateau problem for a Jordan curve $\Gamma \subset \mathbb{R}^3$, in general such a solution does not have to be unique. However, Radó (1930) gave conditions that imply uniqueness (Radó's Theorem, see Theorem 10.8). Nitsche (1989) proved that if $\Gamma \subset \mathbb{R}^3$ is either analytic with total curvature $\leq 4\pi$ or smooth with total curvature $< 4\pi$, then the Douglas-Radó solution for Γ is unique.

5. There are Jordan curves $\Gamma \subset \mathbb{R}^3$, even analytic, whose Douglas-Radó solution to the Plateau problem can never be an embedding: it is enough to take a knot. However, if we can exclude interior self-intersection points in a Douglas-Radó solution, then that solution is embedded. Meeks and Yau (1982) proved that if Γ lies in the boundary of a convex body, then every Douglas-Radó solution with boundary Γ is an embedding. Also recall Theorem 7.2: If $\Gamma \subset \mathbb{R}^3$ is a Jordan curve of class C^2 with total curvature $\leq 4\pi$, then every Douglas-Radó solution with boundary Γ is embedded.

As for the Plateau problem in an arbitrary Riemannian manifold, we will only state the following result (without proof):

Theorem 8.16 *Let (M^3, g) be a complete Riemannian manifold and $\Gamma \subset M$ a rectifiable Jordan curve that is homotopically trivial in M. Then, there exists an immersion $f\colon \overline{\mathbb{D}} \to M$ with boundary Γ that minimizes area among all immersions with boundary Γ.*

9 The isoperimetric inequality for minimal surfaces

Let (M^k, g) be a compact Riemannian manifold with boundary $\partial M \neq \varnothing$. In general, we cannot expect any relation of the type $\mathrm{Vol}(M) \leq C \cdot \mathrm{Area}(\partial M)$ between the volume of M and the area of its boundary (the second one could be arbitrarily small and the first one arbitrarily large), see Figure 15.

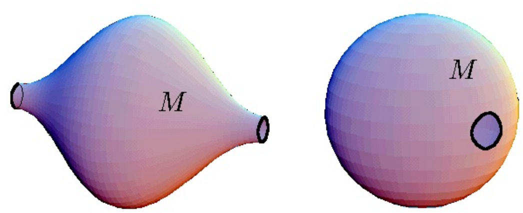


Figure 15: The quotient $\mathrm{Vol}(M)/\mathrm{Area}(\partial M)$ can be arbitrarily large.

Can we say something else in the case of a minimal submanifold of Euclidean space? In $\mathbb{R}^3$, it is expected that minimal surfaces satisfy the Euclidean isoperimetric inequality:

Conjecture 9.1 *If $M \subset \mathbb{R}^3$ is a compact minimal surface with boundary, then $4\pi\mathrm{Area}(M) \leq L(\partial M)^2$, with equality if and only if M is a flat round disk.*

Very recently, Brendle [1] has solved this conjecture. In fact, he has proved a much more general result:

Theorem 9.2 (Brendle, 2019) *Let M be a compact immersed hypersurface in $\mathbb{R}^{n+1}$, with boundary ∂M and mean curvature H (here H means trace of the second fundamental form). Then,*

$$\frac{\text{Area}(\partial M) + \int_M |H|}{\text{Area}(\partial \mathbb{B}^n)} \geq \left(\frac{\text{Vol}(M)}{\text{Vol}(\mathbb{B}^n)} \right)^{\frac{n-1}{n}}$$

Furthermore, equality holds if and only if M is a flat round disk.

Observe that if we take $H = 0$ and $n = 2$, we directly have the validity of Conjecture 9.1.

At the end of this section we will give a proof of Brendle's theorem. Before doing this, we will collect some previous advances on this problem.

Lemma 9.3 *Let $M^k \subset \mathbb{R}^n$ be a compact minimal submanifold with boundary. Take $R > 0$ such that $\partial M \subset \mathbb{B}(R) = \mathbb{B}(\vec{0}, R)$. Then,*

$$\text{Vol}(M) \leq \frac{R}{k} \text{Area}(\partial M).$$

Proof. Consider the variation $M(t) = (1+t)M$ of M by homotheties, whose variational field is $Y_x = x$, $x \in \mathbb{R}^n$. Thus, $\text{Vol}(M(t)) = (1+t)^k \text{Vol}(M)$ and the first variation formula for volume (Proposition 16.1) implies

$$k \cdot \text{Vol}(M) = \left. \frac{d}{dt} \right|_{t=0} [\text{Vol}(M(t))] = -k \int_M \langle Y, \vec{H} \rangle + \int_{\partial M} \langle Y, \eta \rangle = \int_{\partial M} \langle Y, \eta \rangle,$$

$$\leq \left| \int_{\partial M} \langle Y, \eta \rangle \right| \leq \int_{\partial M} \|Y\| \leq \int_{\partial M} R = R \cdot \text{Area}(\partial M).$$

where η is the outward pointing unit conormal vector to M along ∂M. $\square$

Let us particularize Lemma 9.3. After a rigid motion, we can assume $\vec{0} \in \partial M$ hence $\partial M \subset \mathbb{B}(R)$ for $R = L(\partial M)/2$. Thus, Lemma 9.3 tells us that $\text{Area}(M) \leq \frac{R}{2} L(\partial M) = \frac{1}{4} L(\partial M)^2$. This inequality is much weaker than the conjecture (it is like comparing π to 1).

Let $M \subset \mathbb{R}^3$ be a minimal surface and $[\gamma] \in H^1(M)$ a 1-dimensional homology class. Given $\gamma_1, \gamma_2 \in [\gamma]$, there exists a compact domain $M(\gamma_1, \gamma_2) \subset M$ such that $\partial M(\gamma_1, \gamma_2) = \gamma_1 - \gamma_2$ (beware with the orientations). Let η the unit conormal vector to $M(\gamma_1, \gamma_2)$ along its boundary, so that η points outwards (resp. inwards) $M(\gamma_1, \gamma_2)$ along γ_1 (resp. γ_2) o vice versa (see Figure 16).

Let us see that

$$\int_{\gamma_1} \eta = \int_{\gamma_2} \eta. \tag{48}$$

Fix $a \in \mathbb{R}^3$ and consider the tangent vector field to M given by $Y = a^T$. The divergence of Y in M can be computed as follows: take an orthonormal basis ofT_pM (with $p \in M$ arbitrary) and

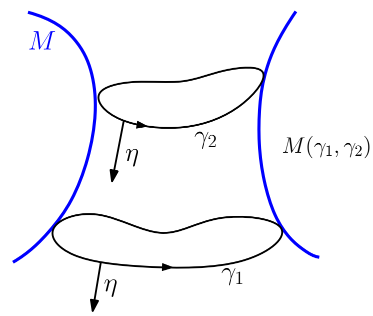


Figure 16: The flux vector does not depend on the representative in the same homology class.

a Gauss map N for M. Then,

$$\operatorname{div}_M(Y) = \sum_{i=1}^{2} \langle \nabla^M_{e_i}(a^T), e_i \rangle = \sum_{i=1}^{2} \langle \nabla^{\mathbb{R}^3}_{e_i} (a - \langle a, N \rangle N), e_i \rangle$$

$$= -\langle a, N \rangle \sum_{i=1}^{2} \langle \nabla^{\mathbb{R}^3}_{e_i} N, e_i \rangle = 2H\langle a, N \rangle = 0,$$

hence the divergence theorem implies

$$0 = \int_{M(\gamma_1,\gamma_2)} \operatorname{div}_M(Y) = \int_{\gamma_1 - \gamma_2} \langle Y, -\eta \rangle = \int_{\gamma_2} \langle Y, \eta \rangle - \int_{\gamma_1} \langle Y, \eta \rangle = \int_{\gamma_2} \langle a, \eta \rangle - \int_{\gamma_1} \langle a, \eta \rangle.$$

It only remains to move a in $\mathbb{R}^3$ to deduce (48). Consequently, $\int_\gamma \eta$ does not depend on the representative $\gamma \in [\gamma]$.

Definition 9.4 (Flux) In the above situation, the *flux* map is the group morphism

$$F\colon H^1(M) \to \mathbb{R}^3, \quad F([\gamma]) = \operatorname{Flux}([\gamma]) = \int_\gamma \eta.$$

Clearly, the flux map does not depend on translations of M in $\mathbb{R}^3$.

Theorem 9.5 *Let $M \subset \mathbb{R}^3$ be a compact minimal surface with boundary $\partial M = \gamma_1 \cup \ldots \cup \gamma_k$, where each $\gamma_i \subset \mathbb{R}^3$ is a Jordan curve. If* $\operatorname{Flux}([\gamma_i]) = 0$ *for each $i = 1, \ldots, k$, then Conjeture 9.1 holds.*

Proof. Consider the vector field on $\mathbb{R}^3$ given y Sea $Y_x = x$. Let us compute the divergence of Y^T in M: take an orthonormal basis e_1, e_2 of T_pM (with $p \in M$ arbitrary) and a Gauss map N for M. Then,

$$\operatorname{div}_M(Y^T) = \sum_{i=1}^{2} \langle \nabla^M_{e_i}(Y^T), e_i \rangle = \sum_{i=1}^{2} \langle \nabla^{\mathbb{R}^3}_{e_i} \left(Y - \langle Y, N \rangle N \right), e_i \rangle$$

$$= 2 - \langle Y, N \rangle \sum_{i=1}^{2} \langle \nabla^{\mathbb{R}^3}_{e_i} N, e_i \rangle = 2 + 2H \langle Y, N \rangle = 2,$$

hence the divergence theorem implies

$$2 \cdot \operatorname{Area}(M) = \int_M \operatorname{div}_M(Y^T) = \int_{\partial M} \langle Y, \eta \rangle = \sum_{i=1}^{k} \int_{\gamma_i} \langle p, \eta \rangle. \tag{49}$$

For each $i = 1, \dots, k$, we choose a point $a_i \in \gamma_i$ and call $\widetilde{\gamma}_i = \gamma_i - a_i$. Then,

$$\int_{p \in \gamma_i} \langle p, \eta \rangle = \int_{q \in \widetilde{\gamma}_i} \langle q + a_i, \eta \rangle = \int_{q \in \widetilde{\gamma}_i} \langle q, \eta \rangle + \left\langle a_i, \int_{\widetilde{\gamma}_i} \eta \right\rangle = \int_{q \in \widetilde{\gamma}_i} \langle q, \eta \rangle + \langle a_i, \operatorname{Flux}([\widetilde{\gamma}_i]) \rangle .$$

The second term above vanishes by hypothesis, hence

$$\int_{\gamma_i} \langle p, \eta \rangle = \int_{\widetilde{\gamma}_i} \langle q, \eta \rangle \leq \int_{\widetilde{\gamma}_i} \|q\| \tag{50}$$

by Schwarz inequality. Using (49) and (50),

$$2 \cdot \operatorname{Area}(M) \leq \sum_{i=1}^{k} \int_{\widetilde{\gamma}_i} \|q\| \overset{(*)}{\leq} \left(\int_{\dot\cup_{i=1}^{k} \widetilde{\gamma}_i} \|q\|^2 \right)^{1/2} L \left(\dot\cup_{i=1}^{k} \widetilde{\gamma}_i \right)^{1/2} = \left(\int_{\dot\cup_{i=1}^{k} \widetilde{\gamma}_i} \|q\|^2 \right)^{1/2} L^{1/2}, \tag{51}$$

where in $(*)$ we have used Schwarz inequality in $L^2(\widetilde{\gamma}_1 \dot\cup \dots \dot\cup \widetilde{\gamma}_k)$ and $L = L(\partial M)$.

Next parameterize $\widetilde{\gamma}_1 \dot\cup \dots \dot\cup \widetilde{\gamma}_k$ by $\phi \colon \mathbb{S}^1_L \to \widetilde{\gamma}_1 \dot\cup \dots \dot\cup \widetilde{\gamma}_k$, where $\mathbb{S}^1_L$ is the circumference of length L and each $\widetilde{\gamma}_i$ is traveled with speed 1 starting and finishing at the origin. ϕ is piecewise smooth (it is continuous and may fail to be smooth when changing of $\widetilde{\gamma}_i$ at the origin). Therefore, we can see ϕ (indeed, each of its components) in the Sobolev space $H^1(\mathbb{S}^1_L)$. Note that can assume $\int_{\mathbb{S}^1_L} \phi = \vec{0} \in \mathbb{R}^3$ (translate ϕ by the vector $-\int_{\mathbb{S}^1_L} \phi \in \mathbb{R}^3$, which does not change the above arguments). The mean zero condition on ϕ allows us to use it to give an estimate of the second eigenvalue of the laplacian for the standard metric on $\mathbb{S}^1_L$ by means of the Rayleigh quotient of ϕ:

$$\int_{\mathbb{S}^1_L} \|\nabla \phi\|^2 \geq \lambda_2(\Delta, \mathbb{S}^1_L) \int_{\mathbb{S}^1_L} \|\phi\|^2 = \left(\frac{2\pi}{L} \right)^2 \int_{\mathbb{S}^1_L} \|\phi\|^2, \tag{52}$$

hence

$$\int_{\dot\cup_{i=1}^k \widetilde{\gamma}_i} \|q\|^2 = \int_{\mathbb{S}^1_L} \|\phi\|^2 \overset{(52)}{\leq} \left(\frac{L}{2\pi}\right)^2 \int_{\mathbb{S}^1_L} \|\nabla\phi\|^2 = \left(\frac{L}{2\pi}\right)^2 \int_{\dot\cup_{i=1}^k \widetilde{\gamma}_i} \|\widetilde{\gamma}_i'\|^2 = \left(\frac{L}{2\pi}\right)^2 L. \tag{53}$$

Finally,

$$2 \cdot \text{Area}(M) \overset{(51)}{\leq} \left(\int_{\dot\cup_{i=1}^k \widetilde{\gamma}_i} \|q\|^2 \right)^{1/2} L^{1/2} \overset{(53)}{\leq} \frac{L^2}{2\pi}.$$

which is the inequality in Conjecture 9.1. If the equality holds in the conjecture, then we will also have equality in each of the estimates that we have done. In particular, ϕ is an eigenfunction for the second eigenvalue of the laplacian for the standard metric on $\mathbb{S}^1_L$. This means that ϕ parameterizes a circle and $k = 1$. Thus, the boundary of M is a circle, and the convex hull property (Corollary 4.7) gives that M is a flat round disk. $\square$

Corollary 9.6 *If $M \subset \mathbb{R}^2$ is a compact minimal surface with connected boundary, Conjeture 9.1 holds.*

Proof. The condition $\text{Flux}([\partial M]) = 0$ holds by the divergence theorem, hence it suffices to apply Theorem 9.5. $\square$

Remark 9.7 Osserman proved Conjecture 9.1 under the additional hypothesis that M is topologically an annulus.

To finish this section, we will give the proof by Brendle of Theorem 9.2. Let M be a compact immersed hypersurface in $\mathbb{R}^{n+1}$, with boundary ∂M and mean curvature H. We recall that for the remainder of this section H will mean trace of the second fundamental form of M.

Lemma 9.8 *After possibly a homothety in $\mathbb{R}^{n+1}$, we can assume*

$$\text{Area}(\partial M) + \int_M |H| = n\,\text{Vol}(M).$$

Proof. Let $f\colon (0,\infty) \to (0,\infty)$ be the function defined by

$$f(\lambda) = \frac{\text{Area}(\partial(\lambda M)) + \int_{\lambda M} |H_{\lambda M}|\, dv_{\lambda M}}{n\,\text{Vol}(\lambda M)},$$

where $dv_{\lambda M}$ denotes the volume element of λM. Note that $\text{Area}(\partial(\lambda M)) = \text{Area}(\lambda\partial(M)) = \lambda^{n-1}\text{Area}(\partial(M))$, $\text{Vol}(\lambda M) = \lambda^n \text{Vol}(M)$ and

$$\int_{\lambda M} |H_{\lambda M}|\, dv_{\lambda M} = \int_M |H_{\lambda M}| \lambda^n\, dv_M = \lambda^{n-1} \int_M |H_M|\, dv_M,$$

hence $f(\lambda) = \frac{1}{\lambda} f(1)$. This implies that $f(\lambda)$ has limit $+\infty$ as $\lambda \to 0^+$ and limits to 0 when $\lambda \to \infty$. By continuity, we conclude that there exists $\lambda_0 \in (0, \infty)$ such that $f(\lambda_0) = 1$. $\square$

Let η be the outward pointing unit conormal vector to M along ∂M.

Lemma 9.9 *Under the normalization in Lemma 9.8, there exists $u \colon M \to \mathbb{R}$ of class $C^{2,\alpha}$ (here α is any number in $(0,1)$) solving the following Neumann type problem:*

$$\left\{ \begin{array}{ll} \Delta u = n - |H| & \textit{in } M, \\ \frac{\partial u}{\partial \eta} = 1 & \textit{on } \partial M, \end{array} \right. \tag{54}$$

Proof. Recall that since M is compact with boundary, the first eigenvalue of the laplacian in M for the Neumann problem is zero, and the associated eigenfunctions are constants. Thus, given $f \in L^2(M)$, the Neumann type problem

$$\left\{ \begin{array}{ll} \Delta v = f & \text{in } M, \\ \frac{\partial v}{\partial \eta} = 0 & \text{on } \partial M \end{array} \right. \tag{55}$$

admits a solution $v \in H^1(M)$ if and only if $\int_M f = 0$. Take a function $\varphi \in C^\infty(M)$ such that $\frac{\partial \varphi}{\partial \eta} = 1$ on ∂M. Then, the Stokes' theorem gives

$$\int_M (n - |H| - \Delta\varphi) = n \operatorname{Vol}(M) - \int_M |H| - \int_{\partial M} \frac{\partial \varphi}{\partial \eta} = n \operatorname{Vol}(M) - \int_M |H| - \operatorname{Area}(\partial M),$$

which vanishes by Lemma 9.8. Since $|H|$ is Lipschitz on M, then $f = n - |H| - \Delta\varphi$ is also Lipschitz, and the above discussion allows us to find a solution $v \in H^1(M)$ of (55). In fact, $v \in C^{2,\alpha}(M)$ for any $\alpha \in (0,1)$ since f is Lipschitz (for this regularity property, see for instance the book of Gilbarg and Trudinger [8]). Finally, the function $u = v + \varphi$ satisfies the conditions of the lemma. This finishes the proof. $\square$

Next we consider the sets

$$\left. \begin{array}{rcl} U & = & \{x \in M \setminus \partial M \mid |\nabla u| < 1\}, \\ \Omega & = & \{(x,s) \in (M \setminus \partial M) \times (-1,1) \mid |\nabla u|^2 + s^2 < 1\}, \\ A & = & \{(x,s) \in \Omega \mid (\nabla^2 u)_x + s\,\sigma_x \text{ is positive semi-definite}\}, \end{array} \right\} \tag{56}$$

where σ is the second fundamental form of M. Clearly $x \in U$ provided that $(x,s) \in \Omega$. Consider also the $C^{1,\alpha}$ map $\Phi \colon M \times \mathbb{R} \to \mathbb{R}^{n+1}$ given by

$$\Phi(x,s) = (\nabla u)(x) - sN_x, \tag{57}$$

where N is the Gauss map of M associated to σ (i.e. $\sigma_x(v_1,v_2) = -\langle dN_x(v_1), v_2\rangle$ $\forall v_1, v_2 \in T_xM$ and $x \in M$). Clearly,

$$|\Phi(x,s)|^2 = |\nabla u|^2(x) + s^2. \tag{58}$$

Lemma 9.10 $\Phi(A) = \mathbb{B}^{n+1}$.

Proof. Since $A \subset \Omega$, then $\Phi(A) \subset \Phi(\Omega) \in \mathbb{B}^{n+1}$.

Let us prove the converse. Take $a \in \mathbb{B}^{n+1}$ and consider the restriction of the height function $\langle a, \cdot\rangle$ to M, whose gradient and hessian are a^T (the tangent part of a to M) and $\langle a, N\rangle\sigma$, respectively. Consider the $C^{2,\alpha}$ function on M given by $f = u - \langle a, \cdot\rangle$, where u is given by Lemma 9.9. We claim that f attains its minimum in $M \setminus \partial M$. To see this, observe that on ∂M we have

$$\frac{\partial f}{\partial \eta} = \frac{\partial u}{\partial \eta} - \langle a^T, \eta\rangle \overset{(54)}{=} 1 - \langle a, \eta\rangle \overset{(\star)}{\geq} 1 - \|a\| > 0,$$

where in $(\star)$ we have used the Schwarz inequality. As f is bounded from below in M (because M is compact), we deduce that f attains its minimum in $M \setminus \partial M$.

Take a point $\overline{x} \in M \setminus \partial M$ where f attains its minimum. Since $(\nabla f)(\overline{x}) = 0$ we have $(\nabla u)(\overline{x}) = a^T(\overline{x})$, from where

$$a = a^T(\overline{x}) + \langle a, N_{\overline{x}}\rangle N_{\overline{x}} = (\nabla u)(\overline{x}) - s\, N_{\overline{x}}, \tag{59}$$

where $s = -\langle a, N_{\overline{x}}\rangle \in (-1, 1)$. Since ∇u and N are orthogonal, (59) implies that

$$|\nabla u|^2(\overline{x}) + s^2 = |a|^2 < 1,$$

i.e., $(\overline{x}, s) \in \Omega$.

Next we will prove that $(\overline{x}, s) \in A$, which amounts to check that $(\nabla^2 u)_{\overline{x}} + s\,\sigma_{\overline{x}}$ is positive semi-definite. As f has a minimum at $\overline{x}$, the hessian $(\nabla^2 f)_{\overline{x}}$ is positive semi-definite. But

$$(\nabla^2 f)_{\overline{x}} = (\nabla^2 u)_{\overline{x}} - \langle a, N_{\overline{x}}\rangle\sigma_{\overline{x}} = (\nabla^2 u)_{\overline{x}} + s\,\sigma_{\overline{x}},$$

and thus, $(\overline{x}, s) \in A$.

Finally, $\Phi(\overline{x}, s) = (\nabla u)(\overline{x}) - s\, N_{\overline{x}} \overset{(59)}{=} a$, from where $a \in \Phi(A)$ and the lemma is proved. $\square$

In the above situation, the *Jacobian determinant* $\operatorname{Jac}\Phi$ of Φ is the determinant of the matrix of the differential $(d\Phi)_{(x,s)}$ in any orthonormal basis of $(T_xM) \times (T_sR) \equiv (T_xM) \times \mathbb{R}$, at every point $(x, s) \in M \times \mathbb{R}$.

Lemma 9.11 *Given* $(x, s) \in A$, $|\operatorname{Jac}\Phi|(x, s) = \det\big((\nabla^2 u)_x + s\,\sigma_x\big)$.

Proof. Take an orthonormal basis $e_1, \ldots, e_n$ of T_xM. Given $i = 1, \ldots, n$, let $\alpha_i = \alpha_i(t)$ be a smooth curve in M with $\alpha_i(0) = x$, $\alpha_i'(0) = e_i$. For $i, j = 1, \ldots, n$, consider the function $f_{ij} : \mathbb{R} \to \mathbb{R}$ given by

$$f_{ij}(t) = \langle \Phi(\alpha_i(t), s), e_j\rangle = \langle (\nabla u)_{\alpha_i(t)} - sN_{\alpha(t)}, e_j\rangle. \tag{60}$$

Thus,

$$\left.\frac{d}{dt}\right|_{t=0} f_{ij}(t) = \langle d\Phi_{(x,s)}(e_i, 0), e_j \rangle,$$

which corresponds to the (i, j)-entry of the Jacobian matrix of Φ. We compute:

$$\begin{aligned}\left.\frac{d}{dt}\right|_{t=0} f_{ij}(t) &= \left.\frac{d}{dt}\right|_{t=0} \langle (\nabla u)_{\alpha_i(t)} - sN_{\alpha_i(t)}, e_j\rangle = \langle \overline{\nabla}_{e_i} (\nabla u - sN), e_j \rangle \\ &= (\nabla^2 u)_x(e_i, e_j) + s\,\sigma_x(e_i, e_j).\end{aligned}$$

Furthermore,

$$d\Phi_{(x,s)}(0, 1) = \left.\frac{d}{dt}\right|_{t=0} \Phi(x, t + s) = \left.\frac{d}{dt}\right|_{t=0} ((\nabla u)_x - (t + s)N_x) = -N_x.$$

Hence, $|\mathrm{Jac}\Phi|(x, s) = |\det((\nabla^2 u)_x + s\,\sigma_x)|$. To finish the proof it suffices to remove the absolute value; this can be done since $(x, s) \in A$. $\square$

Lemma 9.12 *Given $(x, s) \in A$, $(\mathrm{Jac}\,\Phi)(x, s) \le 1$ with equality if and only if $(\nabla^2 u)_x + s\,\sigma_x = g_x$, the induced metric on M at x.*

Proof. Take $(x, s) \in A$. By (54), $(\Delta u)(x) = n - |H|(x)$ hence

$$(\Delta u)(x) + sH(x) \le (\Delta u)(x) + |H|(x) = n. \tag{61}$$

Recall that the arithmetic-geometric mean inequality ensures that

$$\prod_{i=1}^{n} \lambda_i \le \left(\frac{1}{n}\sum_{i=1}^{n} \lambda_i\right)^n,$$

where $\lambda_1, \ldots, \lambda_n \in [0, \infty)$, with equality if and only if all λ_i are equal. This implies that given a real-valued, symmetric, positive semi-definite matrix S of order n, it holds

$$\det S \le \left(\frac{1}{n}\mathrm{trace}(S)\right)^n,$$

with equality if and only if S is a multiple of the identity matrix. Applying this to $S = (\nabla^2 u)_x + s\,\sigma_x$ (which is positive semi-definite because $(x, s) \in A$), we have

$$\begin{aligned}(\mathrm{Jac}\,\Phi)(x, s) &\overset{\text{(Lemma 9.11)}}{=} \det\left((\nabla^2 u)_x + s\,\sigma_x\right) \le \left(\tfrac{1}{n}\mathrm{trace}[(\nabla^2 u)_x + s\,\sigma_x]\right)^n \\ &= \left(\tfrac{1}{n}[(\Delta u)(x) + s\,H(x)]\right)^n \overset{(61)}{\le} 1,\end{aligned}$$

as desired. This proves the first statement of Lemma 9.12.

Now suppose that $(\mathrm{Jac}\,\Phi)(x,s)=1$ and let us prove that $(\nabla^2 u)_x + s\,\sigma_x = g_x$. Since equality holds in the last displayed inequality, we have also equality in our use of the arithmetic-geometric mean inequality, hence $(\nabla^2 u)_x + s\,\sigma_x$ is a multiple of the induced metric g_x at x, say

$$(\nabla^2 u)_x + s\,\sigma_x = \mu\, g_x \tag{62}$$

for some $\mu \in \mathbb{R}$. The lemma will be proved if we check that $\mu = 1$. Observe that equality in (61) must also hold, hence $n = (\Delta u)(x) + sH(x) = \mathrm{trace}[(\nabla^2 u)_x + s\,\sigma_x] \overset{(62)}{=} \mathrm{trace}[\mu\, g_x] = n\mu$. $\square$

Proof. [of Theorem 9.2]
Write each $a \in \mathbb{B}^{n+1}$ as $a = (y,z)$ with $y \in \mathbb{B}^n$ and $z \in \mathbb{R}$. Thus, Fubini's theorem gives

$$\int_{\mathbb{B}^{n+1}} \frac{da}{\sqrt{1-|a|^2}} = \int_{\mathbb{B}^n} \left(\int_{-\sqrt{1-|y|^2}}^{\sqrt{1-|y|^2}} \frac{dz}{\sqrt{1-|y|^2-z^2}} \right) dy = \pi \int_{\mathbb{B}^n} dy = \pi \mathrm{Vol}(\mathbb{B}^n). \tag{63}$$

Using the area formula with the $C^{1,\alpha}$ map Φ on A, we have

$$\begin{aligned}
\int_{\mathbb{B}^{n+1}} \frac{da}{\sqrt{1-|a|^2}} &\overset{\text{(Lemma 9.10)}}{=} \int_{\Phi(A)} \frac{da}{\sqrt{1-|a|^2}} = \int_A \frac{|\mathrm{Jac}\,\Phi|(x,s)}{\sqrt{1-|\Phi(x,s)|^2}} dv_M \times ds \\
&\overset{(\star)}{\leq} \int_{x\in U} \left(\int_{-\sqrt{1-|\nabla u|^2(x)}}^{\sqrt{1-|\nabla u|^2(x)}} \frac{|\mathrm{Jac}\,\Phi|(x,s)\chi_A(x,s)}{\sqrt{1-|\Phi(x,s)|^2}} ds \right) dv_M \\
&\overset{\text{(Lemma 9.12)}}{\leq} \int_{x\in U} \left(\int_{-\sqrt{1-|\nabla u|^2(x)}}^{\sqrt{1-|\nabla u|^2(x)}} \frac{1}{\sqrt{1-|\Phi(x,s)|^2}} ds \right) dv_M \\
&\overset{(58)}{=} \int_{x\in U} \left(\int_{-\sqrt{1-|\nabla u|^2(x)}}^{\sqrt{1-|\nabla u|^2(x)}} \frac{1}{\sqrt{1-|\nabla u|^2(x)-s^2}} ds \right) dv_M \\
&= \pi \int_{x\in U} dv_M = \pi\, \mathrm{Vol}(U),
\end{aligned} \tag{64}$$

where χ_A is the characteristic function of A, and in $(\star)$ we have used that if $(x,s) \in A$ then $x \in U$ and the integrand is non-negative. From (63) and (64) we deduce that $\mathrm{Vol}(\mathbb{B}^n) \leq \mathrm{Vol}(U)$, and thus,

$$\mathrm{Vol}(\mathbb{B}^n) \leq \mathrm{Vol}(M). \tag{65}$$

Next we analyze the quotient that appears in Theorem 9.2. First observe that

$$\mathrm{Area}(\partial\mathbb{B}^n) = n\,\mathrm{Vol}(\mathbb{B}^n), \tag{66}$$

which follows from the divergence theorem in $\mathbb{B}^n$ applied to the vector field $X_p = p$, $p \in \mathbb{B}^n$. Hence,

$$\frac{\mathrm{Area}(\partial M) + \int_M |H|}{\mathrm{Area}(\partial\mathbb{B}^n)} \overset{\text{(Lemma 9.8)}}{=} \frac{n\,\mathrm{Vol}(M)}{\mathrm{Area}(\partial\mathbb{B}^n)} = \frac{\mathrm{Vol}(M)}{\mathrm{Vol}(\mathbb{B}^n)} \overset{(65)}{\geq} \left(\frac{\mathrm{Vol}(M)}{\mathrm{Vol}(\mathbb{B}^n)}\right)^{\frac{n-1}{n}},$$

and the inequality of Theorem 9.2 is proved.

Now suppose that M is a compact immersed hypersurface in $\mathbb{R}^{n+1}$ such that

$$\frac{\text{Area}(\partial M)+\int_M |H|}{\text{Area}(\partial \mathbb{B}^n)}=\left(\frac{\text{Vol}(M)}{\text{Vol}(\mathbb{B}^n)}\right)^{\frac{n-1}{n}}. \tag{67}$$

We claim that M is connected: if not, we decompose M in connected components, $M = M_1 \cup \ldots \cup M_k$ (finitely many because M is compact). Applying the already proven inequality of Theorem 9.2 to M_1 and $M' = M \setminus M_1$, we have the following contradiction with (67):

$$\begin{aligned}
\frac{\text{Area}(\partial M)+\int_M |H|}{\text{Area}(\partial \mathbb{B}^n)} &= \frac{\text{Area}(\partial M_1)+\int_{M_1} |H|}{\text{Area}(\partial \mathbb{B}^n)}+\frac{\text{Area}(\partial M')+\int_{M'} |H|}{\text{Area}(\partial \mathbb{B}^n)} \\
&\geq \left(\frac{\text{Vol}(M_1)}{\text{Vol}(\mathbb{B}^n)}\right)^{\frac{n-1}{n}}+\left(\frac{\text{Vol}(M')}{\text{Vol}(\mathbb{B}^n)}\right)^{\frac{n-1}{n}} \\
&= \frac{\text{Vol}(M_1)^{\frac{n-1}{n}}+\text{Vol}(M')^{\frac{n-1}{n}}}{\text{Vol}(\mathbb{B}^n)^{\frac{n-1}{n}}} > \left(\frac{\text{Vol}(M)}{\text{Vol}(\mathbb{B}^n)}\right)^{\frac{n-1}{n}},
\end{aligned}$$

where the last inequality uses that if $V_1, V_2 > 0$, then

$$V_1^{\frac{n-1}{n}}+V_2^{\frac{n-1}{n}} > (V_1+V_2)^{\frac{n-1}{n}}. \tag{68}$$

For the sake of completeness, we next prove (68): Consider the functions $f(x) = x^{\frac{n-1}{n}}$, $x \geq 0$; it is straightforward to see that $f''(x) < 0$ for each $x > 0$, hence $f'(x) > f'(V_1 + x)$ $\forall x > 0$, which implies that the function $x \in [0, \infty) \mapsto h(x) = f(V_1) + f(x) - f(V_1 + x)$, satisfies $h'(x) > 0$ $\forall x > 0$. Since $h(0) = 0$, then $h(x) > 0$ $\forall x > 0$. In particular, $h(V_2) > 0$ with is (68).

By scaling M in $\mathbb{R}^{n+1}$, we can assume that

$$\text{Area}(\partial M)+\int_M |H| = \text{Area}(\partial \mathbb{B}^n). \tag{69}$$

(Observe that this normalization is a priori, different from the one in Lemma 9.8). From (67) and (69) we have

$$\text{Vol}(M) = \text{Vol}(\mathbb{B}^n). \tag{70}$$

This equality and (66) imply that the normalization to have (69) is in fact the same that gives Lemma 9.8. This allows us to apply Lemma 9.9 and find a $C^{2,\alpha}$ function $u\colon M \to \mathbb{R}$ satisfying (54). Consider the sets U, Ω, A and the map $\Phi\colon M \times \mathbb{R} \to \mathbb{R}^{n+1}$ defined as in (56) and (57) respectively. Then, Lemmas 9.10, 9.11 and 9.12 hold, as well as the chain of inequalities in (64), from where we had deduced the inequality (65). Since the equality holds in (65) by (70), we deduce that all inequalities in (64) must be equalities. In particular, $\text{Vol}(U) = \text{Vol}(M)$ and $\text{Jac}\,\Phi\,\chi_A = 1$ a.e. in Ω. This implies that $\Omega \setminus A$ has measure zero, and $\text{Jac}\,\Phi = 1$ a.e. in Ω. By

Lemma 9.12, we have that $(\nabla^2 u)_x + s\,\sigma_x = g_x$ almost everywhere in Ω. As u is of class $C^{2,\alpha}$, we deduce that

$$(\nabla^2 u)_x + s\,\sigma_x = g_x \quad \text{at each } (x,s) \in \Omega. \tag{71}$$

Since the right-hand-side of (71) does not depend on s, we conclude that $\sigma_x = 0$ whenever $(x,s) \in \Omega$. Since this equation does not depend on s, we deduce that $\sigma_x = 0\ \forall x \in U$. A consequence of $\mathrm{Vol}(U) = \mathrm{Vol}(M)$ is that U is dense in M. Therefore, $\sigma = 0$ in M. This implies that M is contained in a hyperplane $\Pi \subset \mathbb{R}^n$. By (71) we have $(\nabla^2 u)_x = g_x$ for all $(x,s) \in \Omega$. Since this equation does not depend on s, we deduce that $(\nabla^2 u)_x = g_x\ \forall x \in U$, and by density, the same equality holds $\forall x \in M$. As M is contained in a hyperplane, the PDE $\nabla^2 u = g$ can be integrated giving $u(x) = \frac{1}{2}|x-p|^2 + c$ for some $p \in \Pi$ and $c \in \mathbb{R}$. Since $|\nabla u| < 1$ in U and U is dense in M, we deduce that $|\nabla u| \leq 1$ in M. This implies $M \subset \{x \in \Pi\ :\ |x-p| \leq 1\}$. This last containment is in fact an equality because $\mathrm{Vol}(M) = \mathrm{Vol}(\mathbb{B}^n)$. This finishes the proof of Theorem 9.2. □

10 The maximum principle for the mean curvature

Suppose $M_1, M_2 \subset \mathbb{R}^3$ are two surfaces (the argument that follows is purely local,, so we can assume they are embedded) and $p \in M_1 \cap M_2$ is an interior point such that $T_pM_1 = T_pM_2$. We orient M_1, M_2 by choosing Gauss maps N_1, N_2 such that $N_1(p) = N_2(p)$, and let H_1, H_2 be the mean curvature functions of M_1, M_2 with respect to N_1, N_2, respectively (H_1, H_2 are not necessarily constant).

We will say that $M_1 \leq M_2$ around p if $u_1 \leq u_2$ in a neighborhood $\Omega' \subset \Omega$ of the origin. The same notion can be defined if p is a boundary point of both surfaces and we additionally assume $T_p\partial M_1 = T_p\partial M_2$ as oriented vector spaces (in this case, both Ω and Ω' are neighborhoods of $(0,0)$ in $\{(x,y) \in \mathbb{R}^2 \mid y \geq 0\}$).

The maximum principle for the mean curvature is the following local property:

Theorem 10.1 *Let $M_1, M_2 \subset \mathbb{R}^3$ be two surfaces.*

1. *(Interior maximum principle). Let $p \in M_1 \cap M_2$ be an interior point such that $T_pM_1 = T_pM_2$. Take Gauss maps N_1, N_2 on M_1, M_2 so that $N_1(p) = N_2(p)$. Suppose that $M_1 \leq M_2$ in a neighborhood of p, and that the mean curvatures H_1, H_2 with respect to N_1, N_2 satisfy $H_1 \geq H_2$ (meaning that $H_1(x,y,u_1(x,y)) \geq H_2(x,y,u_2(x,y))$ for each $(x,y) \in \Omega'$ with the notation above). Then, there exists a neighborhood O of p in $\mathbb{R}^3$ such that $M_1 \cap O = M_2 \cap O$.*

2. *(Boundary maximum principle). Item 1 holds if $p \in \partial M_1 \cap \partial M_2$ and $T_pM_1 = T_pM_2$, $T_p\partial M_1 = T_p\partial M_2$ as oriented vector spaces.*

A detailed proof of the above theorem can be found in the website
`http://wpd.ugr.es/ jperez/wordpress/wp-content/uploads/todo-2.pdf`

However, we will not follow here the classic analytical PDE approach that one can find in the above URL or in the book by Gilbarg and Trudinger [8]. Instead, we will adopt a more geometric viewpoint; the price we will pay for this is that we will restrict to minimal surfaces, instead of obtaining the maximum principle under more general conditions as in Theorem 10.1 (although the calculations that follow can also be carried out for surfaces of constant mean curvature $H \neq 0$ with certain modifications).

If two $M_1, M_2 \subset \mathbb{R}^3$ intersect transversally, around every intersection point p the local structure of the union is the same as for two planes intersecting along a line; in particular, $M_1 \cap M_2$ consists locally around p of an embedded curve (which is analytic if both surfaces are) and the angle $\sphericalangle(M_1, M_2)$ along this curve is $\geq \varepsilon$ for some $\varepsilon > 0$.

We are now wondering what the intersection of two minimal surfaces M_1, M_2 is like, locally around a point $p \in \text{Int}(M_1) \cap \text{Int}(M_2)$ at which M_1, M_2 intersect tangentially ($T_pM_1 = T_pM_2$). Intuition tells us that since minimal surfaces are locally given as the graph of a 'harmonic' function[4], we will be able to model the 'difference' of M_1 and M_2 by studying the zeros of a function whose principal term is $\text{Re}(z^k)$ for a certain $k \in \mathbb{N}$, $k \geq 2$ (after a change of coordinates centered at p). This zero set has the structure of an equiangular set of segments crossing at the origin, as shown in Figure 17. Consequently, we deduce that M_1 cannot lie at one side of M_2 in

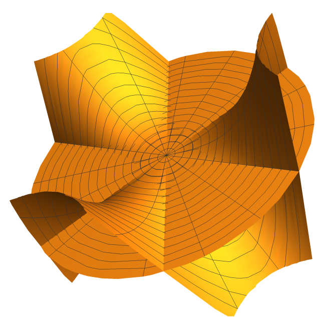


Figure 17: The zero set of $z \mapsto \text{Re}(z^3)$ forms an equiangular system.

any neighborhood of p, which is another way of stating the content of the maximum principle. Next we will develop this idea rigorously.

After possibly a translation and rotation in $\mathbb{R}^3$, we can assume

$$p = \vec{0} \in \mathbb{R}^3, \quad \text{and } T_pM_1 = T_pM_2 = \{z = 0\}.$$

Each surface M_i can be written locally around p as the graph of a smooth function $u_i \colon \mathbb{D}(c) \subset \mathbb{R}^2 \to \mathbb{R}$, for some $c \in (0, 1)$. Let us call $u = u_1 - u_2$. Clearly, $M_1 \cap M_2 = u^{-1}(\{0\})$ locally

[4] We use quotation marks because this property is not actually true: the component functions of the immersion are harmonic, but function that expresses the surface locally as a graph satisfies the minimal surface equation 6).

around p. Furthermore,

$$u_1(0,0) = u_2(0,0) = 0, \quad \nabla u_2(0,0) = \nabla u_2(0,0) = (0,0).$$

Thus, the Taylor expansion of u around $(0,0)$ is of the form:

$$u(x,y) = \sum_{n=2}^{\infty}\sum_{i=0}^{n} \frac{n!}{i!(n-i)!}\frac{\partial^n u}{\partial x^i \partial y^{n-i}}(0,0)x^i y^{n-i} = \sum_{n=k}^{\infty} P_n(x,y), \tag{72}$$

where every $P_n(x,y)$ is a homogeneous polynomial of degree n. By analyticity, either $u \equiv 0$ (i.e. M_1 and M_2 coincide in a neighborhood of p, and so they globally coincide) or the above sum has a first term $P_k(x,y)$ which is non-trivial, of degree $k \geq 2$. From now on we will assume that this last possibility occurs.

Lemma 10.2 *In the above situation, $P_k(x,y)$ is a harmonic polynomial.*

Proof. Taking derivatives term by term in (72) we have $\Delta u = \sum_{n=k}^{\infty} \Delta P_n$, and we want to show that $\Delta P_k = 0$. Reasoning by contradiction, suppose $\Delta P_k \neq 0$. Therefore, ΔP_k is a non-trivial homogeneous polynomial of degree $k-2$. As ΔP_{k+1}, $\Delta P_{k+2}, \ldots$ are homogeneous polynomials of degree $k-1$, $k, \ldots$ (some of them could be identically zero, and in that case they will not affect the argument that follows), the non-trivial term in Δu with lowest degree has degree $k-2$ (there are no cancellations of distinct ΔP_h because they are homogeneous polynomials of different degrees). If we check that

$$\Delta u = A(x,y)u_{xx} + B(x,y)u_{yy} + C(x,y)u_{xy} + D(x,y)u_x + E(x,y)u_y, \tag{73}$$

where each one of the de los five summands above (Au_{xx}, etc) is an analytic function in a neighborhood of $(0,0)$ with a first non-trivial term of degree $\geq k$, then we will have that the term in Δu with lowest degree has at least degree k, which is a contradiction. Hence everything reduces to proving the decomposition (73) with the desired degrees in $Au_{xx}, Bu_{yy}, Cu_{xy}, Du_x, Eu_y$.

As M_i is minimal, we have

$$[1 + (u_i)_y^2](u_i)_{xx} - 2(u_i)_x(u_i)_y(u_i)_{xy} + [1 + (u_i)_x^2](u_i)_{yy} = 0.$$

Therefore,

$$\Delta u_i = -(u_i)_y^2(u_i)_{xx} - (u_i)_x^2(u_i)_{yy} + 2(u_i)_x(u_i)_y(u_i)_{xy}.$$

And

$$\Delta u = \Delta u_1 - \Delta u_2 = -(u_1)_y^2(u_1)_{xx} - (u_1)_x^2(u_1)_{yy} + 2(u_1)_x(u_1)_y(u_1)_{xy}$$
$$+(u_2)_y^2(u_2)_{xx} + (u_2)_x^2(u_2)_{yy} + 2(u_2)_x(u_2)_y(u_2)_{xy}.$$

We maintain the three first terms unchanged. In the fourth one we use that $(u_2)_{xx} = (u_1)_{xx} - u_{xx}$, in the fifth one we replace $(u_2)_{yy} = (u_1)_{yy} - u_{yy}$, and in the sixth one we use that $(u_2)_{xy} = (u_1)_{xy} - u_{xy}$. After grouping the terms by the second order derivatives of u and u_1, we obtain

$$\Delta u = -(u_2)_y^2 u_{xx} - (u_2)_x^2 u_{yy} + 2(u_2)_x(u_2)_y u_{xy}$$

$$+ \left[(u_2)_y^2 - (u_1)_y^2\right](u_1)_{xx} + \left[(u_2)_x^2 - (u_1)_x^2\right](u_1)_{yy} + 2\left[(u_1)_x(u_1)_y - (u_2)_x(u_2)_y\right](u_1)_{xy}.$$

The brackets of the fourth and fifth summands can be written as 'sum times difference', and this difference can be written in terms of u. In the bracket of the sixth summand we add and subtract $(u_1)_x(u_2)_y$ in order to transform this bracket in $(u_1)_x u_y + u_x(u_2)_y$. Writting all together and grouping by $u_{xx}, u_{x,y}, u_x, u_y$ we get the expression (73) where

$$A = -(u_2)_y^2, \quad B = -(u_2)_x^2, \quad C = 2(u_2)_x(u_2)_y,$$

$$D = -\left[(u_1)_x + (u_2)_x\right](u_1)_{yy} + 2(u_2)_y(u_1)_{xy}, \quad E = -\left[(u_1)_y + (u_2)_y\right](u_1)_{xx} + 2(u_1)_x(u_1)_{xy}.$$

A, B, C, D, E are analytic functions in a neighborhood of $(0,0)$, because the surfaces M_1, M_2 are analytic. Thus, each one of the de five summands in (73) is an analytic function in a neighborhood of $(0,0)$. We next analyze the order of the zero of Au_{xx} at the origin: Since ∇u_2 vanishes at $(0,0)$, $(u_2)_y^2$ has a zero at least of order 2 at the origin. By (72), u has a zero of order k at $(0,0)$, hence u_{xx} vanishes at least to order $k-2$ at the origin. This Au_{xx} vanishes at $(0,0)$ at least to order k. The other summands of (73) can be analyzed in an analogous way. □

Theorem 10.3 (Maximum principle for minimal surfaces)
Let $M_1, M_2 \subset \mathbb{R}^3$ two distinct minimal surfaces, and $p \in Int(M_1) \cap Int(M_2)$ such that $T_pM_1 = T_pM_2$. Then, $M_1 \cap M_2$ consists locally around p in the zero set of an analytic function $u(x,y)$ whose first term $h(x,y)$ in the Taylor expansion is a harmonic function with a zero of finite order $k \geq 2$. Therefore:

(1) After possibly a biholomorphism between open subsets of $\mathbb{C}$ that preserves the origin, we can write $h(z) = \mathrm{Re}(z^k)$.

(2) $M_1 \cap M_2$ consists of an equiangular system of k analytic curves crossing at p with an angle of π/k, and on each component of the complement of the union of these curves in a neighborhood of p, the surfaces M_1, M_2 lie at one side of each other, alternating this ordering when changing between adjacent sectors (as in Figure 17).

Proof. We first normalize as explained in the paragraph just before Lemma 10.2. By (72), $M_1 \cap M_2$ consists locally around p of the zero set of the function $u = \sum_{n=k}^{\infty} P_n(x,y)$, and $h := P_k$ is harmonic by Lemma 10.2. Thus, h can be written as $h(x,y) = \mathrm{Re}(z^k f(z))$ for certain holomorphic function $f(z)$ $(z = x + iy)$ with $f(0) \neq 0$. This last condition implies the existence of a k-th holomorphic root $f^{1/k}$ of f, defined in a neighborhood of zero. Hence, $\xi = \xi(z) := zf^{1/k}(z)$ is a biholomorphism around 0, and item (1) of the theorem is proved. Item (2) is a direct consequence of item (1). $\square$

We generalize (without proof, see the discussion at the beginning of this section) the previous theorem for hypersurfaces of constant mean curvature in an arbitrary Riemannian manifold:

Theorem 10.4 (Interior maximum principle for CMC hypersurfaces) *Let Σ_1^n, Σ_2^n two distinct hypersurfaces in an analytic Riemannian manifold (M^{n+1}, g), and $p \in Int(\Sigma_1) \cap Int(\Sigma_2)$ such that $T_p\Sigma_1 = T_p\Sigma_2$. If the mean curvature vectors $\vec{H}_1, \vec{H}_2$ of Σ_1, Σ_2 satisfy $\|\vec{H}_1\| \equiv \|\vec{H}_2\| \equiv c \in [0,\infty)$ and $\vec{H}_1(p) = \vec{H}_2(p)$[5], then $\Sigma_1 \cap \Sigma_2$ consists locally around p of the zero set of an analytic function of n variables, whose first order in the Taylor expansion around the origin in $\mathbb{R}^n$ is a harmonic function of n variables with a zero of finite order $k \geq 2$ at the origin. In particular, Σ_1 cannot lie at one side of Σ_2 in any neighborhood of p.*

We will state, also without proof, the boundary maximum principle (also known as the Hopf maximum principle):

Theorem 10.5 (Boundary maximum principle for CMC hypersurfaces) *Let Σ_1^n, Σ_2^n two distinct hypersurfaces with smooth boundaries in an analytic Riemannian manifold (M^{n+1}, g), and $p \in \partial\Sigma_1 \cap \partial\Sigma_2$ such that $T_p\Sigma_1 = T_p\Sigma_2$ and $T_p\partial\Sigma_1 = T_p\partial\Sigma_2$. If the mean curvature vectors $\vec{H}_1, \vec{H}_2$ of Σ_1, Σ_2 satisfy $\|\vec{H}_1\| \equiv \|\vec{H}_2\| \equiv c \in [0,\infty)$ and $\vec{H}_1(p) = \vec{H}_2(p)$, and with this orientation Σ_1, Σ_2 are local graphs over the same domain with boundary[6] of the common tangent hyperplane, then Σ_1 cannot lie at one side of Σ_2 in any neighborhood of p.*

We will finish this section with two consequences of the maximum principle.

Theorem 10.6 *Let Z be a complete Killing field in a Riemannian manifold (M^3, g), whose integral curves are non-compact. Let $\Sigma \subset M$ be a compact minimal surface with boundary $\partial\Sigma \neq \varnothing$ such that Σ intersects each integral curve of Z at most at one point[7]. Let*

$$\mathcal{C} = \bigcup_{t\in\mathbb{R}} \varphi_t(\Sigma) \subset M$$

be the 'cylinder' obtained by moving Σ by the isometries in the 1-parameter subgroup $\{\varphi_t\}_{t\in\mathbb{R}}$ associated to Z. Then, Σ is the unique compact minimal surface in $\mathcal{C}$ with boundary $\partial\Sigma$.

[5] In the case $c = 0$ this condition is void.

[6] This condition rules out that Σ_1 is graphical over a halfball in $T_p\Sigma_1$ and Σ_2 is graphical over the opposite half-ball, a situation in which the graphing functions cannot be subtracted. This is exactly the same condition expressed by the equality '$T_pM_1 = T_pM_2$,$T_p\partial M_1 = T_p\partial M_2$ as oriented vector spaces' in item 2 of Theorem 10.1.

[7] In this case we say that Σ is a *graph in the direction of Z*.

Proof. Suppose that $\Sigma_1 \subset \mathcal{C}$ is a compact minimal surface with $\partial\Sigma_1 = \partial\Sigma$. Since the integral curves of Z are non-compact and Σ, Σ_1 are compact, we can find $T > 0$ such that $\varphi_T(\Sigma) \cap \Sigma_1 = \varnothing$ (hence the ambient distance between both surfaces is positive). Move continuously $\varphi_t(\Sigma)$ a Σ_1 by decreasing t from T until we find a first contact point between both surfaces; in other words, let

$$t_0 = \inf\{t \in (0, T) \mid \varphi_t(\Sigma) \cap \Sigma_1 = \varnothing\} \in [0, T).$$

Since Σ, Σ_1 are compact, there exists $p \in \varphi_{t_0}(\Sigma) \cap \Sigma_1$ (first contact point between both surfaces). p is interior to $\varphi_{t_0}(\Sigma)$ and to Σ_1, because $\partial\Sigma = \partial\Sigma_1$ is a graph in the direction of Z. Locally around p, $\varphi_{t_0}(\Sigma)$ lies at one side of Σ_1 (because p is the first contact point), hence the maximum principle implies that $\varphi_{t_0}(\Sigma) = \Sigma_1$. This gives that $t_0 = 0$ (just compare the boundaries) and thus, $\Sigma = \Sigma_1$. $\square$

Remark 10.7 We can generalize Theorem 10.6 to the case of non-zero CMC: Under the same hypotheses on (M^3, g) and Z, assume that $\Sigma \subset M$ is a compact surface with boundary $\partial\Sigma \neq \varnothing$, such that $\|\vec{H}_\Sigma\| \equiv c > 0$ and suppose that Σ is a graph in the direction of Z. Σ will divide $\mathcal{C}$ into two components. Let us call $\mathcal{C}^+$ to the component of $\mathcal{C} \setminus \Sigma$ that $\vec{H}_\Sigma$ points to. If $\Sigma_1 \subset \mathcal{C}$ is a compact surface with $\|\vec{H}_\Sigma\| \equiv c$, $\partial\Sigma = \partial\Sigma_1$ and $\vec{H}_{\Sigma_1}$ points towards the end[8] of $\mathcal{C}^+$, then $\Sigma = \Sigma_1$ (for the proof, simply repeat the above arguments by moving Σ 'to the right' in the sense of Figure 18, till the 'translated' copy does not intersect Σ_1, and then move back continuously this translated copy 'to the left' till find a first contact point with Σ_1).

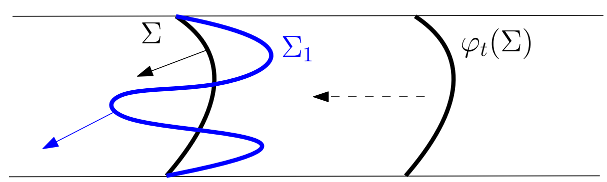


Figure 18: Theorem 10.6 is still true for non-zero CMC.

Theorem 10.8 (Radó) *Let $\Gamma \subset \mathbb{R}^3$ be a Jordan curve that admits a 1-1 projection onto a convex planar curve Γ'. Then, there exists a unique compact minimal surface $M \subset \mathbb{R}^3$ with $\partial M = \Gamma$ (in particular, M is the Douglas-Radó solution for the Plateau problem for Γ). Furthermore, the interior of M is a smooth graph over the convex planar domain bounded by Γ'.*

Proof. [Antonio Ros, Gaceta de la RSME, 2000]
By Theorem 8.1, there exists a compact minimal surface $M \subset \mathbb{R}^3$ with $\partial M = \Gamma$ (in fact, we can take M as a minimizer of area for its boundary, although we will not use this). After possibly

[8]Observe that the three-manifold $\mathcal{C}^+$ has a unique end.

a rigid motion in $\mathbb{R}^3$, we can assume that $\Gamma' \subset \{z = 0\}$. Let $\Omega \subset \mathbb{R}^2 \times \{0\}$ be the convex planar domain bounded by Γ'.

(1) $M \subset \overline{\Omega} \times \mathbb{R}$: This can be deduced from either the maximum principle applied to M and to vertical planes, or the convex hull property (Corollary 4.7, note that since Ω is convex, $\Omega \times \mathbb{R}$ is also convex). We leave the details to the reader.

(2) $M \setminus \Gamma \subset \Omega \times \mathbb{R}$: This is also a consequence of the maximum principle applied to M and to vertical planes.

(3) $M \setminus \Gamma$ is a graph over Ω: Arguing by contradiction, suppose that there exist points $p, q \in M \setminus \Gamma$ with the same vertical projection over Ω and with $x_3(p) < x_3(q)$. Traslate M vertically upwards until $M(t_0) = (M + t_0(0,0,1)) \cap M = \varnothing$ for some $t_0 > 0$ (t_0 exists because M is compact). As $p + t_0(0,0,1) \in M(t_0)$ is strictly above $q \in M$, the first contact point between a translated image of M and M will occur for some $t_1 \in (0, t_0)$. This first contact point cannot lie in the boundary of any of the two surfaces, because $M \setminus \Gamma \subset \Omega \times \mathbb{R}$, and both $\partial M = \Gamma$, $\partial M(t_1) = \Gamma + t_1(0,0,1)$ are graphs over Γ'. Hence, the first contact point is interior to both surfaces, which contradicts the interior maximum principle.

(4) M is the graph of a *smooth* function over Ω: Again by contradiction, let us assume there exists $p \in M \setminus \Gamma$ such that T_pM is vertical. Let Π be the horizontal plane that passes through p. Take an open neighborhood U of p in M small enough so that Π divides U into two connected components U_1, U_2, each one being a surface with boundary. These boundaries share a common curve c lying in Π, that passes through p. Let us denote by U_1^* the image of U_1 by the symmetry with respect to Π. Then, U_1^* and U_2 lie at the same side of Π and they can only intersect along c (otherwise there would exist points in $U_1 \setminus \Pi$ and $U_2 \setminus \Pi$ with the same vertical projection, which is impossible). Therefore, U_1^* lies at one side of U_2 around p. Since $T_pU_1 = T_pU_2$ and $T_p(\partial U_1) = T_p(\partial U_2)$ as oriented vector spaces, we contradict the boundary maximum principle.

(5) M is the unique compact minimal surface with boundary Γ (in particular, M minimizes area among all surfaces with boundary Γ): every compact minimal surface M' with $\partial M' = \Gamma$ satisfies items (1) and (2) of this proof, hence uniqueness follows from Theorem 10.6. $\square$

11 The Douglas criterion for the Plateau problem

Consider two coaxial circles Γ_1, Γ_2 of the same radius $R > 0$, at distance $d > 0$ apart.

(1) If d/R is sufficiently small, there exists a connected compact minimal surface with boundary $\Gamma_1 \cup \Gamma_2$ (a piece of a catenoid).

(2) If d/R is large enough, there are no connected compact minimal surfaces with with boundary $\Gamma_1 \cup \Gamma_2$: to see this, suppose that $M \subset \mathbb{R}^3$ is a connected compact minimal surface with $\partial M = \Gamma_1 \cup \Gamma_2$. After possibly a rigid motion, we can assume that Γ_1, Γ_2 are contained in horizontal planes and their centers are $(0, 0, \pm d/2)$. By the convex hull property, $M \subset \{x_1^2 + x_2^2 \leq R\} \cap \{|x_3| \leq d/2\}$. Now put two coaxial circles C_1, C_2 of radius $r > 0$ to be determined, centered at $(\pm(R + \varepsilon), 0, 0)$ (for a given $\varepsilon > 0$) and contained in planes orthogonal to the x-axis. If r/R is large enough, there exists a compact piece Σ of a catenoid with boundary $\partial\Sigma = C_1 \cup C_2$. From now on we will suppose $d > 2r$ (see Figure 19).

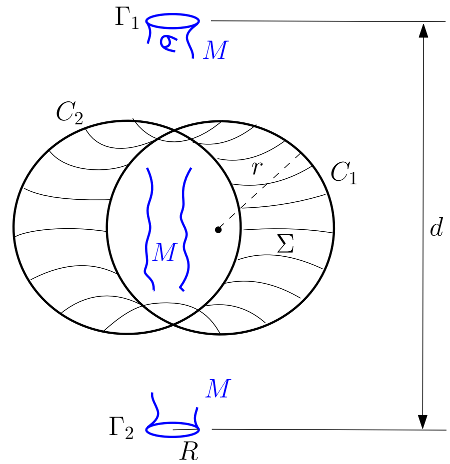


Figure 19: If $R \ll r < \frac{d}{2}$, then we can put a catenoid Σ between Γ_1 and Γ_2.

Now translate Σ by a vector $(0, a, 0)$ with $a > 0$ large enough so that $\Sigma + (0, a, 0)$ does not intersect M (by compactness of M and Σ, this a exists). Also, we can take a large enough so that $\Sigma - (0, a, 0)$ does not intersect M. If we move continuously $\Sigma + (0, t, 0)$ with $t \in [-a, a]$ we will find a first interior contact point between M and a translated image of Σ, which contradicts the interior maximum principle.

¿How can we quantify the dichotomy between (1) and (2)?

Consider the catenoid $C = \{(x, y, z) \mid \cosh^2 z = x^2 + y^2\}$ and its image

$$C_a = \{a^2 \cosh^2(z/a) = x^2 + y^2\}$$

by the homothety of factor $a > 0$. As $a \to 0$, C_a converges to the plane $\{z = 0\}$ with multiplicity 2 away from the origin $\vec{0} \in \mathbb{R}^3$ (we will see how to study rigorously this convergence in Section 12) and as $\lambda \to \infty$, C_a leaves every compact subset of $\mathbb{R}^3$.

Given the vertical one $\mathcal{C}_\lambda = \{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 = \lambda^2 z^2\}$ con $\lambda > 0$, we will denote by

$$\mathrm{Int}(\mathcal{C}_\lambda) = \{(x, y, z \in \mathbb{R}^3 \mid x^2 + y^2 < \lambda^2 z^2\}, \quad \mathrm{Ext}(\mathcal{C}_\lambda) = \{(x, y, z \in \mathbb{R}^3 \mid x^2 + y^2 > \lambda^2 z^2\}.$$

Lemma 11.1 *There exists a cone $\mathcal{C}_\lambda$ that is disjoint from every C_a, $\forall a > 0$.*

Proof. Both $\mathcal{C}_\lambda$ and C_a are surfaces of revolution around the z-axis, thus we can reduce ourselves to analyze the generating curves. This reduces the proof to finding a straight line $y = (\sinh \tau_0)z$ (here $\tau_0 > 0$ is a constant to be determined) that is disjoint from every catenary $y = a \cosh(z/a)$, $a > 0$, see Figure 20.

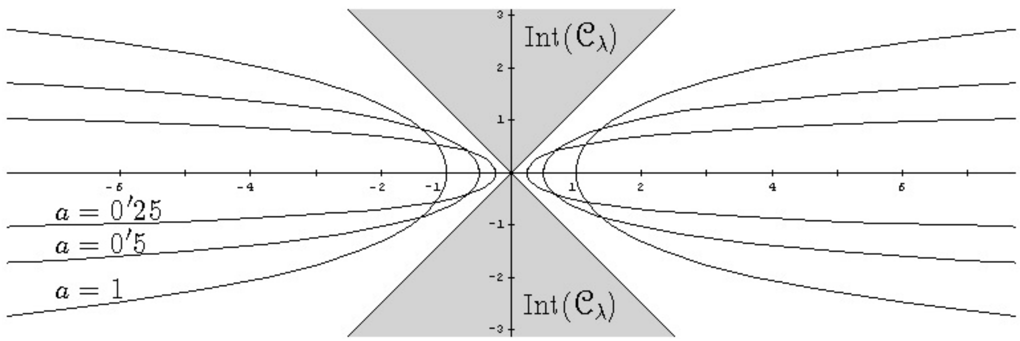


Figure 20: Different catenoids C_a for $a = 1, 0'5$ and $0'25$.

Equivalently, we must find $\tau_0 > 0$ such that $a \cosh(z/a) - (\sinh \tau_0)z \geq 0$ for all $z \in \mathbb{R}$ and $a > 0$ (we have replaced > 0 by ≥ 0, which suffices by increasing slightly the slope of the straight line $y = (\sinh z_0)z$). Consider the function $f(z) = a \cosh(z/a) - (\sinh \tau_0)z$, $z \in \mathbb{R}$. Thus,

$$f'(z) = \sinh(z/a) - \sinh \tau_0, \qquad f''(z) = \frac{1}{a} \cosh(z/a) > 0.$$

hence the unique critical point of f is $z = a\tau_0$, which is a minimum of f. If we find $\tau_0 > 0$ such that $f(a\tau_0) = 0$ for all $a \in \mathbb{R}$, then we will have $f(z) \geq 0$ $\forall z \in \mathbb{R}, \forall a > 0$. But $f(a\tau_0) = a \cosh \tau_0 - a\tau_0 \sinh \tau_0$, that vanishes $\forall a > 0$ if and only if the function $g(\tau) := \cosh \tau - \tau \sinh \tau$ vanishes at τ_0. Observe that

$$g(1) = \cosh 1 = e^{-1} > 0, \quad \lim_{\tau \to +\infty} g(\tau) = -\infty, \quad g'(\tau) = -\tau \cosh \tau < 0 \quad (\text{para } \tau > 0).$$

These three properties imply that there exists a unique $\tau_0 > 1$ such that $g(\tau_0) = 0$. □

Now we can quantify the dichotomy (1)-(2) that appears at the beginning of this section with the following non-existence result.

Theorem 11.2 (Cone Theorem, Hildebrandt [9])
If $\mathcal{C}_\lambda$ is the cone obtained in Lemma 11.1, then there are no connected, compact minimal surfaces $M \subset \mathbb{R}^3$ such that $\partial M \subset \text{Int}(\mathcal{C})$, $\partial M \cap \text{Int}(\mathcal{C})^+ \neq \varnothing$ and $\partial M \cap \text{Int}(\mathcal{C})^- \neq \varnothing$ (here $^+,^-$ denote the portion of the corresponding set above and below the plane $\{z = 0\}$).

Proof. Arguing by contradiction, if such an M exists, then we will find a contradiction applying the interior maximum principle to M and the catenoids C_a with $a > 0$. $\Box$

Next we will give a positive result about existence of minimal surfaces, also related to the dichotomy (1)-(2) of the beginning of this section. Let $\Gamma_1, \Gamma_2 \subset \mathbb{R}^n$ be two disjoint Jordan curves. Let us call $\Gamma = \Gamma_1 \cup \Gamma_2$ and

$$X_\Gamma = \left\{ \psi \colon \mathbb{S}^1 \times [0,1] \to \mathbb{R}^n \mid \psi \text{ is Lipschitz, } \begin{array}{l} \psi|_{\mathbb{S}^1 \times \{0\}} \text{ parameterizes 'monotonically' } \Gamma_1, \text{ and} \\ \psi|_{\mathbb{S}^1 \times \{1\}} \text{ parameterizes 'monotonically' } \Gamma_2 \end{array} \right\}.$$

(here, 'monotonically' means in the sense of the proof of Theorem 8.1). Consider the area functional $A \colon X_\Gamma \to [0, \infty)$ defined as in (35), and the infimum $a(\Gamma)$ of the areas of maps in X_Γ, defined as in (36) with our new family X_Γ of Lipschitz annuli. Consider for $i = 1, 2$ the corresponding infimum $a(\Gamma_i)$ for the area of Lipschitz disks with boundary Γ_i (i.e., $a(\Gamma_i)$ is defined as in (36) in the proof of the Douglas-Radó Theorem).

Theorem 11.3 (Douglas' criterion) *Let $\Gamma_1, \Gamma_2 \subset \mathbb{R}^n$ be two disjoint Jordan curves. If there exists a (not necessarily minimal) compact annulus $\Sigma \subset \mathbb{R}^n$ with $\partial \Sigma = \Gamma_1 \cup \Gamma_2$ such that $A(\Sigma) < a(\Gamma_1) + a(\Gamma_2)$, then there exists a compact minimal annulus $M \subset \mathbb{R}^n$ with $\partial M = \Gamma_1 \cup \Gamma_2$ that minimizes area in X_Γ, i.e., $A(\Sigma) = a(\Gamma)$.*

Remark 11.4 The Douglas' criterion is valid in any Riemannian manifold.

To prove the Douglas' criterion, take $\{\psi_k\}_k \subset X_\Gamma$ with $A(\psi_k) \searrow a(\Gamma)$ (a minimizing sequence). We will follow as much as possible the proof of the Douglas-Radó Theorem. A priori, every ψ_k is only Lipschitz, but we can assume it to be of class C^2 by a density argument. Given $r > 0$, define

$$\psi_{k,r} \colon \mathbb{S}^1 \times [0,1] \to \mathbb{R}^{n+2}, \quad \psi_{k,r}(x,y) = (\psi_k(x,y), rx, ry).$$

$\psi_{k,r}$ is injective, C^2 and both $A(\psi_{k,r})$, $E(\psi_{k,r})$ (energy) depend continuously on r. In particular, taking $\{r_k\}_k \to 0$ we can assume that ψ_{k,r_k} has area and energy arbitrarily close to those of ψ_k.

Since $\psi_{k,r_k}(A)$ is a C^2 embedded disk in $\mathbb{R}^{n+2}$, the induced metric by ψ_{k,r_k} in A is Riemannian, hence admits isothermal coordinates. This means that we can compose each ψ_{k,r_k} with a conformal diffeomorphism, obtaining a new map $\overline{\psi}_k \colon \mathbb{S}^1 \times [0, T_k] \to \mathbb{R}^{n+2}$ with energy equal to twice its area ($\overline{\psi}_k(A)$ is conformally equivalent to $\mathbb{S}^1 \times [0, T_k]$). As $A(\phi_k) = A(\psi_{k,r_k})$ (because area is invariant under reparameterizations), we obtain that $\{\phi_k\}_k$ is an A-minimizing and E-minimizing sequence (for this step to hold we must prove that the infimum of energies of maps in X_Γ coincides with twice the infimum of areas of maps in in X_Γ, which follows from a straightforward adaptation of the proof of Proposition 8.6). After projecting again $\mathbb{R}^{n+2}$ to $\mathbb{R}^n$ forgetting the last two components, we get for each $k \in \mathbb{N}$ a Lipschitz map, denoted again by $\psi_k \colon \mathbb{S}^1 \times [0, T_k] \to \mathbb{R}^n$, such that $\psi_k|_{\mathbb{S}^1 \times \{0\}}$ is a 'monotone' parameterization of Γ_1

and $\psi_k|_{\mathbb{S}^1\times\{T_k\}}$ is a 'monotone' parameterization of Γ_2, and $\{\psi_k\}_k$ is an A-minimizing and E-minimizing sequence in X_Γ.

Now replace each ψ_k by the unique harmonic map $f_k\colon \mathbb{S}^1 \times [0, T_k] \to \mathbb{R}^n$ with the same boundary values as ψ_k. Thus, $A(f_k)$ converges as $k \to \infty$ to $a(\Gamma)$ and $E(f_k)$ converges to twice the infimum of the energies of maps in X_Γ. The last step of the proof consists of finding a convergent subsequence of $\{f_k\}_k$. The reasoning in our current situation is more delicate than in the proof of the Douglas-Radó Theorem, because in that case we dealt with disks (with fixed conformal structure) and a three-point normalization condition in the boundary allowed us to extract a convergent subsequence. We will now to work some more to control the conformal structures of the annuli.

Lemma 11.5 *$\{T_k\}_k$ is bounded from above.*

Proof. Take $t_0 \in [0, T_k]$ such that $C_k := f_k(\mathbb{S}^1 \times \{t_0\})$ is a Jordan curve in $\mathbb{R}^n$ (for example, $t_0 = 0$). Let $D_k \subset \mathbb{R}^n$ be a Douglas-Radó solution disk to the Plateau problem with boundary C_k (see Figure 21).

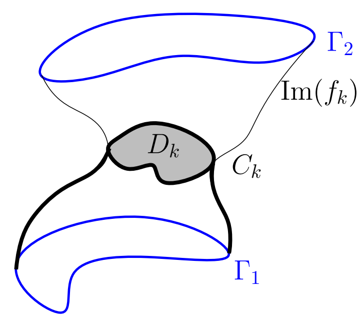


Figure 21: D_k is a disk with boundary C_k, and it is a Douglas-Radó solution to the Plateau problem with this boundary.

Since $f_k(\mathbb{S}^1 \times [0, t_0]) \cup D_k$ is a Lipschitz disk with boundary Γ_1, then

$$a(\Gamma_1) \le A\left[f_k(\mathbb{S}^1 \times [0, t_0])\right] + A(D_k).$$

Analogously,

$$a(\Gamma_2) \le A\left[f_k(\mathbb{S}^1 \times [t_0, T_k])\right] + A(D_k).$$

Hence,

$$a(\Gamma_1) + a(\Gamma_2) \le A(f_k) + 2A(D_k) \overset{(*)}{\le} A(f_k) + \frac{1}{2\pi} L(C_k)^2,$$

where in $(*)$ we have used the isoperimetric inequality for minimal disks (Theorem 9.5). Taking $k \to \infty$,

$$a(\Gamma_1) + a(\Gamma_2) \leq a(\Gamma) + \frac{1}{2\pi}\underline{\lim} L(C_k)^2,$$

By hypothesis, $a(\Gamma_1) + a(\Gamma_2) - a(\Gamma) > 0$, hence we conclude that

$$L(C_k)^2 \geq \varepsilon > 0 \quad \text{for some } \varepsilon > 0. \tag{74}$$

Note that (74) has been proven for each C_k being a Jordan curve in $\mathbb{R}^n$ (this condition depends on t_0, given k), but the lower bound for the square of the length of C_k is independent on t_0. By a perturbation argument of the curve $C_k(t) := h_k(\mathbb{S}^1 \times \{t\})$, the above lower bound holds independently of $t \in [0, T_k]$.

On the other hand, using the Cauchy-Schwarz inequality,

$$L(C_k(t))^2 = \left(\int_0^{2\pi} \left\| \frac{\partial f_k}{\partial \theta}(\theta, t) \right\| d\theta \right)^2 \leq 2\pi \int_0^{2\pi} \left\| \frac{\partial f_k}{\partial \theta}(\theta, t) \right\|^2 d\theta, \tag{75}$$

and integrating from 0 to T_k,

$$\varepsilon T_k = \int_0^{T_k} \varepsilon \, dt \overset{(74)}{\leq} \int_0^{T_k} L(C_k(t))^2 \, dt \overset{(75)}{\leq} 2\pi \int_0^{T_k} \left(\int_0^{2\pi} \left\| \frac{\partial f_k}{\partial \theta}(\theta, t) \right\|^2 d\theta \right) dt = 2\pi E(h_k),$$

which is bounded from above. Now the lemma is proved. □

Remark 11.6 The proof of (74) can be adapted to show that given any generator C_k of the fundamental group of $f_k(\mathbb{S}^1 \times [0, T_k])$, the length of C_k is not less than some $\delta > 0$ independent of $k \in \mathbb{N}$.

Lemma 11.7 *$\{T_k\}_k$ is bounded from below by some positive number.*

Proof. Since Γ_1, Γ_2 are disjoint, $d := \mathrm{dist}_{\mathbb{R}^n}(\Gamma_1, \Gamma_2) > 0$. Fix $k \in \mathbb{N}$ and $\theta \in \mathbb{S}^1$. Since the arc $t \in [0, T_k] \mapsto f_k(\theta, t)$ joins Γ_1 and Γ_2, we have that $d \leq L\left(t \in [0, T_k] \mapsto f_k(\theta, t)\right)$. Squaring and using Cauchy-Schwarz,

$$d^2 \leq \left(\int_0^{T_k} \left\| \frac{\partial f_k}{\partial t}(\theta, t) \right\| dt \right)^2 \leq T_k \int_0^{T_k} \left\| \frac{\partial f_k}{\partial t}(\theta, t) \right\|^2 dt$$

hence integrating from 0 to 2π,

$$2\pi d^2 \leq T_k \int_0^{2\pi} \left(\int_0^{T_k} \left\| \frac{\partial f_k}{\partial t}(\theta, t) \right\| dt \right) d\theta = T_k \, E(f_k).$$

Thus, $T_k \geq \frac{2\pi d^2}{E(f_k)}$, that converges as $k \to \infty$ to $\frac{\pi d^2}{a(\Gamma)} > 0$. □

Remark 11.8 By Lemmas 11.5 and 11.7, after passing to a subsequence we can assume that $\{T_k\}_k$ converges to some $T > 0$.

Next step consists of modifying the proof of the Courant-Lebesgue Lemma (Lemma 8.10) to adapt it to annuli (we will not give ths proof).

Lemma 11.9 (Courant-Lebesgue para anillos) *Let $E_0 > 0$, and $f : A \to \mathbb{R}^n$ be a Lipschitz map defined in an annulus A, with $E(f) \leq E_0$. Given $p \in \partial A$, there exists $\{r_m\}_m \to 0$ such that if $L(r_m)$ denotes the length of $f(C(p, r_m) \cap A)$, then $L(r_m)$ tends to zero as $m \to \infty$ (see Figure 22).*

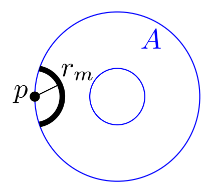


Figure 22: The image by f of the thick arc tends to zero as $m \to \infty$ (for all $p \in \partial A$).

Coming back to our proof of the Douglas criterion, given f_k, we can assume that given $p \in \mathbb{S}^1 \times \{0\}$ there exists $\{r_{k,m}\}_m \to 0$ such that

$$L(r_{k,m}) = L\left[f_k(C(p, r_{k,m}) \cap (\mathbb{S}^1 \times [0, T_k])\right) \ll \delta/2 \quad \text{for } m \text{ large enough,}$$

where $\delta > 0$ is the number that appear in Remark 11.6.

Abusing slightly of the notation, we will call $[p - r_{k,m}, p + r_{k,m}]$ to the arc inside $\mathbb{S}^1 \times \{0\}$ with extrema $C(p, r_{k,m}) \cap (\mathbb{S}^1 \times \{0\})$ of least length, and call I to the complementary interval (see Figure 23). Thus, $\mathbb{S}^1 \times \{0\} = [p - r_{k,m}, p + r_{k,m}] \dot{\cup} I$.

$f_k(p - r_{k,m}), f_k(p + r_{k,m})$ are two points of Γ_1, and

$$\operatorname{dist}_{\mathbb{R}^n}\left(f_k(p - r_{k,m}), f_k(p + r_{k,m})\right) \leq L\left[f_k(C(p, r_{k,m}) \cap (\mathbb{S}^1 \times [0, T_k]))\right] \ll \delta/2. \tag{76}$$

Furthermore, we can choose $\delta > 0$ arbitrarily small (see Remark 11.6).

Since Γ_1 is a Jordan curve in $\mathbb{R}^n$, the chord-arc ratio is bounded in Γ_1. More precisely, we can choose $\delta > 0$ small enough so that for any $P, Q \in \Gamma_1$ with $0 < |P - Q| < \delta/2$, it holds that P, Q separate Γ_1 into two arcs $\Gamma_1(P, Q), \widetilde{\Gamma}_1(P, Q)$ such that the diameter in $\mathbb{R}^n$ of $\Gamma_1(P, Q)$ is less than $\delta/2$ (see the first paragraph after the proof of Lemma 8.10 for the same argument).

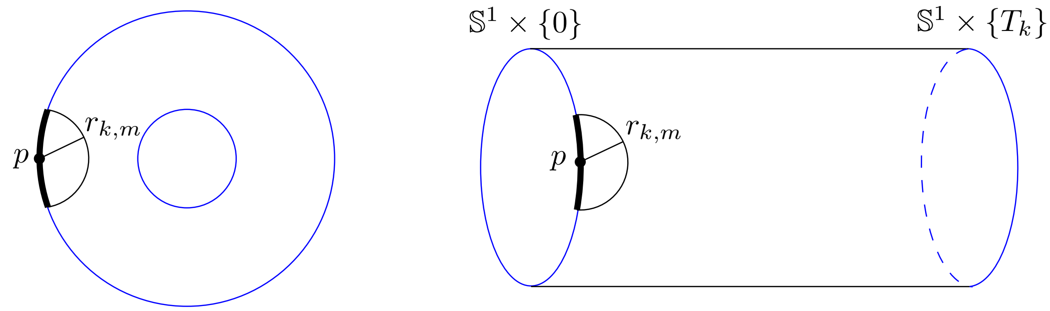


Figure 23: Two conformal representations of the annulus A.

Lemma 11.10 $\left\{h_n|_{\mathbb{S}^1\times\{0\}}\right\}_k$ *is equicontinuous.*

Proof. Following the above notation and reasoning as in the proof of Lemma 8.9, the lemma will be proved if we check that

$$f_k\left([p_k - r_{k,m}, p_k + r_{k,m}]\right) = \Gamma_1(P, Q) \quad \text{(independently of } p \in \mathbb{S}^1 \times \{0\}), \tag{77}$$

where $P = f_k(p_k - r_{k,m})$, $Q = f_k(p_k + r_{k,m})$ (compare to equation (46)). Arguing ba contradiction, suppose that (77) fails to hold. Then $f_k\left([p_k - r_{k,m}, p_k + r_{k,m}]\right) = \widetilde{\Gamma}_1(P, Q)$, hence $f_k(I) = \Gamma_1(P, Q)$. But

$$L\left[f_k\left[I \dot\cup \left[C(p, r_{k,m}) \cap (\mathbb{S}^1 \times [0, T_k])\right]\right]\right] = L(f_k(I)) + L\left[f_k\left(C(p, r_{k,m}) \cap (\mathbb{S}^1 \times [0, T_k])\right)\right]$$

$$\overset{(76)}{\leq} L(f_k(I)) + \delta/2 = L(\Gamma_1(P, Q)) + \delta/2 < \delta,$$

which contradicts that $I \dot\cup \left[C(p, r_{k,m}) \cap (\mathbb{S}^1 \times [0, T_k])\right]$ is an embedded generator of the fundamental group of $f_k(\mathbb{S}^1 \times [0, T_k])$, se Remark 11.6. This finishes the proof of the Lemma. $\square$

Replacing f_k by $\widehat{f}_k(\theta, t) = f_k(\theta, T_k - t)$ and using the above argument, we obtain:

Lemma 11.11 $\left\{h_n|_{\mathbb{S}^1\times\{T_n\}}\right\}_k$ *is equicontinuous.*

Lemma 11.12 *There exists* $(f : \mathbb{S}^1 \times [0, T] \to \mathbb{R}^n) \in X_\Gamma$, *(this* $T > 0$ *is the number that appeared in Remark 11.8) harmonic in* $\mathbb{S}^1 \times (0, T)$, *and there exists a subsequence of* $\{f_k\}_k$ *that converges on compact subsets of* $\mathbb{S}^1 \times (0, T)$ *to* f.

Proof. By Lemmas 11.10 and 11.11 and the Arzelá-Ascoli Theorem, after extracting a subsequence we can assume that

$$\left\| f_k|_{\mathbb{S}^1\times\{0,T_k\}} - f_h|_{\mathbb{S}^1\times\{0,T_k\}} \right\|_\infty \to 0 \quad \text{if } k, h \to \infty.$$

Since f_k and f_h are harmonic and $T_k \to T$, the maximum principle for harmonic functions implies

$$\|f_k|_K - f_h|_K\|_{L^\infty(K)} \to 0 \quad \text{if } k, h \to \infty,\ \forall K \subset \mathbb{S}^1 \times [0,T] \text{ compact subset.}$$

Given a compact subset $K \subset \mathbb{S}^1 \times [0,T]$, the above displayed property implies the existence of a harmonic map $h\colon K \to \mathbb{R}^n$ such that $\{f_k|_K\}_k \to f$ uniformly in K (again after passing to a subsequence). A diagonal argument varying K in an increasing exhaustion of $\mathbb{S}^1 \times [0,T]$ by compact subsets proves that there exists a harmonic map $h\colon \mathbb{S}^1 \times [0,T] \to \mathbb{R}^n$ and a subsequence of $\{f_k\}_k$ that satisfies the conditions of the Lemma. □

Now we can finish the proof of the Douglas criterion in the same way as we did for the Douglas-Radó theorem.

12 Limits of embedded minimal surfaces

Our next goal is to understand how to take limits of a sequence of embedded minimal surfaces, and the different objects we may encounter in the limit process.

12.1 Motivation

Suppose that Γ is a polygon in $\mathbb{R}^3$, i.e. a C^0 Jordan curve consisting of with finitely points joined by straight line segments. We want to produce a compact embedded minimal surface $\Sigma \subset \mathbb{R}^3$ with boundary Γ.

Note that Douglas-Radó's Theorem is not enough to produce the desired surface, as it only produces a disc that minimizes the area among all surfaces with boundary Γ, but such a disk could have self-intersections and/or branch points. However, if Γ satisfies the hypotheses of Radó's Theorem (Theorem 10.8), then we can ensure the existence a unique compact minimal surface $\Sigma \subset \mathbb{R}^3$ with this boundary, which is embedded and has no branch points.

This application ot Radó's Theorem (and of its proof) can be generalized by replacing $\mathbb{R}^3$ by a homogeneous, simply connected three-manifold $\mathbb{E}(\kappa,\tau)$ admitting a Riemannian submersion $\pi\colon \mathbb{E}(\kappa,\tau) \to \mathbb{M}^2(\kappa)$ with bundle curvature $\tau \in \mathbb{R}$ onto the simply connected surface of constant curvature $\kappa \in \mathbb{R}$. These spaces are $\mathbb{S}^2(\kappa) \times \mathbb{R}$ if $\kappa > 0$, $\mathbb{H}^2(\kappa) \times \mathbb{R}$ if $\kappa < 0$ and $\mathbb{R}^3$ in the case of product spaces, and if the fibration π is non-trivial (equivalently, $\tau \neq 0$), we have the Heisenberg space $\mathbb{E}(0,\tau)$ fibering over $\mathbb{R}^2$, the Berger spheres fibering over $\mathbb{S}^2(\kappa)$ with $\kappa > 0$, and the universal cover $\widetilde{\mathrm{SL}}(2,\mathbb{R})$ of the special linear group fibering over $\mathbb{H}^2(\kappa)$ when $\kappa < 0$. In all these cases we can extend Radó's Theorem exchanging the orthogonal projection from $\mathbb{R}^3$ into $\mathbb{R}^2$ by the Riemannian submersion $\pi\colon \mathbb{E}(\kappa,\tau) \to \mathbb{M}(\kappa)$. In order the proof of Theorem 10.8 to be valid in $\mathbb{E}(\kappa,\tau)$, we need the following ingredients (compare to the steps of the proof of Theorem 10.8):

(A) Vertical 'planes', that is, $\pi^{-1}(\gamma)$ with $\gamma \subset \mathbb{M}(\kappa)$ any geodesic arc, must be minimal. This follows since given a geodesic $\gamma \subset \mathbb{M}(\kappa)$, γ is invariant under the rotation of angle π around any of the points of γ, and such a rotation lifts via π to an isometry of $\mathbb{E}(\kappa, \tau)$ that inverts the orientation of $\pi^{-1}(\gamma)$. This allows us to reproduce steps (1) and (2) of the proof of Theorem 10.8.

(B) The fibers $\pi^{-1}(\{p_0\})$ of π, with p_0 any point in $\mathbb{M}(\kappa)$, are diffeomorphic to $\mathbb{R}$. This is valid for every $\mathbb{E}(\kappa, \tau)$ except for the Berger spheres), and that vertical translations (i.e., elements in the 1-parameter group generated by the unit vertical field E_3 on $\mathbb{E}(\kappa, \tau)$ that generates $\ker(d\pi)$) are isometries (equivalently, E_3 is a Killing field). In this way, we can move a minimal surface vertically in $\mathbb{E}(\kappa, \tau)$ and produce a 1-parameter family of minimal surfaces depending continuously on the parameter. This allows us to extend the arguments of step (3) of the proof of Theorem 10.8.

(C) The argument we gave in step (4) of the proof of Theorem 10.8 is no longer valid now, since we do not have reflections about horizontal 'planes' (in the case $\tau = 0$ we do have these reflections), but we can overcome this problem by the following argument: if there exists $p \in M$ with T_pM vertical, then we can find a vertical 'plane' $\pi^{-1}(\gamma) \subset \mathbb{E}(\kappa, \tau)$ passing through p, which is also a minimal surface, such that as the intersection of M and $\pi^{-1}(\gamma)$ produces an equiangular system of curves that intersect at p and the two minimal surfaces cross each other along these curves, which contradicts the fact that M does not have two points on the same vertical.

The above observations tell us that we can extend Radó's Theorem to spaces $\mathbb{E}(\kappa, \tau)$ not being a Berger sphere (in this case there is also a Radó's type Theorem provided that we give additional conditions in order step (B) above to hold).

Next we choose a geodesic polygon $\Gamma \subset \mathbb{E}(\kappa, \tau)$ whose edges are vertical (i.e., the velocity vector of an edge lies in $\ker(d\pi)$) or horizontal (the velocity vector of an edge is orthogonal to $\ker(d\pi)$), and we want as before to find an embedded minimal surface with boundary Γ. The problem is that for a contour Γ like this, the restriction $\pi|_\Gamma$ fails to produce a 1-1 projection onto a convex polygonal Jordan curve in $\mathbb{M}^2(\kappa)$: vertical segments of Γ apply to points of $\pi(\Gamma)$. Therefore we cannot directly apply Radó's Theorem, see Figure 24.

To solve this problem, we will approximate Γ by polygons Γ_n to which we can apply Radó's Theorem and we will take limits in the corresponding sequence of minimal surfaces with boundary Γ_n.

Let us simplify the situation by going back to $\mathbb{R}^3$, with the geodetic polygon Γ of Figure 24. By the Theorem of Douglas-Radó, there exists a compact minimal disk Σ (possibly immersed and with branch points) with $\partial\Sigma = \Gamma$. By the convex hull property, Σ is contained in the solid cylinder $\pi^{-1}(R)$, where $\pi(x, y, z) = (x, y)$ and $R \subset \{z = 0\}$ is the rectangle bounded by $\pi(\Gamma)$.

Given $n \in \mathbb{N}$, take another two polygons $\widetilde{\Gamma}_n$ and Γ'_n as in Figure 25, determined by the vertices a_n, b_n.

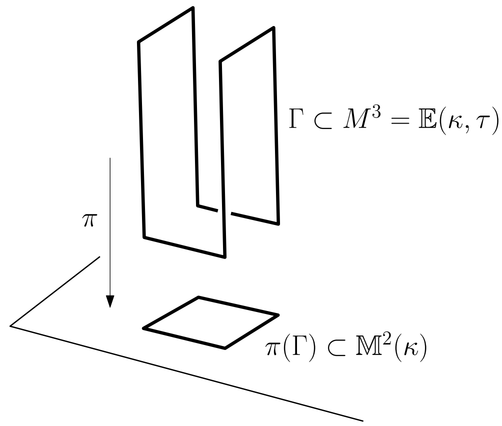


Figure 24: Radó's Theorem does not apply in this situation.

Both $\widetilde{\Gamma}_n$ and Γ'_n satisfy the hypotheses of Radó's Theorem, hence there exist compact minimal surfaces $\widetilde{\Sigma}_n$, $\Sigma'_n \subset \mathbb{R}^3$ with $\partial\widetilde{\Sigma}_n = \widetilde{\Gamma}_n$, $\partial\Sigma'_n = \Gamma'_n$, such that $\widetilde{\Sigma}_n$, Σ'_n are graphs over the convex domain $\pi(\Gamma)$. Moreover, $\widetilde{\Sigma}_n$, $\Sigma'_n \subset \mathbb{R}^3$ are the unique compact minimal surfaces with these boundaries.

Now compare $\widetilde{\Sigma}_n$ with Σ. Given $t > 0$, llet $\widetilde{\Sigma}_n(t) = \widetilde{\Sigma}_n + te_3$. For $T > 0$ large enough, $\widetilde{\Sigma}_n(T)$ is disjoint from Σ. Start moving $\widetilde{\Sigma}_n(t)$ down from $t = T$ to $t = 0$, until we find a first contact point. Analyzing the boundaries of $\widetilde{\Sigma}_n(t)$ and Σ we conclude that this first contact point occurs at $t = 0$, i.e., $\widetilde{\Sigma}_n \geq \Sigma$ (this ordering refers to the vertical direction). Reasoning in an analogous way with translations of the type $\Sigma'_n - te_3$, $t > 0$, we get that $\Sigma \geq \Sigma'_n$. In summary:

$$\Sigma'_n \leq \Sigma \leq \widetilde{\Sigma}_n, \quad \text{for all } n \in \mathbb{N}. \tag{78}$$

If we know that Σ'_n and $\widetilde{\Sigma}_n$ converge to the same compact minimal surface and that such limit is a graph over its projection by π, then we will have proven the conclusions of Radó's Theorem for the boundary Γ. In fact, Σ'_n and $\widetilde{\Sigma}_n$ are respectively the graphs of solutions $u_n, v_n \colon R \to \mathbb{R}$ of the PDE (6), and the above arguments prove that $\{u_n\}_n$ is increasing and $\{v_n\}_n$ is decreasing. From here and (78) we can conclude that $\{u_n\}_n$ and $\{v_n\}_n$ converge in the uniform topology on compact subsets of R to a minimal graph (observe that the space of solutions of (6) is closed under this topology). In more general situations we will not have as good conditions as the above monotonicity, so we will need more general results that allow us to take limits of sequences of embedded minimal surfaces.

To finish with the example of the contour Γ, let us see that in this particular case everything reduces to know that there exists $\Sigma_\infty := \lim_n \Sigma'_n$ (in what follows, it is not necessary to know

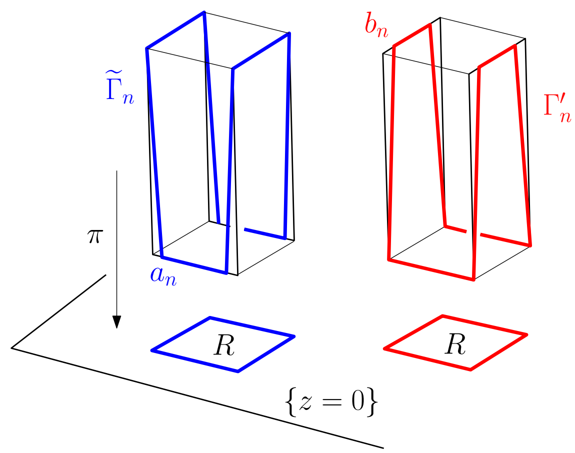


Figure 25: Radó's Theorem de Radó can be applied to the polygons $\widetilde{\Gamma}_n$ and Γ'_n.

that $\Sigma_\infty = \Sigma$). Clearly

$$\partial\Sigma_\infty := \partial\left(\lim_{n\to\infty} \Sigma'_n\right) = \lim_{n\to\infty} \partial\Sigma'_n = \lim_{n\to\infty} \Gamma'_n = \Gamma = \partial\Sigma.$$

We claim that if η^{Σ_∞} (resp. η^Σ) denotes the inward pointing unit conormal vector to Σ_∞ (resp. Σ) along Γ, then

$$\langle \eta^{\Sigma_\infty}, e_3\rangle \leq \langle \eta^\Sigma, e_3\rangle : \tag{79}$$

In the vertical part of Γ, both members of (79) are zero. In the horizontal part of Γ, the fact that $\Sigma_\infty \leq \Sigma$ (this can be deduced from (78) taking limits) and the maximum principle for minimal surfaces ensures that (79) holds.

Lemma 12.1 *Let Σ be a compact minimal surface with boundary in a Riemannian manifold (M^3, g) and let Y be a Killing field on M. Then,*

$$\int_{\partial\Sigma} \langle Y, \eta\rangle = 0.$$

where η is a unit conormal field to Σ along $\partial\Sigma$.

Proof. $\mathrm{div}_\Sigma(Y^T) = 0$ because $H = 0$ and Y is a Killing field on M. The lemma now follows from integrating on Σ and applying the divergence theorem. $\square$

Applying Lemma 12.1 to our situation with $\Sigma_\infty, \Sigma \subset \mathbb{R}^3$ and $Y \equiv e_3$, we deduce that

$$\int_\Gamma \langle \eta^{\Sigma_\infty}, e_3\rangle = 0 = \int_\Gamma \langle \eta^\Sigma, e_3\rangle,$$

hence (79) implies that $\langle \eta^{\Sigma_\infty}, e_3\rangle = \langle \eta^{\Sigma}, e_3\rangle$ along the horizontal part of Γ. This implies that $\Sigma_\infty = \Sigma$ by the boundary maximum principle.

Note that we could have taken $\widetilde{\Gamma}_n$ as the image of Γ'_n by the composition ϕ of the rotation of angle $\pi/2$ with respect to the vertical axis that passes through the center of R with the rotation of angle π with respect to a certain horizontal straight line whose height is half the height of Γ'_n. By the uniqueness given by Radó's Theorem, we deduce that $\widetilde{\Sigma} = \phi(\Sigma'_n)$ for each n. Therefore,

$$\lim_{n\to\infty} \widetilde{\Sigma}_n = \lim_{n\to\infty} \phi(\Sigma'_n) = \phi\left(\lim_{n\to\infty} \Sigma'_n\right) = \phi(\Sigma_\infty) = \phi(\Sigma),$$

that is, there exists the limit of $\widetilde{\Sigma}_n$ and equals $\phi(\Sigma)$. Applying the above argument we conclude that $\phi(\Sigma) = \Sigma$, i.e., Σ is invariant by ϕ.

Although we will not use it, it is worth mentioning that the minimal surface Σ bounded by Γ (we already know that Σ is unique) generates a properly embedded, triply periodic[9] minimal surface without boundary, by successive applications of the Schwarz's reflection principle below with respect to any segment in the boundary.

Proposition 12.2 (Schwarz's reflection principle) *Let Σ be a minimal surface in a Riemannian manifold (M^3, g) and $\alpha \subset \partial\Sigma$ a geodesic arc. Suppose there exists an order two isometry $\phi\colon M \to M$ such that $\phi(p) = p$ for each $p \in \alpha$ and $d\phi_p$ is a rotation of angle π in T_pM with respect to the straight line tangent to α at p. Then, $\Sigma \cup \phi(\Sigma)$ is a minimal surface is a neighborhood of each point of α.*

Proof. (Only in the case $(M, g) = (\mathbb{R}^3, \langle,\rangle)$)
Apply the classical Schwarz's reflection principle for harmonic functions to each of the coordinate functions of Σ with respect to a coordinate system having α as one of the coordinate axes. □

12.2 Limits of sequences of minimal surfaces

Take a sequence $\{\Sigma_n\}_n$ of properly embedded minimal surfaces without boundary in a complete Riemannian manifold (M^3, g).

> Under which conditions there exists a subsequence of $\{\Sigma_n\}_n$ that converges to a complete embedded minimal surface inside M?

Let us see some examples.

(1) Let Σ_1 be a catenoid with axis the z-axis. If we define $\Sigma_n = \frac{1}{n}\Sigma_1$ (homothety) for each $n \in \mathbb{N}$, then $\{\Sigma_n\}_n$ converges to the plane $\{z = 0\}$ on compact subsets of $\mathbb{R}^3 \setminus \{\vec{0}\}$ with multiplicity 2. The Gauss curvature K_{Σ_n} of Σ_n blows up in any neighborhood of $\vec{0}$.

[9] That is, invariant under a group of translations of $\mathbb{R}^3$ of rank three.

(2) Suppose that Σ_1 is a helicoid with axis the z-axis. Let $\Sigma_n = \frac{1}{n}\Sigma_1$. Then, $\{\Sigma_n\}_n$ converges to the foliation of $\mathbb{R}^3$ by horizontal planes, away from the z-axis, where the Gauss curvature K_{Σ_n} blows up, see Figure 26.

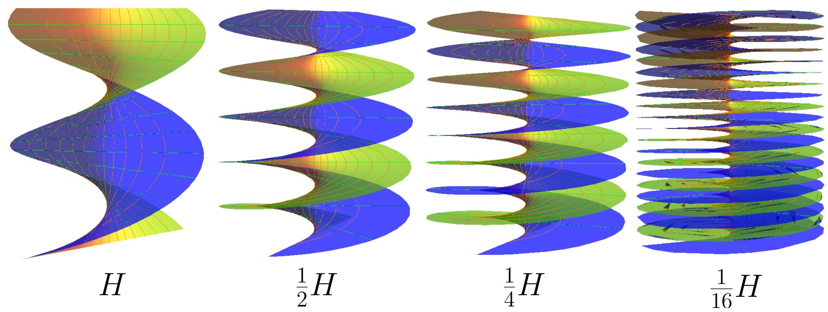


Figure 26: When we take limits of $\frac{1}{n}H$ where H is a vertical helicoid, we obtain the foliation of $\mathbb{R}^3$ by horizontal planes, but convergence fails to holds along the z-axis.

(3) Suppose that Σ_1 is a properly embedded triply periodic minimal surface without boundary, as for instance the one we have just constructed in the previous section. Let $\Sigma_n = \frac{1}{n}\Sigma_1$. Then $\overline{\{\Sigma_n\}_n} = \mathbb{R}^3$ (closure) and $\{K_{\Sigma_n}\}_n$ blows up in any neighborhood of every point of $\mathbb{R}^3$. In other words, the limit of $\{\Sigma_n\}_n$ (or of any subsequence) has no structure.

We will prove several compactness results, always assuming the existence of an accumulation point of the sequence of embedded minimal surfaces and different assumptions on uniform local bounds for the sequence.

12.3 Limits of minimal graphs

Let Ω be a connected open subset in a Riemannian surface (M^2, g), and

$$\mathcal{M}(\Omega) = \left\{ u \in C^\infty(\Omega) \ : \ \mathrm{div}_M \left(\frac{\nabla_M u}{W} \right) = 0 \right\}$$

the set of minimal graphs over Ω (they are minimal in $(M \times \mathbb{R}, g \times dt^2)$), where $W = \sqrt{1 + |\nabla_M u|^2}$.

Theorem 12.3 *Let $\{u_n\}_n \subset \mathcal{M}(\Omega)$. If there exist $p \in \Omega$ and $c > 0$ such that $|u_n(p)| \leq c$ for all $n \in \mathbb{N}$, and for each compact subset $K \subset \Omega$ there exists $C(K) > 0$ such that $\|\nabla_M u_n\| \leq C(K)$ in K for all $n \in \mathbb{N}$, then there exists $u \in \mathcal{M}(\Omega)$ and a subsequence of $\{u_n\}_n$ that converges on compact subsets of Ω to u in the topology C^k for each $k \in \mathbb{N}$.*

Proof. Fix a connected compact subset $K \subset \Omega$ that contains p. Given $q \in K$ and $n \in \mathbb{N}$,

$$|u_n(p) - u_n(q)| \leq \|\nabla_M u_n\|_{L^\infty(K)} \cdot \mathrm{dist}_M(p,q) \leq C(K) \cdot \mathrm{dist}_M(p,q),$$

hence $\{u_n\}_n$ is uniformly Lipschitz in K and so, uniformly equicontinuous in K. By the triangle inequality, $\{u_n\}_n$ is also uniformly bounded in K. By the Arzelá-Ascoli Theorem, there exists $u \in \mathcal{M}(K)$ and a subsequence of $\{u_n\}_n$ that converges uniformly to u in K (and in C^k for each $k \in \mathbb{N}$ by a standard argument in elliptic theory). After moving K in an exhaustion of Ω and applying a diagonal argument we conclude the proof. $\square$

Lemma 12.4 (Uniform graph lemma) *Let $O \subset \mathbb{R}^3$ an open set. Take $C > 0$ and let $\Sigma \subset O$ be a properly embedded surface*[10] *with $|A_\Sigma| \leq C$. Then, given $p \in \Sigma$ and $R \in (0, R_0)$ where*

$$R_0 = \min\left\{\frac{1}{8C}, \frac{1}{2}\mathrm{dist}(p, \partial O)\right\}, \tag{80}$$

there exists a differentiable function $u\colon D(p,R) = \{p + v \mid v \in T_p\Sigma,\ \|v\| < R\} \to \mathbb{R}$ such that $u(p) = 0$ and Σ can be written locally around p as the graph of u. Furthermore, given $q \in D(p,r)$,

(1) $|u(p) - u(q)| \leq 8C\|p-q\|^2$.

(2) $\|\nabla u\|(q) \leq 8C\|p-q\|$ (in particular, $(\nabla u)(p) = 0$).

(3) $\|\nabla^2 u\|(q) \leq 8C$.

Proof. After possibly a rigid motion, we can assume $p = \vec{0} \in \mathbb{R}^3$ and $T_p\Sigma = \{z = 0\}$. Let $\mathcal{I}$ be the set of the numbers $R \in (0, R_0)$ such that Σ is locally around p the graph of a differentiable function $u \in C^\infty(D(p,R))$ with $u(\vec{0}) = 0$, $(\nabla u)(\vec{0}) = \vec{0} \in \mathbb{R}^2$. Consider the parameterization of Σ given by $\psi(x,y) = (x, y, u(x,y))$, with Gauss map $N = \frac{1}{W}(-u_x, -u_y, 1)$, where $W = \sqrt{1 + \|\nabla u\|^2}$. Let $N_3 = \frac{1}{W}$. Since $N_3(\vec{0}) = 1$, we can assume taking $R \in \mathcal{I}$ small enough that

$$N_3 \geq \frac{1}{\sqrt{2}} \quad \text{in } D(\vec{0}, R). \tag{81}$$

Let

$$\mathcal{I}_1 = \{R \in \mathcal{I} \mid (81) \text{ holds}\} \quad \text{and} \quad R_1 = \sup \mathcal{I}_1.$$

Thus, $\mathcal{I}_1$ is an interval of the form $(0, R_1]$, and Σ is locally around p the graph of a differentiable function $u \in C^\infty(D(p,R_1))$ with $u(\vec{0}) = 0$, $(\nabla u)(\vec{0}) = \vec{0} \in \mathbb{R}^2$ and $N_3 \geq \frac{1}{\sqrt{2}}$ in $D(\vec{0}, R_1)$. If we check that $R_1 = R_0$ and that u satisfies (1),(2),(3), we will have proven the lemma.

Note that exactly one of the following possibilities holds:

[10] Not necessarily minimal.

(A) u can be extended to $\overline{D}(\vec{0}, R_1)$ (in particular $\partial D(p, R_1) \cap \partial O = \varnothing$), and there exists $q_0 \in \partial D(p, R_1)$ such that $N_3(q_0) = \frac{1}{\sqrt{2}}$.

(B) Case (A) does not hold, and there exists $\{q_n\}_n \subset D(p, R_1)$ such that $\|q_n - p\| \to R_1$ and $\text{dist}(\psi(q_n), \partial O) \to 0$.

(C) Neither (A) or (B) occur. In this case, either $R_1 < R_0$ (this is impossible because it contradicts that $R_1 = \sup \mathcal{I}$), or $R_1 = R_0$.

Claim 12.5 *If (A) holds, then $R_1 = R_0$.*

Proof. $|(N_3)_x| = |(\langle N, e_3 \rangle)_x| = \left|\langle \overline{\nabla}_{\psi_x} N, e_3 \rangle\right| = |\langle A\psi_x, e_3\rangle| \leq \|A\psi_x\| \leq \|A\| \cdot \|\psi_x\| \leq C\|\psi_x\| = C\sqrt{1+u_x^2} \leq CW \overset{(81)}{\leq} \sqrt{2}C$. Analogously, $|(N_3)_y| \leq \sqrt{2}C$, hence $\|\nabla N_3\|^2 \leq 4C^2$ and

$$\|\nabla N_3\| \leq 2C \text{ in } D(\vec{0}, R_1). \tag{82}$$

given $q \in D(\vec{0}, R_1)$,

$$|N_3(p) - N_3(q)| \overset{(82)}{\leq} 2C\|p - q\| \leq 2CR_1. \tag{83}$$

On the other hand, from $W \leq \sqrt{2}$ (i.e. (81)) we have $1 + \|\nabla u\|^2 \leq 2$ hence

$$\|\nabla u\| \leq 1 \text{ in } D(\vec{0}, R_1). \tag{84}$$

Therefore, $|u(p) - u(q)| \overset{(84)}{\leq} \|p - q\| \leq R_1$. Now evaluate (83) at the point $q_0 \in \partial D(p, R_1)$ that appears in (A) (note that (84) can be used in the boundary of $D(p, R_1)$ because u extends to $\overline{D}(\vec{0}, R_1)$ and by continuity).

$$R_1 \overset{(83)}{\geq} \frac{1}{2C}|N_3(p) - N_3(q_0)| = \frac{1}{2C}\left(1 - \frac{1}{\sqrt{2}}\right) \geq \frac{1}{8C} \overset{(80)}{\geq} R_0, \tag{85}$$

hence $R_1 = R_0$ as desired. □

Claim 12.6 *(B) cannot occur.*

Proof. By the triangle inequality,

$$\text{dist}(p, \partial O) \leq \text{dist}(p, \psi(q_n)) + \text{dist}(\psi(q_n), \partial O) \leq \int_p^{q_n} W + \text{dist}(\psi(q_n), \partial O)$$

$$\leq \int_p^{q_n} \sqrt{2} + \text{dist}(\psi(q_n), \partial O) = \sqrt{2}\|q_n - p\| + \text{dist}(\psi(q_n), \partial O) \leq \sqrt{2}R_1 + \text{dist}(\psi(q_n), \partial O),$$

hence taking $n \to \infty$ we have $\mathrm{dist}(p, \partial O) \leq \sqrt{2} R_1$. But $R_0 \leq \frac{1}{2}\mathrm{dist}(p, \partial O)$, hence $2R_0 \leq \sqrt{2}R_1 \leq \sqrt{2}R_0$, which is a contradiction. $\square$

We now finish the proof of Lemma 12.4. From Claims 12.5 and 12.6 we deduce that $R_1 = R_0$. It remains to prove that (1), (2) and (3) hold.

$\frac{u_{xx}}{W} = \langle N, \psi_{xx} \rangle = -\langle N_x, \psi_x \rangle = \langle A\psi_x, \psi_x \rangle$, hence $\frac{|u_{xx}|}{W} \leq \|A\| \cdot \|\psi_x\|^2 \leq C(1 + u_x^2) \leq CW^2$. Thus, $|u_{xx}| \leq CW^3 \leq 2^{3/2}C$. Analogously,

$$|u_{xy}| \leq 2^{3/2}C, \quad |u_{yy}| \leq 2^{3/2}C.$$

thus, $\|\nabla^2 u\|^2 = u_{xx}^2 + u_{yy}^2 + 2u_{xy}^2 \leq 32C^2$, and $\|\nabla^2 u\| \leq 4\sqrt{2}C \leq 8C$, which is (3).

Given $q \in D(p, R_1)$, $\|\nabla u\|(q) = \|(\nabla u)(p) - (\nabla u)(q)\| \leq 8C\|p - q\|$, and we have (2). Finally, $|u(p) - u(q)| \leq \|\nabla u\|_\infty \|p - q\| \leq 8C\|p - q\|^2$ and we have (1). $\square$

12.4 Compactness theorems for sequences of embedded minimal surfaces

Definition 12.7 We call a Riemannian manifold (M^n, g) *homogeneously regular* if its injectivity radius $\mathrm{Inj}(M)$ is positive (hence given $R \in (0, \mathrm{Inj}(M))$, the exponential map $\exp_p^M$ provides geodesic coordinates in the geodesic ball $B(p, R)$ centered at p of radius R) and the absolute sectional curvature of M is bounded in M.

Every Riemannian manifolds is locally homogeneously regular, and every compact Riemannian manifold is homogeneously regular. By using harmonic coordinates, it can be proven that the local distortion in a homogeneously regular Riemannian manifold is uniformly $C^{1,\alpha}$-controlled[11]. This property allows us to generalize Lemma 12.4 to these manifolds:

Theorem 12.8 (Uniform graph lemma)
Let (M^3, g) be a homogeneously regular Riemannian manifold. Given $C > 0$, there exists $\delta > 0$ such that if $\Sigma \subset M$ is a properly embedded surface[12] with $|A_\Sigma| \leq C$, then Σ is a local graph over the disk

$$D(p, \delta) := \exp_p^M \left(D_{T_p\Sigma}(p, \delta) \right)$$

and such a graph has bounded geometry independently of Σ[13] (see Figure 27).

Remark 12.9
(1) As M is homogeneously regular, we can assume that $\delta < \mathrm{Inj}(M)$; in particular, $\exp_p^M$ is a diffeomorphism in the ball $B_M(p, \delta)$.

[11]This means that the entries g_{ij} of the metric with respct to harmonic coordinates centered at any point p of M are controlled in the $C^{1,\alpha}$ topology for any $\alpha \in (0, 1)$, and this control only depends on $\mathrm{Inj}(M)$ and of the bound of the absolute sectional curvature of M.

[12]Not necessarily minimal.

[13]This means that all derivatives of u in $D(p, \delta)$ are bounded by a constant that does not depend on Σ.

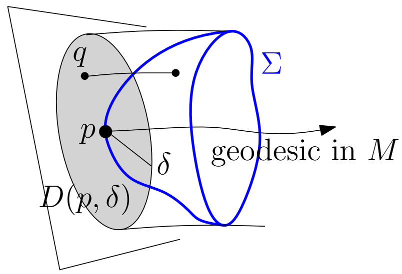


Figure 27: graph in exponential coordinates.

(2) Also because (M^3, g) is homogeneously regular, there exist $\delta, \varepsilon > 0$ small enough so that

$$W(p,\delta,\varepsilon) := \left\{ \exp_q^M \left(t \cdot \widetilde{N}_q \right) \ : \ q \in D(p,\delta), |t| \leq \varepsilon \right\}$$

is a tubular neighborhood of the surface $D(p,\delta)$, where $\widetilde{N} \colon D(p,\delta) \to TM$ is a unit normal field for $D(p,\delta)$. That Σ is a local graph over the disk $D(p,\delta)$ means that Σ can be written locally around p as the image of the map

$$q \in D(p,\delta) \mapsto \exp_q^M \left(u(q) \widetilde{N}_q \right), \tag{86}$$

where $u \in C^\infty(D(p,\delta))$ satisfies $u(p) = 0$ and $(\nabla u)(p) = 0$. In fact, the map defined in (86) parameterizes a neighborhood of p in the component $\Sigma(p)$ of $\Sigma \cap W(p,\delta,\varepsilon)$ that contains p (see Figure 28).

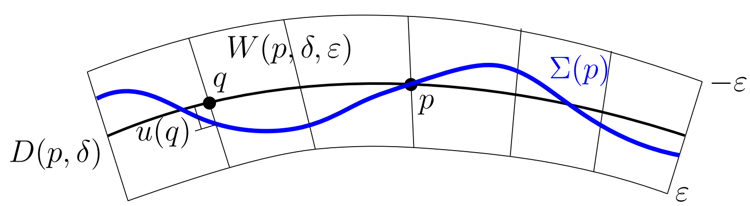


Figure 28: The component $\Sigma(p)$ of $\Sigma \cap W(p,\delta,\varepsilon)$ that contains p can be written as a graph in exponential coordinates.

Definition 12.10 A surface Σ in a Riemannian three-manifold (M^3, g) is called *two-sided* it Σ admits a globally defined, smooth unit normal field $N \colon \Sigma \to U_1(M)$.

Take a sequence $\{\Sigma_n\}_n$ of complete two-sided minimal surfaces without boundary, embedded in a complete and homogeneously regular Riemannian manifold (M^3, g). Suppose that $\{\Sigma_n\}_n$ has an accumulation point $p \in M$; thus, $p = \lim_n p_n$ for some sequence $p_n \in \Sigma_n$. We will assume that the second fundamental forms of the Σ_n has uniform local bounds (we will not assume uniform local bounds for the area). Under these conditions, the sequence $(p_n, N_n(p_n))$ converges to an element $(p, N_p) \in U_1(M)$ after passing to a subsequence. Since (M, g) is homogeneously regular, there exists $\delta > 0$ such that

$$D(p,\delta) := \left\{\exp_p^M(tw) \mid w \in T_pM,\ g_p(w,w) = 1,\ g_p(w,N_p) = 0, |t| < \delta\right\}$$

is a surface passing through p. Analogously, define for each $n \in \mathbb{N}$

$$D(p_n,\delta) := \left\{\exp_{p_n}^M(tw) \mid w \in T_p\Sigma_n,\ g_p(w,w) = 1,\ |t| < \delta\right\}.$$

Note that $D(p_n, \delta)$ converges to $D(p, \delta)$ uniformly as $n \to \infty$.

By Theorem 12.8, we can choose $\delta > 0$ such that a neighborhood of p_n

$$\Sigma_n(p_n) := \mathrm{Gr}[u_n \colon D(p_n,\delta) \to \mathbb{R}] \subset \Sigma_n$$

is the graph of a differentiable function with bounded geometry independently of n (think of $\Sigma_n(p)$ as the component of $\Sigma_n \cap W(p_n, \delta, \varepsilon)$ that contains p_n, where $\varepsilon > 0$ is independent of n). Since $D(p_n, \delta)$ converges uniformly to $D(p, \delta)$, for n large enough $\Sigma_n(p_n)$ is also the graph of a differentiable function $v_n \colon D(p, \delta) \to \mathbb{R}$ with bounded geometry independently of n (we should take here $\delta/2$ instead of δ but we will not do this for the sake of simplicity). Moreover, each v_n is a solution of an elliptic PDE that is independent of n (since Σ_n is minimal). By Theorem 12.3, after passing to a subsequence the v_n converge on compact subsets of $D(p, \delta)$ to a smooth function $v_\infty \colon D(p, \delta) \to \mathbb{R}$ in the C^k topology for each k. In particular, v_∞ satisfies the same PDE as the v_n, i.e., v_∞ produces a minimal graph $\Sigma(p) := \mathrm{Gr}(v_\infty)$ over $D(p, \delta)$. By construction, $\Sigma(p)$ has unit normal N_p at p.

Next we will construct a global object that contains $\Sigma(p)$. Note that $\Sigma(p)$ has the topology of a disk. Take $q \in \partial\Sigma(p)$. As the convergence of $\Sigma_n(p_n)$ to $\Sigma(p)$ is C^1 on compact subsets, there exists $\{q_n\}_n \in \Sigma_n(p_n)$ such that

$$N_n(q_n) = N^{\Sigma_n(p_n)}(q_n) \to N^{\Sigma(p)}(q),$$

where el super-index denote the related surface. Theorem 12.8 ensures that around q_n, $\Sigma_n(p_n)$ can be written as the graph of a differentiable function defined in a disk of radius δ (this is the same δ before). Repeating the same reasoning, a subsequence of these graphs converges uniformly on compact subsets of $D(q, \delta)$ to a smooth function $w_\infty \colon D(q, \delta) \to \mathbb{R}$ in the C^k topology for each k. Thus, the limits $\mathrm{Gr}(v_\infty)$ and $\mathrm{Gr}(w_\infty)$ overlap and their union produces a minimal surface that contains q in its interior and which lies in the closure of the Σ_n. Applying repeatedly this continuation argument we will obtain a minimal surface $\Sigma \subset M$ without self-intersections[14]. The intersection of Σ with each compact subset of M has bounded second

[14] Recall that if Σ has self-intersections, then we can fin a point of *transversal* self-intersection of Σ; this produces self-intersection points in the Σ_n for n large enough, which is a contradiction.

fundamental form, and Σ is a closed subset of M (Σ has no boundary by the above continuation argument). This last property and the fact that (M^3, g) is complete imply that Σ is also complete (every Cauchy sequence in Σ is a Cauchy sequence in M, hence it is convergent in M, and its limit lies in the closure of Σ, which equals Σ). In summary, we have proved the following result:

Theorem 12.11 *Let (M^3, g) be a complete and homogeneously regular Riemannian manifold, and let $\{\Sigma_n \subset M\}_n$ be a sequence of complete embedded, two-sided minimal surfaces without boundary. If $\{\Sigma_n\}_n$ has an accumulation point in M and local uniform bounds for the second fundamental form, then*

$$\mathrm{Lim}(\{\Sigma_n\}_n) = \{p = \lim_{n\to\infty} p_n \in M \mid p_n \in \Sigma_n \ \forall n \in \mathbb{N}\}$$

contains a complete embedded minimal surface $\Sigma \subset M$ such that after passing to a subsequence, the Σ_n converge on compact subsets of M to Σ in the C^k topology for each k.

Note that under the conditions of Theorem 12.11, we do not ensure that $\mathrm{Lim}(\{\Sigma_n\}_n)$ equals Σ. Next we will develop this idea further.

Assume that in addition to the hypotheses of Theorem 12.11, the surfaces Σ_n have uniform local area bounds[15]. In this case, in the previous local argument to produce graphs $\Sigma_n(p_n)$ converging to a minimal graph $\Sigma(p)$, we can assume that $\Sigma_n \cap W(p_n, \delta, \varepsilon)$ has only a finite number (independent of n) of components, of which $\Sigma_n(p_n)$ is the one that contains p_n. In particular, the convergence of $\Sigma_n \cap W(p_n, \delta, \varepsilon)$ to $\Sigma(p)$ has finite multiplicity. This finite multiplicity is constant in each connected component of Σ, and around each $p \in \mathrm{Lim}(\{\Sigma_n\}_n)$, the local structure of the limit set

$$\mathrm{Lim}(\{\Sigma_n\}_n) = \{p = \lim_{n\to\infty} p_n \in M \mid p_n \in \Sigma_n \ \forall n \in \mathbb{N}\}$$

consists of a properly embedded minimal surface inside M. In summary:

Theorem 12.12 *Let (M^3, g) be a complete and homogeneously regular Riemannian manifold, and let $\{\Sigma_n \subset M\}_n$ be a sequence of properly embedded, two-sided minimal surfaces without boundary. If $\{\Sigma_n\}_n$ has an accumulation point in M and uniform local bounds for the second fundamental form and area, then after passing to a subsequence we have:*

(1) The limit set $\mathrm{Lim}(\{\Sigma_n\}_n)$ consists of a properly embedded minimal surface $\Sigma \subset M$.

(2) The Σ_n converge on compact subsets of M to Σ in the C^k topology for each k, with finite multiplicity that is constant in every component of Σ.

If under the hypotheses of Theorem 12.11, the surfaces Σ_n fail to have uniform local bounds of the area, there is still a reasonable structure in the set $\mathrm{Lim}(\{\Sigma_n\}_n)$: the surfaces Σ_n could accumulate at the same time that converge to $\Sigma(p)$ with infinite multiplicity. This produces a minimal lamination of codimension 1 in (M^3, g), in the following sense:

[15]That is, for each $p \in M$ there exists $R > 0$ such that $\mathrm{Area}(\Sigma_n \cap B_M(p, R))$ is bounded by a constant that only depends on p and R. In particular, each Σ_n is proper.

Definition 12.13 A *lamination of codimension 1* in a Riemannian manifold (M^n, g) is the union of a collection of pairwise disjoint, connected, injectively immersed hypersurfaces in M, with a certain local product structure. More precisely, it is a pair $(\mathcal{L}, \mathcal{A})$ that satisfies:

1. $\mathcal{L}$ is a closed subset of M;
2. $\mathcal{A} = \{\varphi_\beta \colon \mathbb{D}^{n-1} \times (0,1) \to U_\beta\}_\beta$ is an atlas of M (here $\mathbb{D}^{n-1}$ denotes the unit ball of $\mathbb{R}^{n-1}$ and U_β is an open subset of M);
3. For each β, there exists a closed subset C_β of $(0,1)$ such that $\varphi_\beta^{-1}(U_\beta \cap \mathcal{L}) = \mathbb{D} \times C_\beta$, see Figure 29.

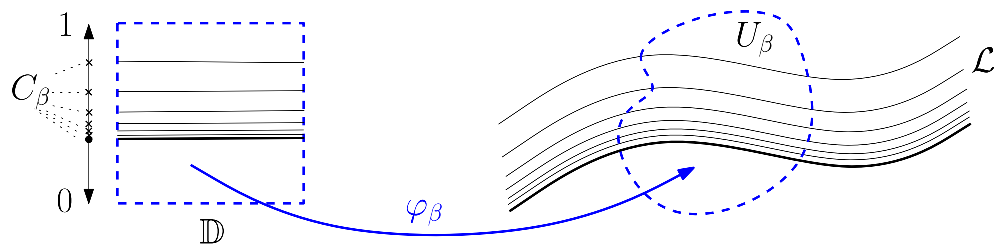


Figure 29: Local product structure of a lamination.

We will denote laminations simply by $\mathcal{L}$, omitting the local charts φ_β in $\mathcal{A}$. A lamination $\mathcal{L}$ is called a (codimension 1) *foliation* of M if $\mathcal{L} = M$. Every lamination $\mathcal{L}$ decomposes naturally into pairwise disjoint connected hypersurfaces (given locally by $\varphi_\beta(\mathbb{D}^{n-1} \times \{t\})$, $t \in C_\beta$, with the notation above), which are called the *leaves* of $\mathcal{L}$.

A codimension 1 lamination is called *minimal* if its leaves are minimal hypersurfaces.

The simplest example of a minimal lamination of $\mathbb{R}^3$ is a collection of parallel planes $\mathcal{L} = \{z = c\}_{c \in C}$, where $C \subset \mathbb{R}$ is a closed subset of the real line.

With the above discussion at hand, we have the following result.

Theorem 12.14 *Let (M^3, g) be a complete and homogeneously regular Riemannian manifold, and let $\{\Sigma_n \subset M\}_n$ be a sequence of complete embedded, two-sided minimal surfaces without boundary. If $\{\Sigma_n\}_n$ has an accumulation point in M and uniform local bounds for the second fundamental form, then after passing to a subsequence the limit set* $\mathrm{Lim}(\{\Sigma_n\}_n)$ *has the structure of a codimension 1 minimal lamination $\mathcal{L}$ of (M^3, g), and given any leaf Σ of $\mathcal{L}$, the Σ_n converge on compact subsets of M to Σ in the C^k topology for each k.*

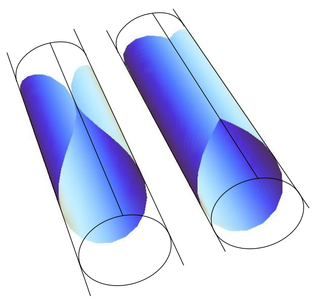
13 Complete minimal surfaces with finite total curvature

Definition 13.1 Let (M^2, g) be a connected, complete Riemannian surface with Gauss curvature K. The *total curvature* of (M, g) is

$$C(M) = \int_M K \, dA.$$

If (M^2, g) is a connected, orientable, complete non-compact Riemannian surface What is the relationship between the total curvature of (M, g) and its topological and/or conformal type?[16]?

In the above situation, take a point $p_0 \in M$ and let $\overline{D}_M(p_0, r)$ be a metric ball of radius $r > 0$ (neither r is necessarily a geodesic radius nor $\overline{D}_M(p_0, r)$ is topologically a disk). In $\overline{D}_M(p_0, r)$ we have radial geodesics starting from p, that might fail to produce local coordinates since r is not necessarily a geodesic radius. $\partial\overline{D}_M(p_0, r)$ consists of a curve or collection of them, with a finite number of vertices $V_1, \ldots, V_{n_r}$, $n_r \in \mathbb{N}$. Since we are assuming M is oriented, we also have an induced orientation in $\partial\overline{D}_M(p_0, r)$. This orientation produces angles $\theta_1^r, \ldots, \theta_{n_r}^r < 0$ at the vertices, as in Figure 30.



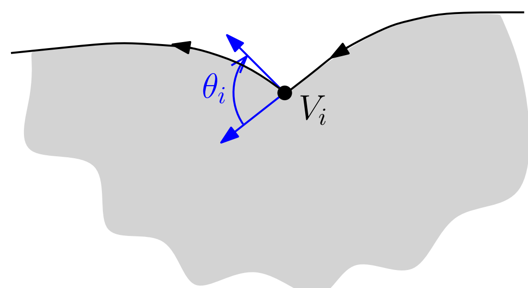


Figure 30: Left: The topology of a metric ball can change as the radius varies, and the boundary does not have to be smooth. Right: angle at a vertex.

Let us call $L(r)$ to the length of $\partial\overline{D}_M(p_0, r)$.

Theorem 13.2 (First variation of length formula) *In the above situation, the function* $r > 0 \mapsto L(r)$ *is of class* C^1 *a.e. in* r, *and*

$$l'(r) = \int_{\partial\overline{D}_M(p_0,r)} \kappa_g(s) \, ds + 2\sum_{i=1}^{n_r} \tan\left(\theta_i^r/2\right), \tag{87}$$

where $l'(r)$ *makes sense, where* κ_g *is the geodesic curvature of* $\overline{D}_M(p_0, r)$ *and* s *is arclength parameter.*

[16] In other words, to what possible models is M homeomorphic and/or conformally equivalent?

Proof. We can view $l(r)$ as the 1-dimensional volume of $\partial D_M(p,r)$, which in turn can be considered to be a variation by curves in (M,g). This variation fails to be smooth at the vertices of $\partial D_M(p_0,r)$. In absence of these vertices, we could apply the first variation of volume formula (Proposition 16.1) and conclude that

$$l'(r) = -\int_{\partial D_M(p_0,r)} \langle X, \vec{H}\rangle ds$$

where X is the variational field of the variation of $\partial D_M(p_0,r)$ by the boundaries of metric balls. We can write this variation in the form $F(r,\xi) = \exp_{p_0}(r\xi)$, where ξ parameterizes the unit sphere $U_{p_0}M = \{\xi \in T_{p_0}M \mid \|\xi\| = 1\}$. Thus, the variational field of F is $X(\xi) = \frac{\partial F}{\partial r}(r,\xi) = \gamma_\xi'(r)$, where γ_ξ is the unique geodesic in M with initial conditions $\gamma_{xi}(0) = p_0$, $\gamma_\xi'(0) = \xi$. By Gauss' Lemma, $X(\xi) = J\Gamma'(s)$, where J is the rotation of angle $\pi/2$ on $T_{\Gamma(s)}M$ and $\Gamma(s)$ is a parameterization by arclength of $\partial D_M(p_0,r)$ around the point of intersection of γ_ξ and $\partial D_M(p_0,r)$ (furthermore, X points outwards $D_M(p_0,r)$). As for $\vec{H} = \vec{H}(s)$, it equals the geodesic curvature vector of Γ:

$$\vec{H}(s) = \vec{\kappa_g}(s) = \nabla^M_{\dot{\Gamma}}\dot{\Gamma},$$

hence

$$\langle X, \vec{H}\rangle = \langle J\Gamma', \vec{\kappa_g}\rangle \overset{(*)}{=} -\kappa_g,$$

where in $(*)$ we have used that $J\Gamma'$ points outwards $D_M(r_0,M)$ and the geodesic curvature vector of $\partial D_M(p_0,r)$ points inwards $D_M(p_0,r)$. Thus, if $\partial D_M(p_0,r)$ had no vertices we would have

$$l'(r) = \int_{\partial D_M(p_0,r)} \kappa_g(s)\, ds.$$

In order to compute the infinitesimal variation of length produced by the vertices, first note that this a order 1 information for $l(r)$ around the vertices, hence we can replace the two arcs meeting at a vertex by straight line segments L_1, L_2 as in Figure 31.

Given $t > 0$, we next analyze how to compute the length $l(r+t)$. As we are not interested in what happens far from the vertex, we can think of just measuring the variation of $l(r+t)$ by measuring the lengths of the new segments $L_1(t), L_2(t)$ parallel to L_1, L_2 at distance t from these last ones, so that the vertices $A(t), B(t)$ of $L_1(t)$ satisfy that the segment $\overline{A, A(t)}$ is perpendicular to L_1, and analogously the vertices $B(t), C(t)$ of $L_2(t)$ make the segment $\overline{C, C(t)}$ perpendicular to L_2, see Figure 31.

Consider the points $B_1 \in L_1$, $B_2 \in L_2$ such that $A, A(t), B(t), B_1$ are the vertices of a rectangle based on L_1; and $C, C(t), B(t), B_2(t)$ are the vertices of another rectangle based on L_2. The points $B(t), B_1, B$ are the vertices of a right triangle (with right angle at B_1); let us call β to the angle of this triangle at B and $\varphi = \frac{\pi}{2} - \beta$ to the angle at $B(t)$. By construction, the points $B(t), B_2, B$ are the vertices of another right triangle, with a right angle at B_2 and

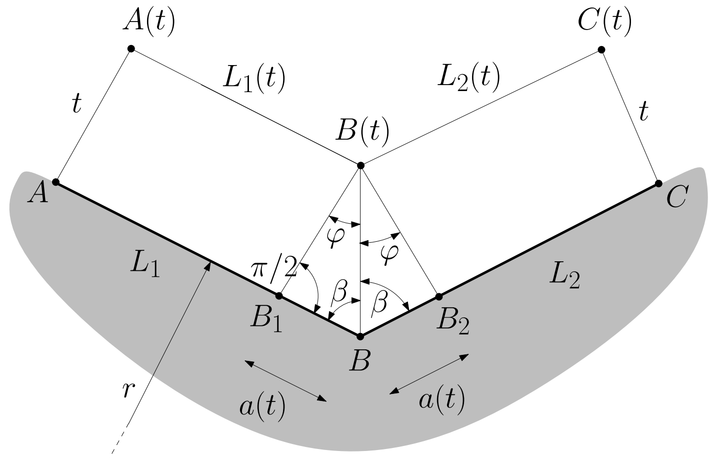


Figure 31: Computing the infinitesimal variation of $l(r)$ around a vertex B. The gray region represents a portion of $D_M(p_0, r)$.

angles β at in B and φ at $B(t)$. If l_0 denotes the length of the polygonal line $L_1 \cup L_2$ and $l(t)$ the length of the polygonal line $L_1(t) \cup L_2(t)$, then

$$l(t) = l_0 - 2a(t),$$

where $a(t)$ is the length of $\overline{B_1, B}$. On the other hand, the external angle θ at B (with the orientation that goes from B_2 to B_1 passing through B) satisfies $2\beta + |\theta| = \pi$. Thus, $\varphi = \frac{\pi}{2} - \beta = \frac{\pi}{2} - \frac{\pi - |\theta|}{2} = \frac{|\theta|}{2}$. As $\tan\varphi = \frac{a(t)}{t}$, we have $a(t) = t\,\tan\varphi = t\,\tan\left(|\theta|/2\right)$ and

$$l(t) = l_0 - 2t\,\tan\left(|\theta|/2\right),$$

hence $l'(0) = -2\tan\left(|\theta|/2\right)$. This proves the formula of Theorem 13.2. $\square$

We keep working with a metric ball $\overline{D}_M(p_0, r)$ is a non-compact, connected, orientable, complete Riemannian surface (M^2, g). The function $x \mapsto \tan(x) - x$ is non-decreasing, as it has non-negative derivative. Since the value of this function at zero is zero, it takes non-positive values when $x < 0$. Since each angle θ_i^r in (87) is negative, we have $\tan\left(\theta_i^r/2\right) \leq \theta_i^r/2$, hence

$$\begin{aligned} l'(r) &\overset{(87)}{=} \int_{\partial \overline{D}_M(p_0,r)} \kappa_g(s)\,ds + 2\sum_{i=1}^{n_r} \tan\left(\theta_i^r/2\right) \leq \int_{\partial \overline{D}_M(p_0,r)} \kappa_g(s)\,ds + \sum_{i=1}^{n_r} \theta_i^r \\ &\overset{(*)}{=} 2\pi\chi\left(D_M(p_0,r)\right) - \int_{D_M(p_0,r)} K\,dA, \end{aligned} \tag{88}$$

where in $(*)$ we have used the Gauss-Bonnet formula, $\chi(D_M(p_0, R))$ is the Euler characteristic of $D_M(p_0, R)$ and K the Gauss curvature of (M^2, g).

Note that $l(r) > 0\ \forall r > 0$ (because since M is connected, complete and non-compact, we have $\partial D_M(p_0, r) \neq M$).

Lemma 13.3 *In the above situation,* $\overline{\lim}_{r\to\infty} l'(r) \geq 0$.

Proof. Arguing by contradiction, suppose that there exists $r_0 > 0$ such that $l'(r) \leq \delta < 0$ a.e. in $[r_0, \infty)$. This implies that there exists $R > r_0$ such that $l(R) = 0$, which contradicts that $l(r) > 0\ \forall r > 0$. $\square$

Using Lemma 13.3 and (88),

$$0 \leq \overline{\lim_{r\to\infty}}\, l'(r) \leq \overline{\lim_{r\to\infty}} \left[2\pi\chi(D_M(p_0, R)) - \int_{D_M(p_0,r)} K\, dA \right] \tag{89}$$

$$= \overline{\lim_{r\to\infty}} \left[2\pi(2 - 2g(r) - C(r)) - \int_{D_M(p_0,r)} K^+\, dA + \int_{D_M(p_0,r)} K^-\, dA \right], \tag{90}$$

where

$$\left. \begin{array}{c} g(r) = \text{genus}(D_M(p_0, r)), \quad C(r) = \#[\text{connected components of } \partial D_M(p_0, r)], \\ K^+ = \max(K, 0), \quad K^- = -\min(K, 0). \end{array} \right\}$$

From now on, we will assume that

$$\int_M K^-\, dA < \infty. \tag{91}$$

(91) gives an upper bound for every positive term of the bracket in (90), independent of r. As the right-hand-side of (90) is ≥ 0, we conclude that all non-positive terms of the bracket in (90) must be bounded from below by a number that is independent of r. Therefore:

$$g(r),\ C(r),\ \int_M K^+\, dA \text{ are bounded from above as functions of } r. \tag{92}$$

A first consequence is:

Lemma 13.4 *If (M^2, g) is non-compact, connected, orientable, complete Riemannian surface with $\int_M K^-\, dA < \infty$, then:*

(1) (M, g) has finite total curvature.

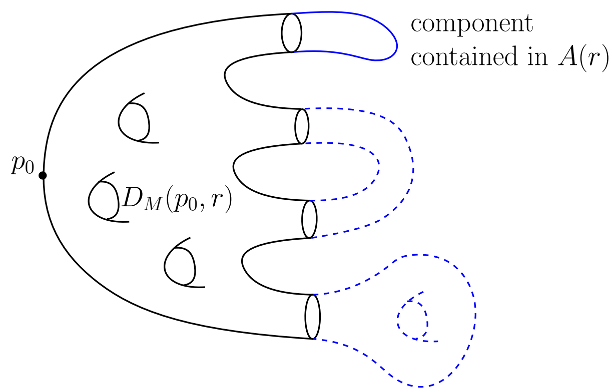


Figure 32: The dotted components of $M \setminus D_M(p_0, r)$ cannot exist.

(2) There exists $r_0 > 0$ such that $g(r) \equiv g(r_0)$ $\forall r \geq r_0$.

From now on we will use the number $r_0 > 0$ that appears in Lemma 13.4. Given $r \geq r_0$, we define

$$A(r) = D_M(p_0, r) \cup [\text{compact components of } M \setminus D_M(p_0, r)] \,.$$

Let us analyze the topology of $A(r)$ for $r \geq r_0$. Given $r \geq r_0$, $\overline{A(r)}$ is compact hence it has finite genus (independent of r by item (2) of Lemma 13.4) and a finite number of boundary components.

Lemma 13.5 *Given $r \geq r_0$, every compact component of $M \setminus D_M(p_0, r)$ is topologically a disk.*

Proof. Let C be a compact component of $M \setminus D_M(p_0, r)$. If ∂C has more then one component, then $D_M(p_0, r) \cup C$ has genus strictly greater than $g(r_0)$. This contradicts that $D_M(p_0, r) \cup C \subset D_M(p_0, r_C)$ for some $r_C > r_0$ and item (2) of Lemma 13.4; therefore, ∂C has only one component. If the genus of C is positive, then $D_M(p_0, r) \cup C$ has genus strictly greater than $g(r_0)$ and we have the same contradiction. $\square$

The only way that the topology of $D_M(p_0, r)$ gets more complicated for $r \geq r_0$ is by means of pieces of genus zero and at least three boundary components (pair of pants), as in Figure 33.

Observe that once a piece bifurcates into two pieces by means of a pair of pants, the two newly created pants cannot join in a posterior stage (i.e., for some $r' > r$) because genus cannot increase ($g(r') = g(r)$). Also note that once a pair of pants appears, the number of components $C(r)$ of $D_M(p_0, r)$ increases by at least one. Since $C(r)$ is bounded from above independently of r), we conclude that there exists $\{r_m\}_m \nearrow \infty$ such that $C(r_m) = k \in \mathbb{N}$ is constant in m. This implies that $A(r_{m+1})$ is the union of $A(r_m)$ with m cylinders.

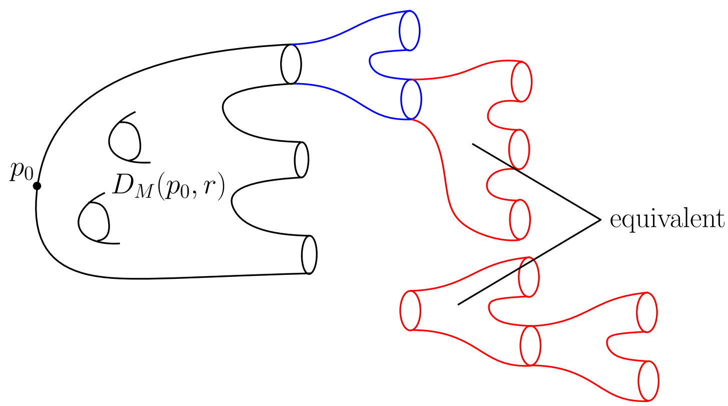


Figure 33: Red components are topologically equivalent.

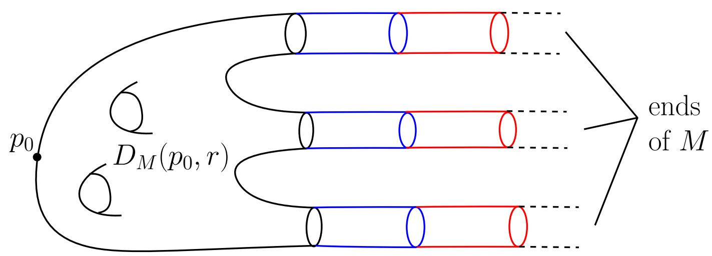


Figure 34: From a radius on, the topology of $D_M(p_0, r)$ remains the same.

If we take $m \to \infty$, we will deduce that M has *finite topology,* i.e., finite genus $g(r_0)$ and a finite number of ends k. Furthermore, taking $r = r_m \to \infty$ in (89) we obtain

$$0 \leq 2\pi\chi(M) - \int_M K\, dA.$$

In summary, we have proven the following result.

Theorem 13.6 (Cohn-Vossen) *Let (M^2, g) be a connected, orientable, complete Riemannian surface with Gauss curvature K. If $\int_M K^-\, dA < \infty$, then M has finite topology, finite total curvature and*

$$\int_M K\, dA \leq 2\pi\chi(M).$$

In view of the above proof, any surface M as above is homeomorphic to a compact surface of genus $g \in \mathbb{N} \cup \{0\}$ with $k \in \mathbb{N}$ points removed, each one forming an end of M. In particular, the k ends of M are *annular*[17]. Regarding the conformal structure of each end, we have two options: either the end is conformally equivalent to a punctured disk $\mathbb{D}^* = \{z \in \mathbb{C} \mid 0 < |z| \leq 1\}$ (*parabolic end*), or to an annulus $A(0; R, 1) = \{z \in \mathbb{C} \mid R < |z| \leq 1\}$ for some $R \in (0, 1)$ (*hyperbolic* end). The following result rules out the hyperbolic case.

Theorem 13.7 (Huber) *Let (M^2, g) be a connected, orientable, complete Riemannian surface with Gauss curvature K. If $\int_M K^- \, dA < \infty$, then each end of M is parabolic and M is conformally equivalent to a compact Riemann surface of genus $g \in \mathbb{N} \cup \{0\}$ with $k \in \mathbb{N}$ points removed.*

Proof. Let E be an end of M, which is annular by the previous discussion. Suppose that E is conformally equivalent to $\{1 \leq |z| < R\}$ for some $R > 1$, let us see that $R = \infty$. Using the same notation son far, we can take $r > r_0$ so that $\partial D_M(p_0, r) \cap E = \Gamma(r)$, where $\Gamma(r)$ is a generator of $\pi_1(E) \equiv \mathbb{Z}$ (note that $\Gamma(r)$ might fail to be smooth).

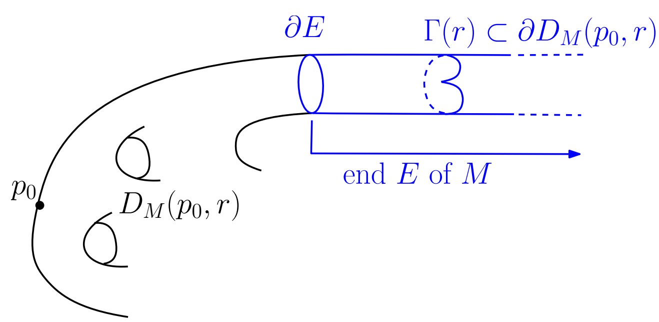


Figure 35: Annular end E of M.

Let $\phi\colon E \to \{1 \leq |z| < R\}$ be a conformal diffeomorphism. $\phi(\Gamma(r))$ is a generator of the fundamental group of $\{|z| \geq 1\}$, hence

$$2\pi = L(\{|z| = 1\}) \leq L[\phi(\Gamma(r))] = \int_{\Gamma(r)} \|d\phi_{\Gamma(r)}(\Gamma(r)')\| ds$$

where s is the arclength parameter of $\Gamma(r)$ (defined except at a finite number of points) and $' = \frac{d}{ds}$. By Schwarz' inequality,

$$4\pi^2 \leq L[\phi(\Gamma(r))]^2 \leq L[\Gamma(r)] \int_{\Gamma(r)} \|d\phi_{\Gamma(r)}(\Gamma(r)')\|^2 ds$$

[17] Diffeomorphic to an annulus.

$$\stackrel{(*)}{=} L[\Gamma(r)] \int_{\Gamma(r)} (\text{Jac}\phi) \|\Gamma(r)'\|^2 ds = L[\Gamma(r)] \int_{\Gamma(r)} \text{Jac}\phi \, ds, \tag{93}$$

where in $(*)$ we have used that ϕ is conformal.

Recall (88):

$$l'(r) \leq 2\pi\chi\left(D_M(p_0, r)\right) - \int_{D_M(p_0,r)} K \, dA,$$

for almost all $r > 0$. By Cohn-Vossen's Theorem, $\chi\left(D_M(p_0, r)\right) = \chi(M)$ for each $r \geq r_1 > 0$, and since the limit of $\int_{D_M(p_0,r)} K \, dA$ as $r \to \infty$ exists and equals $C(M) = \int_M K \, dA \in \mathbb{R}$, we conclude that $l'(r) \leq 2\pi\chi(M) - C(M) + 1$ for all r large enough. Thus, there exists $C > 0$ such that

$$l(r) \leq Cr \quad \text{for all } r > 0. \tag{94}$$

Finally, from (93) and (94) we have

$$\frac{4\pi^2}{Cr} \leq \frac{4\pi^2}{l(r)} \leq \frac{4\pi^2}{L[\Gamma(r)]} \leq \int_{\Gamma(r)} \text{Jac}\phi \, ds$$

for almost all $r > 0$, hence integrating from $r_1 > 1$ to ∞ we get

$$\infty = \frac{4\pi^2}{C} [\log r]_{r_1}^{\infty} = \int_{r_1}^{\infty} \frac{4\pi^2}{Cr} dr \leq \int_{r_1}^{\infty} \left(\int_{\Gamma(r)} \text{Jac}\phi \, ds \right) dr.$$

As the last right-hand-side is the area of $\text{Im}(\phi) = \{1 \leq |z| < R\}$ with the standard flat metric in $\mathbb{R}^2$, we conclude that $R = \infty$. This proves that every end of M is conformally parabolic, and the theorem is proved. □

Lemma 13.8 *Let $\Sigma \subset \mathbb{R}^3$ be an orientable minimal surface. Then, $|C(\Sigma)|$ equals the spherical area of image of Σ by its Gauss map counting multiplicities.*

Proof. $C(\Sigma) = \int_M K \, dA = -\int_\Sigma |K| \, dA = -\int_\Sigma |\text{Jac}N| \, dA$, which is the opposite of the spherical area of $N(\Sigma)$ (counting multiplicities) by the area formula. □

Theorem 13.9 (Osserman) *Let $\Sigma \subset \mathbb{R}^3$ be an orientable complete minimal surface without boundary. Then, either $C(\Sigma) = -\infty$ or $C(\Sigma)$ is a non-positive integer multiple of 4π. In particular, if $C(\Sigma) \in (-4\pi, 0]$ then Σ is a plane.*

Proof. Suppose that $C(\Sigma) > -\infty$. By Huber's Theorem, M is conformally equivalent to a compact Riemann surface $\mathbb{M}_g$ of genus g minus a finite number of points $p_1, \ldots, p_k \in \mathbb{M}_g$. Let $g\colon \Sigma \to \overline{\mathbb{C}}$ be the Gauss map of Σ, stereographically projected from the North pole of $\mathbb{S}^2$. Recall from Section 5 that g is a meromorphic function defined on Σ. Up to a conformal biholomorphism, we can view $g\colon \mathbb{M}_g \setminus \{p_1, \ldots, p_k\} \to \overline{\mathbb{C}}$ as a meromorphic function.

Let us prove that g extends meromorphically across each p_j (i.e., g has at most a pole at p_j, not an essential singularity). Take $j \in \{1, \ldots, k\}$ and let $(U = \mathbb{D}^* = \{0 < |z| \leq 1\}, z)$ be a local isothermal coordinate for $\mathbb{M}_g$ around p_j. We can assume that p_j is the unique end of Σ in $\mathbb{D}^*$. We have $g\colon \mathbb{D}^* \to \overline{\mathbb{C}}$, a meromorphic function. Given $r \in (0,1)$, denote by $\gamma_r(s)$ an arclength parameterization of $\{|z| = r\}$. Then, $L\left(g(\gamma_r)\right) = \int_{\gamma_r} |dg_{\gamma_r}(\gamma_s')|\, ds$, hence by Schwarz' inequality,

$$[L\left(g(\gamma_r)\right)]^2 \leq 2\pi r \int_{\gamma_r} \left\|dg_{\gamma_r}(\gamma_s')\right\|^2 ds.$$

Dividing by $2\pi r$ and integrating in r from 0 to 1,

$$\int_0^1 \frac{[L\left(g(\gamma_r)\right)]^2}{2\pi r} dr \leq \int_0^1 \left(\int_{\gamma_r} \left\|dg_{\gamma_r}(\gamma_s')\right\|^2 ds \right) dr \overset{(*)}{=} 2 \text{ Area}(g(\mathbb{D}^*)),$$

where in $(*)$ we have used that g is conformal. The last right-hand-side is finite by Lemma 13.8. Since $1/r$ is not integrable in $(0,1]$, there exists $\{r_m\}_m \subset (0,1]$ tending to zero such that $L\left(g(\gamma_{r_m})\right) \to 0$ as $m \to \infty$. Take an accumulation point $w_0 \in \overline{\mathbb{C}}$ of $\{g(\gamma_{r_m})\}_m$. Then, for any $\delta > 0$ small, the closed curve $g(\gamma_{r_m})$ is contained in $\mathbb{D}(w_0, \delta)$ para m large enough depending on δ. By the maximum principle for holomorphic functions, the image by g of the annulus bounded by $\gamma_{r_m} \cup \gamma_{r_{m+1}}$ is contained in $\mathbb{D}(w_0, \delta)$ for all m large enough. This implies that g is bounded in a neighborhood of the end p_j. By Liouville's Theorem, g extends meromorphically across p_j, and the same holds for each $j = 1, \ldots, k$.

Finally, the above arguments show that the Gauss map N of Σ extends as a meromorphic map to $\widetilde{N}\colon \mathbb{M}_g \to \mathbb{S}^2$, hence the spherical area of $\widetilde{N}(\mathbb{M}_g)$ is $4\pi \deg(\widetilde{N})$. By Lemma 13.8, this spherical area equals $|C(M)|$ and the proof is complete. □

14 The Gauss map of a complete minimal surface

Recall that the Picard's little theorem asserts that for a non-constant entire function f, the set of values that $f(z)$ assumes is either the whole complex plane or the plane minus a single point. Taking into account the relationship between the Gauss of a complete, orientable minimal surface $\Sigma \subset \mathbb{R}^3$ and the holomorphic function $g\colon \Sigma \to \overline{\mathbb{C}}$ given by (18), the central problem of this section arises naturally:

> ¿How many points of $\mathbb{S}^2$ can the Gauss map $N\colon \Sigma \to \mathbb{S}^2$ of a complete, non-flat, orientable minimal surface $\Sigma \subset \mathbb{R}^3$ omit?

The Gauss map of the catenoid misses two points of $\mathbb{S}^2$, as well as the one of the helicoid. The Gauss map of the singly (or doubly) periodic Scherk minimal surface omits four points, and the one of the Enneper surface omits one point. For a long time, it was conjectured that the answer to the above problem is that the maximum number possible of points omitted for $N(\Sigma)$ is four. This problem boosted the research on minimal surfaces via Complex Analysis. Using the classic Liouville theorem, Osserman deduced that $N(\Sigma)$ must be dense in $\mathbb{S}^2$; later, Osserman improved his own result by proving that $\mathbb{S}^2 \setminus N(\Sigma)$ has zero logarithmic capacity[18]; Xavier proved in 1981 that $N(\Sigma)$ cannot miss seven points; in 1988 López and Ros reduced seven to six, and finally in 1988, Fujimoto gave the final solution proving that $N(\Sigma)$ cannot miss five points. In this section we will give the first result of Osserman mentioned above.

Closely related to the above discussion we have the following open problem:

> ¿How many points of $\mathbb{S}^2$ can the Gauss map $N\colon \Sigma \to \mathbb{S}^2$ of a complete, non-flat, orientable minimal surface with finite total curvature $\Sigma \subset \mathbb{R}^3$ omit?

Some known facts about this problem:

(1) In 1964, Osserman proved that N cannot omit four points of $\mathbb{S}^2$.

(2) In 1987, Weitsman and Xavier proved that ii N omits three points, then $C(\Sigma) \geq -16\pi$ (total curvature), or equivalently, $\deg(N) \geq 4$. In 1993, Fang improved this result proving under the same hypotheses that $\deg(N) \geq 5$.

There are no known examples of total finite curvature whose application of Gauss omits three points of $\mathbb{S}^2$. So far, it is not known if two or three is the maximum number of points that can be omitted by the Gauss map of a complete, non-flat, orientable minimal surface with finite total curvature.

Theorem 14.1 (Osserman) *Let $\Sigma \subset \mathbb{R}^3$ be a complete, non-flat, orientable minimal surface without boundary. Then, the Gauss map image $N(\Sigma)$ is dense in $\mathbb{S}^2$.*

Proof. Let $\pi\colon \widetilde{\Sigma} \to \Sigma$ be the universal cover de Σ. If we endow $\widetilde{\Sigma}$ with the covering metric, π becomes a local isometry and $\widetilde{\Sigma}$ can be considered to be an immersed surface in $\mathbb{R}^3$ in the same conditions of the theorem. Thus, in the sequel we will assume that Σ is simply connected. This implies that Σ is conformally equivalent to $\mathbb{C}$ or to the open unit disk $\mathbb{D}$. Let $g\colon \Sigma \to \overline{\mathbb{C}}$ be the Gauss map of Σ, after stereographic projection from the North pole of $\mathbb{S}^2$.

Arguing by contradiction, suppose that $N(\Sigma)$ is not dense in $\mathbb{S}^2$. Thus, $g(\Sigma)$ omits an open disk of $\overline{\mathbb{C}}$. After a rotation of Σ in $\mathbb{R}^3$, we can assume that thus missed disk is centered at ∞, i.e., there exists $C > 0$ such that

$$|g(z)| \leq C \quad \text{for all } z \in \Sigma. \tag{95}$$

[18] This concept is technical and we will not define it here; we will only say that it refers to the conformal structure of the open set $N(\Sigma)$, and that the complement of any subset of $\mathbb{S}^2$ with zero logarithmic capacity is dense in $\mathbb{S}^2$.

If Σ is conformally $\mathbb{C}$, then g is entire and bounded, hence constant by Liouville's Theorem. This contradicts that Σ is not flat. Hence Σ is conformally equivalent to $\mathbb{D}$.

Let $(g, f dz)$ be the Weierstrass pair of Σ (defined in Section 5). Completeness of the induced metric ds^2 on Σ and equation (17) imply that for any divergent curve $\gamma \subset \Sigma$, it holds

$$\infty = L(\gamma, ds^2) = \int_\gamma ds = \frac{1}{2}\int_\gamma (1+|g|^2)|f||dz| \overset{(95)}{\leq} \frac{(1+C^2)}{2}\int_\gamma |f||dz|,$$

from where we deduce that $|f|^2|dz|^2$ is a complete metric (globally defined) on $\mathbb{D}$. As $|f|^2|dz|^2$ is flat (because f is holomorphic, as can be deduced from the equation $K'e^{2u} = K - \Delta u$ for the Gauss curvatures K, K' of two conformal metrics $g, g' = e^{2u}g$ on a surface), there exists an isometry $\psi\colon (\mathbb{R}^2, g_0) \to (\mathbb{D}, |f|^2|dz|^2)$ (here g_0 denotes the standard inner product on $\mathbb{R}^2$). The composition $\psi \circ g$ is holomorphic in $\mathbb{C}$ and bounded, hence constant by Liouville's Theorem. Hence, g is also constant, which is a contradiction. $\square$

15 Compactness results under uniform local bounds of the total curvature

As in Section 12, we will consider possible limits of a sequence $\{\Sigma_n\}_n$ of embedded minimal surfaces in a Riemannian manifold (M^3, g). Let $\Omega \subset M^3$ a homogeneously regular open subset. Suppose that

(1) $(\Sigma_n, \partial\Sigma_n) \subset (\Omega, \partial\Omega)$ for each $n \in \mathbb{N}$ (Σ_n is properly embedded in Ω).

(2) $\{\Sigma_n\}_n$ has an accumulation point in Ω.

(3) $\{\Sigma_n\}_n$ has uniform local bounds of the second fundamental form in Ω:

$$\forall p \in \Omega, \exists r(p), C(p) > 0 \text{ such that } B_M(p, r(p)) \subset \Omega \text{ and } \left|A_{\Sigma_n \cap B_M(p,r(p))}\right| \leq C(p), \forall n \in \mathbb{N}.$$

In these conditions, the techniques explained in Section 12 allow us to prove that the limit set (after passing to a subsequence) $\mathrm{Lim}(\{\Sigma_n\}_n) := \mathcal{L}$ has the structure of a minimal lamination of codimension 1 of Ω, and that given a leaf Σ of $\mathcal{L}$, the surfaces Σ_n converge on compact subsets of Ω to Σ in the C^k topology for each k. Furthermore, if we additionally have uniform local bounds for the area of $\{\Sigma_n\}_n$ in Ω, then $\mathcal{L}$ reduces to a (possibly non-connected) surface $(\Sigma, \partial\Sigma) \subset (\Omega, \partial\Omega)$ and the convergence of $\{\Sigma_n\}_n$ has finite multiplicity that is constant on every component of Σ.

Theorem 15.1 *Given $C \in (0, 8\pi)$ there exists $C_0 = C_0(C) > 0$ such that if $\{\Sigma_n\}_n$ is a sequence of complete embedded minimal surfaces in $\mathbb{R}^3$ with boundary*[19] *$\partial\Sigma_n \neq \varnothing$, such that*

$$\int_{\Sigma_n} |A_{\Sigma_n}|^2 \leq C,$$

then

$$|A_{\Sigma_n}| \cdot \mathrm{dist}_{\Sigma_n}(\cdot, \partial\Sigma_n) \leq C_0 \quad \text{in } \Sigma_n, \text{ for each } n \in \mathbb{N}. \tag{96}$$

Proof. We claim that we can suppose that Σ_n is compact $\forall n \in \mathbb{N}$: to see this, take $n \in \mathbb{N}$ and consider an exhaustion

$$\Sigma_n^1 \subset \Sigma_n^2 \subset \ldots \subset \Sigma_n^k \subset \ldots \nearrow \Sigma_n$$

If we prove that there exists $C_0 = C_0(C)$ such that (96) holds $\forall n, k \in \mathbb{N}$, then the theorem will be proven.

Given $n \in \mathbb{N}$, let $p_n \in \Sigma_n$ be a maximum of the continuous function

$$f_n \colon \Sigma_n \to [0, \infty), \quad f_n(q) = |A_{\Sigma_n}|(q) \cdot \mathrm{dist}_{\Sigma_n}(q, \partial\Sigma_n), \tag{97}$$

which exists by compactness of Σ_n. Since $f_n = 0$ in $\partial\Sigma_n$, we have $p_n \in \Sigma_n \setminus \partial\Sigma_n$ (for this we need Σ_n to be non-flat, in which case the theorem is trivial).

Assume that (96) fails to hold. This produces a sequence of surfaces Σ_n as in the theorem, and points $p_n \in \Sigma_n$ defined as above, such that $f_n(p_n) \to \infty$.

Translate Σ_n in $\mathbb{R}^3$ in such a way that $p_n = \vec{0}$ for each n (we will keep denoting this point by p_n, we will see the reason when dealing with the general case of (M^3, g)). Let us call

$$\Sigma_n' = |A_{\Sigma_n}(p_n)|\Sigma_n \quad \text{(homothety in } \mathbb{R}^3), \ \forall n \in \mathbb{N}.$$

$\{\Sigma_n'\}_n$ is a sequence of compact embedded minimal surfaces in $\mathbb{R}^3$ with non-empty boundary, each one passing through the origin, with $|A_{\Sigma_n}(\vec{0})| = 1$. Since f_n is invariant under changes of scale, we have

$$\mathrm{dist}_{\Sigma_n'}(\vec{0}, \partial\Sigma_n') = |A_{\Sigma_n'}|(\vec{0}) \cdot \mathrm{dist}_{\Sigma_n'}(\vec{0}, \partial\Sigma_n') = |A_{\Sigma_n}|(p_n) \cdot \mathrm{dist}_{\Sigma_n}(p_n, \partial\Sigma_n) = f_n(p_n) \to \infty, \tag{98}$$

in other words, the boundary of Σ_n' diverges intrinsically. If we knew that $\{\Sigma_n'\}_n$ converges (after passing to a subsequence) to a complete minimal surface $\Sigma_\infty' \subset \mathbb{R}^3$ (without boundary), then we would have $\vec{0} \in \Sigma_\infty'$, $|A_{\Sigma_\infty'}(\vec{0})| = 1$, and

$$\int_{\Sigma_\infty'} |A_{\Sigma_\infty'}|^2 \leq \lim_{n\to\infty} \int_{\Sigma_n'} |A_{\Sigma_n'}|^2 \overset{(*)}{=} \lim_{n\to\infty} \int_{\Sigma_n} |A_{\Sigma_n}|^2 \leq C < 8\pi,$$

[19]This means that the length of each divergent curve in Σ_n is infinite, for each $n \in \mathbb{N}$.

where $(*)$ follows from the change of variables formula applied to the homothety that relates Σ_n and Σ_n'.

Note that for a minimal surface $\Sigma \subset \mathbb{R}^3$, it holds $|A_\Sigma|^2 = -2K$ hence the above inequality is equivalent to $C(\Sigma_\infty') \in (-4\pi, 0]$. By Osserman's Theorem (Theorem 13.9), Σ_∞' is a plane, which contradicts that $|A_{\Sigma_\infty'}(\vec{0})| = 1$.

Thus, in order to finish the proof it suffices to check that a subsequence of $\{\Sigma_n'\}_n$ converges to a complete minimal surface $\Sigma_\infty' \subset \mathbb{R}^3$. To do this, take $R > 0$ and let us see that $|A_{\Sigma_n'}| \leq 2$ in the intrisic ball $B_{\Sigma_n'}(\vec{0}, R)$ for n large enough (depending on R). By (98), for n large enough we have

$$B_{\Sigma_n'}(\vec{0}, R) \subset \Sigma_n' \setminus \partial\Sigma_n'. \tag{99}$$

Take $q' \in B_{\Sigma_n'}(\vec{0}, R)$. Thus, q' is of the form $q' = |A_{\Sigma_n}(p_n)|\, q$ for some $q \in B_{\Sigma_n}(p_n, \frac{R}{|A_{\Sigma_n}|(p_n)})$. The invariance of f_n under homotheties implies

$$|A_{\Sigma_n'}|(q') \cdot \mathrm{dist}_{\Sigma_n'}(q', \partial\Sigma_n') = f_n(q) \leq f_n(p_n) = |A_{\Sigma_n'}|(\vec{0}) \cdot \mathrm{dist}_{\Sigma_n'}(\vec{0}, \partial\Sigma_n') = \mathrm{dist}_{\Sigma_n'}(\vec{0}, \partial\Sigma_n'). \tag{100}$$

On the other hand,

$$\mathrm{dist}_{\Sigma_n'}(\vec{0}, \partial\Sigma_n') \leq \mathrm{dist}_{\Sigma_n'}(\vec{0}, q') + \mathrm{dist}_{\Sigma_n'}(q', \partial\Sigma_n') \leq R + \mathrm{dist}_{\Sigma_n'}(q', \partial\Sigma_n'),$$

hence

$$\mathrm{dist}_{\Sigma_n'}(q', \partial\Sigma_n') \geq \mathrm{dist}_{\Sigma_n'}(\vec{0}, \partial\Sigma_n') - R \overset{(99)}{>} 0. \tag{101}$$

Therefore,

$$|A_{\Sigma_n'}|(q') \overset{(100)}{\leq} \frac{\mathrm{dist}_{\Sigma_n'}(\vec{0}, \partial\Sigma_n')}{\mathrm{dist}_{\Sigma_n'}(q', \partial\Sigma_n')} \overset{(101)}{\leq} \frac{\mathrm{dist}_{\Sigma_n'}(\vec{0}, \partial\Sigma_n')}{\mathrm{dist}_{\Sigma_n'}(\vec{0}, \partial\Sigma_n') - R} \leq 2$$

provided that $\mathrm{dist}_{\Sigma_n'}(\vec{0}, \partial\Sigma_n') \geq 2R$, which holds if n is sufficiently large. As $q' \in B_{\Sigma_n'}(\vec{0}, R)$ is arbitrary, we have $|A_{\Sigma_n'}| \leq 2$ in $B_{\Sigma_n'}(\vec{0}, R)$ for each n large enough. In other words, $\{B_{\Sigma_n'}(\vec{0}, R)\}_n$ admits uniform (global) bounds for their second fundamental forms. Note that Theorem 12.11 assumes uniform local-extrinsic[20] bounds for the second fundamental forms (it also assumes the surfaces in the sequence to be complete without boundary). It is possible to adapt Theorem 12.11 to our current setting in $\{B_{\Sigma_n'}(\vec{0}, R)\}_n$, although we will not see this modified result here. This allows us to conclude that after passing to a subsequence, $\{B_{\Sigma_n'}(\vec{0}, R)\}_n$ converges to a compact minimal surface $\Sigma_\infty'(R)$ with boundary, in the C^k topology for each $k \in \mathbb{N}$. In particular, $\vec{0} \in \Sigma_\infty'(R)$ and $\mathrm{dist}_{\Sigma_\infty'(R)}(\vec{0}, \partial\Sigma_\infty'(R)) = R$.

Now take $R = R_m$ with $m \in \mathbb{N}$ and $R_m \nearrow \infty$, and repeat the above argument. The uniqueness of the limit (more precisely, the identity principle for minimal surfaces[21]) and a

[20]By this we mean that the bound of the norm of the second fundamental form must be independent of n in an extrinsic ball centered at any point (its extrinsic radius depends of the point).

[21]If two minimal surfaces $\Sigma_1, \Sigma_2 \subset \mathbb{R}^3$ coincide in a set with non-empty interior, then $\Sigma_1 \cup \Sigma_2$ is a minimal surface.

diagonal argument imply the convergence of a subsequence of $\{\Sigma'_n\}_n$ (denoted in the same way) to the minimal surface

$$\Sigma'_\infty = \bigcup_{m\in\mathbb{N}} \Sigma'_\infty(R_m) \subset \mathbb{R}^3.$$

This finishes the proof. □

Next we will explain how to generalize Theorem 15.1 to the case in that the ambient space $\mathbb{R}^3$ is replaced by a homogeneously regular 3-manifold.

Theorem 15.2 *Given $C \in (0, 8\pi)$ and $K_0 > 0$, there exists $C_0 = C_0(C, K_0) > 0$ such that if (M_n, g_n) is a sequence of homogeneously regular 3-manifolds with $|K_{\mathrm{sec}}(M_n, g_n)| \leq K_0$ and $\{\Sigma_n \subset M_n\}_n$ is a sequence of complete embedded minimal surfaces with boundary $\partial\Sigma_n \neq \varnothing$, such that*

$$\int_{\Sigma_n} |A_{\Sigma_n}|^2 \leq C,$$

then,

$$|A_{\Sigma_n}| \cdot \min\left\{\mathrm{dist}_{\Sigma_n}(\cdot, \partial\Sigma_n), \frac{\pi}{\sqrt{K_0}}\right\} \leq C_0 \quad \textit{in } \Sigma_n, \textit{ for each } n \in \mathbb{N}. \tag{102}$$

Remark 15.3

1. It is worth commenting about differences between (96) and (102). Let K_0 be a nonnegative real number and let (M, g) be a homogeneously regular 3-manifold with $|K_{\mathrm{sec}}| \leq K_0$. Observe that if for some $C > 0$ there exists $C_0 > 0$ such that an inequality of the type (96) holds for any complete minimal surface $\Sigma \subset M$ with boundary $\partial\Sigma \neq \varnothing$ satisfying $\int_\Sigma |A_\Sigma|^2 \leq C$, then the unique complete minimal surface Σ without boundary such that $\int_\Sigma |A_\Sigma|^2 \leq C$ is totally geodesic (take any point $x_0 \in \Sigma$, any intrinsic radius $r > 0$ at x_0, and apply (96) to the intrinsic ball $B_\Sigma(x_0, r)$ to conclude that $|A_\Sigma|(x_0)r \leq C_0$, and now take $r \to \infty$). This is too restrictive for such an (M, g) (Is the standard Hyperbolic space $\mathbb{H}^3$ a counterexample due to the existence of complete stable minimal surfaces which are not totally geodesic?).

2. For an explanation of the technical point in which the proof of Theorem 15.1 fails when we replace $\mathbb{R}^3$ by a (sequence of) homogeneously regular 3-manifold (M, g) with $|K_{\mathrm{sec}}| \leq K_0$, see Remark 15.4 after the proof of Theorem 15.2.

Proof. Arguing again by contradiction, assume there exist $C \in (0, 8\pi)$, $K_0 > 0$, a sequence (M_n, g_n) of homogeneously regular 3-manifolds with $|K_{\mathrm{sec}}(M_n, g_n)| \leq K_0$ and a sequence of compact[22] minimal surfaces Σ_n in M_n satisfying the hypotheses of the theorem, such that

$$|A_{\Sigma_n}|(q_n) \cdot \min\left\{\mathrm{dist}_{\Sigma_n}(q_n, \partial\Sigma_n), \frac{\pi}{\sqrt{K_0}}\right\} \to \infty \quad \text{as } n \to \infty, \tag{103}$$

[22]The first step of the proof in the case $(M^3, g) = (\mathbb{R}^3, g_0)$ still holds in this setting.

for some points $q_n \in \mathrm{Int}(\Sigma_n)$.

Define $s_n = \mathrm{dist}_{\Sigma_n}(q_n, \partial\Sigma_n)$. We have the following two possibilities for each $n \in \mathbb{N}$:

$(A)_n$ $\frac{\pi}{\sqrt{K_0}} < s_n$.

$(B)_n$ $\frac{\pi}{\sqrt{K_0}} \geq s_n$.

We will explain how to replace each Σ_n by a subset so that after the replacement (and with the same notation), we do not change the value of the second fundamental form at q_n, we do not increase the value of $\int_{\Sigma_n} |A_{\Sigma_n}|^2$, $(B)_n$ holds and $\Sigma_n = \overline{B}_{\Sigma_n}(q_n, s_n)$ for each $n \in \mathbb{N}$: For those $n \in \mathbb{N}$ such that $(A)_n$ holds, we replace Σ_n by $\overline{B}_{\Sigma_n}(q_n, \frac{\pi}{\sqrt{K_0}})$. For those $n \in \mathbb{N}$ such that $(B)_n$ occurs, we replace Σ_n by $\overline{B}_{\Sigma_n}(q_n, s_n)$. Therefore, we can assume from now on that $(B)_n$ holds and $\Sigma_n = \overline{B}_{\Sigma_n}(q_n, s_n)$ for each $n \in \mathbb{N}$.

In particular, (103) reduces to

$$|A_{\Sigma_n}|(q_n) \cdot s_n \to \infty \quad \text{as } n \to \infty. \tag{104}$$

Consider the continuous function (compare with (97)):

$$f_n \colon \Sigma_n = \overline{B}_{\Sigma_n}(q_n, s_n) \to [0, \infty), \quad f_n(q) = |A_{\Sigma_n}|(q) \cdot \mathrm{dist}_{\Sigma_n}(q, \partial\Sigma_n). \tag{105}$$

Let $p_n \in \Sigma_n$ be a point where f_n attains its maximum (observe that $f_n = 0$ at Σ_n).

Define $r_n = \mathrm{dist}_{\Sigma_n}(p_n, \partial\Sigma_n) \in (0, s_n]$. Since case $(B)_n$ holds for each $n \in \mathbb{N}$, then $\{r_n\}_n$ is a bounded sequence. We claim that $|A_{\Sigma_n}|(p_n) \to \infty$:

$$|A_{\Sigma_n}|(p_n) \cdot r_n = f_n(p_n) \geq f_n(q_n) = |A_{\Sigma_n}|(q_n) \cdot s_n \overset{(104)}{\to} \infty. \tag{106}$$

Since $\{r_n\}_n$ is bounded, (106) implies that $|A_{\Sigma_n}|(p_n) \to \infty$ as desired.

Note that for every $x \in B_{\Sigma_n}(p_n, \frac{r_n}{2})$, we have $\mathrm{dist}_{\Sigma_n}(x, \partial\Sigma_n) \geq \frac{r_n}{2}$, and thus,

$$\frac{1}{2}|A_{\Sigma_n}|(x) \cdot r_n \leq |A_{\Sigma_n}|(x) \cdot \mathrm{dist}_{\Sigma_n}(x, \partial\Sigma_n) = f_n(x) \leq f_n(p_n) \overset{(106)}{=} |A_{\Sigma_n}|(p_n) \cdot r_n,$$

from where we deduce that

$$|A_{\Sigma_n}| \leq 2|A_{\Sigma_n}|(p_n) \quad \text{in } B_{\Sigma_n}(p_n, \tfrac{r_n}{2}). \tag{107}$$

Now we make a change of scale in the ambient metric: consider for each $n \in \mathbb{N}$ the metric

$$g'_n = |A_{\Sigma_n}(p_n)|\, g_n, \tag{108}$$

and let

$$\Sigma'_n = \left(B_{\Sigma_n}(p_n, \frac{r_n}{2}), g'_n\right) \subset (M^3, g'_n), \tag{109}$$

which is again a compact minimal surface, embedded in (M^3, g_n') and with non-empty boundary. Furthermore, $p_n \in \Sigma_n'$, $|A_{\Sigma_n'}|(p_n) = 1$ and

$$|A_{\Sigma_n'}| \cdot \mathrm{dist}_{\Sigma_n'}(\cdot, \partial \Sigma_n') = |A_{\Sigma_n}| \cdot \mathrm{dist}_{\Sigma_n}(\cdot, \partial B_{\Sigma_n}(p_n, \frac{r_n}{2})). \tag{110}$$

Next we adapt the argument of (98) to conclude that $\mathrm{dist}_{\Sigma_n'}(p_n, \partial \Sigma_n') \to \infty$ as $n \to \infty$:

$$\begin{array}{rcl} \mathrm{dist}_{\Sigma_n'}(p_n, \partial \Sigma_n') & = & |A_{\Sigma_n'}|(p_n) \cdot \mathrm{dist}_{\Sigma_n'}(p_n, \partial \Sigma_n') \\ & \stackrel{(110)}{=} & |A_{\Sigma_n}|(p_n) \cdot \mathrm{dist}_{\Sigma_n}(p_n, \partial B_{\Sigma_n}(p_n, \frac{r_n}{2})) \\ & = & |A_{\Sigma_n}|(p_n) \cdot \frac{r_n}{2}, \end{array}$$

which tends to ∞ as $n \to \infty$ by (106).

If we knew that a subsequence of $\{\Sigma_n'\}_n$ converges to a complete minimal surface $\Sigma_\infty' \subset \mathbb{R}^3$, we could finish like in the proof of Theorem 15.1. Again let us set $R > 0$. As (M^3, g) is homogeneously regular and the scaling factors $|A(p_n)|(p_n) \to \infty$, after re-scaling the ambient metric we have that the metric ball $B_{(M, g_n')}(p_n, R)$ is arbitrarily close to $\mathbb{B}(\vec{0}, R) \subset \mathbb{R}^3$ for n large enough (depending on R). Therefore, we can consider $\Sigma_n' \cap B_{(M, g_n')}(p_n, R)$ to be a surface in $\mathbb{B}(\vec{0}, R)$, non necessarily minimal, with mean curvature arbitrarily close to zero, and with $|A_{\Sigma_n'}| \leq 2$ by (107). Under these conditions, a modification of the argument of the case $(\mathbb{R}^3, g)$ gives that after passing to a subsequence, the sequence $\left\{\Sigma_n' \cap B_{(M, g_n')}(p_n, R)\right\}_n$ converges in the C^k topology for each $k \in \mathbb{N}$ to a minimal surface (with respect to the limit ambient metric, i.e., the standard flat inner product restricted to $\mathbb{B}(\vec{0}, R)$) denoted by $\Sigma_\infty'(R)$, which is compact with boundary. Now we repeat the diagonal argument with $R = R_m$ and $R_m \nearrow \infty$ and we finish the proof of the theorem. □

Remark 15.4 With the proof of Theorem 15.2 at hand, we can now explain why the proof of Theorem 15.1 fails in the general setting of a (sequence of) homogeneously regular 3-manifold (M, g) with $|K_{\mathrm{sec}}| \leq K_0$, see Remark 15.3.

1. One of the problems that the proof of Theorem 15.1 has in this general setting is that $|A_\Sigma|(p_n)$ could be bounded independently of n (we are using the notation of the proofs of Theorems 15.1 and 15.2), which of course can only happen if $\mathrm{dist}_{\Sigma_n}(p_n, \partial \Sigma_n) \to \infty$ as $n \to \infty$. And if the sequence $\{|A_\Sigma|(p_n)\}_n$ is bounded, when rescaling both the ambient metric as in (108) and the minimal surface as in (109), we cannot insure that the limit of (a subsequence of) the Σ_n' is a minimal surface in $\mathbb{R}^3$.

2. Also note that in the case of $\mathbb{R}^3$ we can apply homoteties, and so we could have normalized the surfaces Σ_n in the proof of Theorem 15.1 so that $\mathrm{dist}_{\Sigma_n}(p_n, \partial \Sigma_n) = 1$. With this normalization, $|A_\Sigma|(p_n) = f_n(p_n) \to \infty$, which avoids the problem mentioned in the previous item of this remark.

Let us continue with our sequence $\{\Sigma_n\}_n$ of complete embedded minimal surfaces with boundary in $\mathbb{R}^3$. Theorem 15.1 tells us that every uniform bound $C < 8\pi$ for the 'total curvature' of the Σ_n produces a bound $C_0 = C_0(C)$ for the function $|A_{\Sigma_n}| \cdot \mathrm{dist}_{\Sigma_n}(\cdot, \partial\Sigma_n)$. It is clear that $C_0(C)$ can be taken non-increasing as a function of C (the pointwise estimate of $|A_{\Sigma_n}| \cdot \mathrm{dist}_{\Sigma_n}(\cdot, \partial\Sigma_n)$ improves if the bound of the total curvature also improves).

Proposition 15.5 *In the above situation, if $C = C_n \to 0$ then $C_0(C_n) \to 0$.*

Proof. Take as before a maximum $p_n \in \Sigma_n$ of the function f_n defined in (97) (we can again start assuming that Σ_n is compact $\forall n$). We know that $\{f_n(p_n)\}_n$ is bounded from above (because $C_0(C)$ is non-increasing), and we want to check that $f_n(p_n) \to 0$ as $n \to \infty$.

Recall that all surfaces Σ_n are contained in $\mathbb{R}^3$. Translate so that $p_n = \vec{0}$ $\forall n \in \mathbb{N}$. Instead of re-scaling by the norm of the second fundamental form at p_n, we will use a different change of scale: consider for each $n \in \mathbb{N}$ the surface

$$\Sigma_n' = \frac{1}{d_{\Sigma_n}(p_n, \partial\Sigma_n)} \Sigma_n.$$

Σ_n' is again a compact minimal surface with boundary in $\mathbb{R}^3$, which now satisfies $\mathrm{dist}_{\Sigma_n'}(\vec{0}, \partial\Sigma_n') = 1$ $\forall n \in \mathbb{N}$. We work slightly away from the boundary of Σ_n': take $\delta \in (0, 1/2)$ and define

$$\overline{\Sigma}_n := \Sigma_n' \setminus B_{\Sigma_n'}(\partial\Sigma_n', \delta). \tag{111}$$

Given $\overline{q} \in \overline{\Sigma}_n$,

$$|A_{\Sigma_n'}|(\overline{q}) \cdot \mathrm{dist}_{\Sigma_n'}(\overline{q}, \partial\Sigma_n') \leq |A_{\Sigma_n'}|(\vec{0}) \cdot \mathrm{dist}_{\Sigma_n'}(\vec{0}, \partial\Sigma_n') = f_n(p_n).$$

But $\mathrm{dist}_{\Sigma_n'}(\overline{q}, \partial\Sigma_n') \geq \delta$, hence

$$\delta |A_{\overline{\Sigma}_n}|(\overline{q}) \leq |A_{\Sigma_n'}|(\overline{q}) \cdot \mathrm{dist}_{\Sigma_n'}(\overline{q}, \partial\Sigma_n') \leq f_n(p_n)$$

Since the above argument holds $\forall \overline{q} \in \overline{\Sigma}_n$ and $\{f_n(p_n)\}_n$ is bounded from above, we conclude like in the proof of Theorem 15.1 that a subsequence of $\{\overline{\Sigma}_n\}_n$ converges to a minimal surface $\overline{\Sigma}_\infty \subset \mathbb{R}^3$ (which is not complete in this case), with $\vec{0} \in \overline{\Sigma}_\infty$, and

$$\int_{\Sigma_\infty} |A_{\overline{\Sigma}_\infty}|^2 = \lim_{n\to\infty} \int_{\overline{\Sigma}_n} |A_{\overline{\Sigma}_n'}|^2 = \lim_{n\to\infty} \int_{\Sigma_n} |A_{\Sigma_n}|^2 \leq C_n \to 0.$$

Therefore, $\overline{\Sigma}_\infty$ is totally geodesic, hence it is contained in a plane. Finally,

$$|A_{\overline{\Sigma}_n}|(\vec{0}) = |A_{\Sigma_n'}|(\vec{0}) = |A_{\Sigma_n'}|(\vec{0}) \cdot \mathrm{dist}_{\Sigma_n'}(\vec{0}, \partial\Sigma_n') = f_n(p_n) = C_0(C_n).$$

Taking limits as $n \to \infty$, the left-hand-side converges to $|A_{\overline{\Sigma}_\infty}|(\vec{0}) = 0$, which concludes the proof of the proposition. $\square$

Let us see what we can conclude in the conditions of Theorem 15.1 if the uniform bound C of the 'total curvature' is arbitrary, not necessarily small: the simplest example is when $\Sigma_n = \frac{1}{n}$Catenoid, which converges with multiplicity 2 to a plane minus one point.

Theorem 15.6 (Weak compactness) *Let (M^3, g) be a complete, homogeneously regular Riemannian manifold, and let $\{\Sigma_n \subset M\}_n$ be a sequence of complete embedded minimal surfaces (possibly with boundary) such that there exists $C > 0$ such that*

$$\int_{\Sigma_n} |A_{\Sigma_n}|^2 \leq C \qquad \forall n \in \mathbb{N}.$$

If $\{\Sigma_n\}_n$ has an accumulation point in M, then there exists a complete embedded minimal surface $\Sigma \subset M$ (possibly with boundary) and a finite number of points $p_1, \ldots, p_k \in \mathrm{Int}(\Sigma)$ such that after passing to a subsequence, $\{\mathrm{Int}(\Sigma_n)\}_n$ converges on compact subsets of $\Sigma \setminus \{p_1, \ldots, p_k\}$ to Σ in the C^k topology for each k. Furthermore, of Σ is not totally geodesic, then the multiplicity of this convergence is finite.

Proof. Given $n \in \mathbb{N}$, consider the measure on M given by

$$\mu_n(B) := \int_{\Sigma_n \cap B} |A_{\Sigma_n}|^2$$

for each measurable subset $B \subset M$. By hypothesis, $\{\mu_n\}_n$ is a sequence of measures on M with bounded total mass $\mu(M) \leq C$ $\forall n \in \mathbb{N}$, hence after passing to a subsequence $\{\mu_n\}_n$ converges weakly[23] to a measure μ over M with total mass $\mu(M) \leq C$. This bound implies that there exist at most a finite number of points $p_1, \ldots, p_k \in M$ such that

$$\lim_{r \to 0^+} \mu\left(B_M(p_j, r)\right) \geq 7\pi, \tag{112}$$

for each $j = 1, \ldots, k$. Now take $q \in M \setminus \{p_1, \ldots, p_m\}$. By (112), there exists $r = r(q) > 0$ such that $\mu\left(B_M(q, r)\right) < 7\pi$, hence for n large enough, $\mu_n\left(B_M(q, r)\right) \leq c < 8\pi$ for some constant $c > 0$ independent of n. Applying Theorem 15.1 to the sequence of surfaces $\{\Sigma_n \cap B_M(q, r/2)\}_n$, we conclude that there exists $C_0(q) > 0$ such that for all $n \in \mathbb{N}$,

$$|A_{\Sigma_n \cap B_M(q,r/2)}| \cdot \mathrm{dist}_{\Sigma_n \cap B(q,r/2)}(\cdot, \partial[\Sigma_n \cap B_M(q, r/2)]) \leq C_0 \quad \text{in } \Sigma_n \cap B_M(q, r/2).$$

As the diameter of $\Sigma_n \cap B_M(q, r/2)$ is bounded independently of $n \in \mathbb{N}$, the above implies that $\{\Sigma_n \cap B_M(q, r/2)\}_n$ admits uniform local bounds of the second fundamental form. Under these conditions, we can adapt Theorem 12.11 (see the proof of Theorem 15.1 for a similar argument) to conclude that after passing to a subsequence, the $\Sigma_n \cap B_M(q, r/2)$ converge to a minimal surface (with boundary) contained in $B_M(q, r/2)$. Finally, move q in $M \setminus \{p_1, \ldots, p_k\}$ and use a diagonal argument to conclude the proof of the theorem. The finite multiplicity of the convergence provided that Σ is not totally geodesic follows from the fact that the total curvature of Σ_n is bounded. $\square$

[23] In certain weak topology; this is essentially a consequence of the Banach-Alaoglú Theorem. We will not enter in details here.

16 Second variation of area formula. Stability

We already know that minimal submanifolds are critical points of the area functional for normal variations with compact support (first variation of area formula). It is then natural to study the second derivative of the area functional at critical points. In this section we will assume that hypersurfaces are two-sided.

Proposition 16.1 (Second variation of area formula) *Let Σ^n be a minimal hypersurface of a Riemannian manifold (M^{n+1}, g). Suppose that Σ admits a unit normal field $N\colon \Sigma \to U_1(M)$ (i.e. Σ is two-sided). Let $f \in C_0^\infty(\Sigma)$ and $F\colon \Sigma \times (-\varepsilon, \varepsilon) \to M$ be a variation of Σ with variational field $\frac{\partial F}{\partial t}(p, 0) = f(p)N_p$, $\forall p \in \Sigma$. Then, the n-dimensional volume function $t \mapsto A(F_t)$ of the variation satisfies $A'(0) = 0$ and*

$$A''(0) = \left.\frac{d^2}{dt^2}\right|_{t=0} A(F_t) = -\int_\Sigma f L f \, dA, \tag{113}$$

where $L = \Delta_\Sigma + |A_\Sigma|^2 + \mathrm{Ric}_M(N)$ is the Jacobi operator de Σ.

(We omit the proof, which is similar to the one of Proposition 16.1).

Definition 16.2 Let Σ^n be a two-sided minimal hypersurface of a Riemannian manifold (M^{n+1}, g), with unit normal field $N\colon \Sigma \to U_1(M)$. Σ is called *stable* if for every normal variation of Σ with compact support, the n-dimensional volume function of the variation satisfies $A''(0) \geq 0$. Equivalently,

$$-\int_\Sigma f L f \, dA \geq 0, \qquad \text{for all } f \in C_0^\infty(\Sigma). \tag{114}$$

Every local minimum Σ^n of the n-dimensional volume satisfies $A(t) \geq A(0)$ for all t sufficiently close to zero, and hence Σ is stable.

L is a *Schrödinger operator* with potential $q = |A_\Sigma|^2 + \mathrm{Ric}_M(N) \in C^\infty(\Sigma)$. This is a particular case of a second order elliptic operator, that is self-adjoint with respect to the usual inner product in the Hilbert space $L^2(\Sigma)$. In this section we will apply some basic results of classical spectral theory for elliptic self-adjoint operators of second order. For instance, (114) is equivalent to

$$-\int_\Sigma f L f \, dA \geq 0, \qquad \text{for all } f \in H_0^1(\Sigma), \tag{115}$$

where $H_0^1(\Sigma)$ denotes the closure of $C_0^\infty(\Sigma)$ in the topology generated by the *Sobolev norm,*

$$\|f\|_{H^1(\Sigma)} = \left(\|f\|^2_{L^2(\Sigma)} + \|\nabla f\|^2_{\mathcal{L}^2(\Sigma)}\right)^{1/2},$$

where f varies in the Sobolev space $H^1(\Sigma)$, i.e., the Hilbert space of all functions in $L^2(\Sigma)$ that admit a L^2 vector field (called *weak gradient* of f) $\nabla f \in \mathcal{L}^2(\Sigma)$ satisfying the following *compatibility condition:*

$$\int_\Sigma (g(\nabla f, X) + f\mathrm{div}(X)) = 0,$$

for every compactly supported, smooth vector field X on Σ.

The *quadratic form* associated to L is $Q\colon H_0^1(\Sigma)\times H_0^1(\Sigma)\to\mathbb{R}$ given by

$$Q(u,v) = -\int_\Sigma uLv = -\int_\Sigma vLu, \quad \forall u,v\in H_0^1(\Sigma). \tag{116}$$

Using the expression of L, we can re-write (115) for $f\in C_0^\infty(\Sigma)$ as

$$\int_\Sigma \left[|A_\Sigma|^2 + \mathrm{Ric}_M(N)\right] f^2 \leq -\int_\Sigma f\Delta f \overset{(*)}{=} \int_\Sigma \|\nabla f\|^2, \tag{117}$$

where in $(*)$ we have used the Divergence Theorem. The inequality (117) is continuous with respect to the norm $\|\cdot\|_{H^1}$, hence (117) is equivalent to the same inequality for all $f\in H_0^1(\Sigma)$.

Theorem 16.3 *If (M^{n+1}, g) is a Riemannian manifold with non-negative Ricci curvature, then every compact stable minimal hypersurface without boundary in M is totally geodesic. If the Ricci curvature of M is positive, then there are no compact stable minimal hypersurfaces without boundary in M.*

Proof. Suppose that Σ^n is a compact stable minimal hypersurface without boundary in M. As Σ is compact, we can take $f\equiv 1$ in (117), hence $\int_\Sigma \left[|A_\Sigma|^2 + \mathrm{Ric}_M(N)\right] \leq 0$. As $\mathrm{Ric}_M(N)\geq 0$, the left-hand-side of the last inequality is ≥ 0, and thus, $|A_\Sigma|^2 + \mathrm{Ric}_M(N)\equiv 0$ in Σ. From here we have $|A_\Sigma|^2\equiv 0$ in Σ (i.e., Σ is totally geodesic) and $\mathrm{Ric}_M(N)\equiv 0$ in Σ. This last property is impossible if M has positive Ricci curvature. □

The last result applies to $\mathbb{S}^{n+1}$ and $\mathbb{P}^{n+1}$ (every equator of $\mathbb{S}^{n+1}$ is a compact minimal hypersurface, but it cannot be stable because we can always deform it by decreasing its n-dimensional volume).

Let Σ be a surface in a Riemannian manifold (M^3, g). Let us call $S = \mathrm{trace}(\mathrm{Ric}_M)$ to the scalar curvature of g. Take a local orthonormal basis of TM of the type $\{e_1, e_2, N\}$, where e_1, e_2 is a local orthonormal basis of $T\Sigma$ and N is a unit normal field to Σ. Then,

$$\left.\begin{array}{rcl} S & = & \mathrm{Ric}_M(e_1) + \mathrm{Ric}(e_2) + \mathrm{Ric}_M(N) \\ & = & [K(e_1,e_2) + K(e_1,N)] + [K(e_1,e_2) + K(e_2,N)] + \mathrm{Ric}_M(N) \\ & = & 2[(K(T\Sigma) + \mathrm{Ric}_M(N)], \end{array}\right\} \tag{118}$$

where $K(T\Sigma)$ is the sectional curvature of $T\Sigma$. Assume that Σ is minimal and denote by $K_e = \det A_\Sigma$ the *extrinsic curvature* of Σ. The Gauss equation can be written as

$$K_\Sigma = K(T\Sigma) + \det A_\Sigma = K(T\Sigma) + K_e \overset{(*)}{=} K(T\Sigma) - \frac{1}{2}|A_\Sigma|^2, \tag{119}$$

where in $(*)$ we have used that Σ is minimal. From (118) and (119) we have

$$S = 2K_\Sigma + 2\mathrm{Ric}_M(N) + |A_\Sigma|^2.$$

Thus, (117) and the above equation imply that stability of Σ in (M^3, g) is equivalent to

$$\int_\Sigma \left[\frac{1}{2}\left(S + |A_\Sigma|^2\right) - K_\Sigma\right] f^2 \leq \int_\Sigma \|\nabla f\|^2, \quad \text{for all } f \in H_0^1(\Sigma). \tag{120}$$

Theorem 16.4 *Let Σ be a compact stable minimal surface without boundary in a Riemannian manifold (M^3, g) whose scalar curvature is non-negative. Then, $\chi(\Sigma) \geq 0$, hence Σ is diffeomorphic to a sphere, torus, projective plane or Klein bottle. Furthermore, if Σ is diffeomorphic to a torus or Klein bottle, then Σ is totally geodesic and the scalar curvature of M vanishes along Σ.*

Proof. Since Σ is compact, we can take $f \equiv 1$ in (120), thus

$$\frac{1}{2}\int_\Sigma \left(S + |A_\Sigma|^2\right) \leq \int_\Sigma K_\Sigma \overset{\text{(Gauss-Bonnet)}}{=} 2\pi\chi(\Sigma),$$

The rest of the argument is analogous to that of the end of the proof of Theorem 16.3. $\square$

16.1 Jacobi functions

Recall that the stability of a two-sided minimal hypersurface Σ^n in a Riemannian manifold (M^{n+1}, g) is equivalent to the fact that its Jacobi operator $L = \Delta_\Sigma + |A_\Sigma|^2 + \mathrm{Ric}_M(N)$ satisfies $-L \geq 0$, in the sense that any of the equivalent inequalities (114), (115) or (117). A *Jacobi function* is a function in the kernel of L.

In what follows we will use some results of the classical spectral theory of self-adjoint elliptic operators of second order, adapted to the our situation. Whenever possible, we will work in the following general situation: Let (Σ^n, g) be a Riemannian manifold, $q \in C^\infty(\Sigma)$ and $L = \Delta + q$.

Given a relatively compact domain $\Omega \subset \Sigma$, we define the *spectrum* of L in Ω as the sequence of real numbers (called *eigenvalues* of L in Ω)

$$\mathrm{Spec}(L, \Omega) = \{\lambda_1 < \lambda_2 \leq \ldots \leq \lambda_k \leq \ldots\} \nearrow \infty$$

such that $\forall k \in \mathbb{N}$ there exists $\varphi_k \in C^\infty(\Omega) \cap H_0^1(\Omega)$ (eigenfunction associated to the eigenvalue λ_k) such that $L\varphi_k + \lambda_k\varphi_k = 0$ in Ω, with $\|\varphi_k\|_{L^2} = 1$ and $\varphi_k \perp_{L^2} \varphi_h$ whenever $k \neq h$. Furthermore, $\{\varphi_k\}_k$ is a Hilbert basis of $L^2(\Omega)$.

Given $\lambda_k \in \mathrm{Spec}(L, \Omega)$, we will call

$$V_{\lambda_k} = \{\varphi \in C^\infty(\Omega) \cap H_0^1(\Omega) \mid L\varphi + \lambda_k \varphi = 0\}$$

to the (linear) *eigenspace* associated to the eigenvalue λ_k. The dimension of V_{λ_k} coincides with the multiplicity of λ_k in the spectrum $\mathrm{Spec}(L, \Omega)$, and the *first eigenvalue* λ_1 admits the following variational characterization in terms of the *Rayleigh quotient,*

$$\lambda_1(L, \Omega) = \inf \left\{ \frac{Q(u,u)}{\|u\|_{L^2}^2} \ : u \in H_0^1(\Omega) \setminus \{0\} \right\} = \inf \left\{ - \int_\Omega uLu \ : \ u \in H_0^1(\Omega), \ \|u\|_{L^2}^2 = 1 \right\}, \tag{121}$$

where Q is the index form given by (116).

Lemma 16.5 *In the above situation,*

(1) Σ is stable if and only if $\lambda_1(L, \Omega) \geq 0$, $\forall \Omega \subset\subset \Sigma$.

(2) $Q(u, u) \geq \lambda_1(L, \Omega) \int_\Omega u^2$, $\forall u \in H_0^1(\Omega)$ and equality holds if and only if $u \in V_{\lambda_1}$.

(3) Si $u \in V_{\lambda_1}$, then u does not change sign in Ω.

Proof. Item (1) can be directly deduced from (121), as well as the inequality in item (2). Equality in the inequality of item (2) implies that given $v \in H_0^1(\Omega)$, the function $f(t) = Q(u + tv, u + tv) - \lambda_1(L, \Omega) \int_\Omega (u + tv)^2$, satisfies $f(0) = 0$, $f(t) \geq 0$ $\forall t \in \mathbb{R}$, hence $f'(0) = 0$. Since

$$f'(0) = 2 \left(Q(u, v) - \lambda_1(L, \Omega) \int_\Omega uv \right),$$

we deduce that $u \in V_{\lambda_1(L,\Omega)}$.

regarding item (3), if $u \in V_{\lambda_1}$ then $u \in H_0^1(\Omega)$ and $|u| \in H_0^1(\Omega)$. Since $Q(|u|, |u|) = Q(u, u)$, then equality holds in the inequality of item (2) for $|u|$, and so, $|u| \in V_{\lambda_1}$. By elliptic regularity, every eigenfunction is smooth, hence $|u|$ cannot have zeros in Ω unless $u = 0$. □

Take $\Omega \subset\subset \Sigma$. For $k > 1$, every function $u \in V_{\lambda_k(\Omega)}$ changes sign: this follows from the fact that $u \perp_{L^2} V_{\lambda_1(\Omega)}$ and from item (3) of Lemma 16.5. Therefore, if there exists $u \in V_{\lambda=0}$ (a Jacobi function) such that $u > 0$, then $\lambda_1(L, \Omega) = 0$ and Ω is stable. The next lemma extends this result without assuming that $u \in H_0^1(\Omega)$ (u does not have to vanish at $\partial\Omega$).

Lemma 16.6 *Let $\Omega \subset \Sigma$ be a relatively compact domain. If there exists $u \in C^\infty(\Omega)$ such that $u > 0$ and $Lu = 0$, then $\lambda_1(L, \Omega) = 0$ and Ω is stable.*

Proof. Take $f \in C_0^\infty(\Omega)$. Let us see that $\int_\Omega qf^2 \leq \int_\Omega \|\nabla f\|^2$ and Ω will be stable.

Define $w = \log u \in C^\infty(\Omega)$. Thus, $\nabla w = \frac{\nabla u}{u}$ and $\Delta w = \operatorname{div}\left(\frac{\nabla u}{u}\right) = \frac{\Delta u}{u} - \frac{\|\nabla u\|^2}{u^2} = \frac{\Delta u}{u} - \|\nabla w\|^2 = -q - \|\nabla w\|^2$, where $L = \Delta + q$. Integrating $f^2 \Delta w = -qf^2 - \|\nabla w\|^2 f^2$ and using the Divergence Theorem with the vector field $f^2 \nabla w$ we obtain

$$\int_\Omega qf^2 + \int_\Omega \|\nabla w\|^2 f^2 = -\int_\Omega f^2 \Delta w = 2\int_\Omega f \langle \nabla f, \nabla w \rangle$$
$$\leq 2\int_\Omega f \|\nabla f\| \|\nabla w\| \leq \int_\Omega f^2 \|\nabla w\|^2 + \int_\Omega \|\nabla f\|^2.$$

and now we only have to cancel $\int_\Omega \|\nabla w\|^2 f^2$ at both sides. □

Proposition 16.7 *Given $\Omega \subset\subset \Sigma$, the dimension of V_{λ_1} is 1.*

Proof. If $u, v \in V_{\lambda_1}$, then $u - av \in V_{\lambda_1}$ for each $a \in \mathbb{R}$. By item (3) of Lemma 16.5, $u - av$ does not change sign in Ω. Choose $a_0 \in \mathbb{R}$ so that $u - a_0 v$ has a zero at a prescribed point of Ω. Hence we conclude that $u = a_0 v$. □

The following *monotonicity property* for the first eigenvalue is easy to check:

Lemma 16.8 *If $\Omega_1 \subset \Omega_2$ are relatively compact open subsets of Σ, then $\lambda_1(L, \Omega_2) \leq \lambda_1(L, \Omega_1)$.*

Definition 16.9 The *first eigenvalue* of L in Σ is defined by

$$\lambda_1(L, \Sigma) = \lim_{\substack{\Omega \nearrow \Sigma \\ \Omega \subset\subset \Sigma}} \lambda_1(L, \Omega) = \inf \{\lambda_1(L, \Omega) \ : \ \Omega \subset\subset \Sigma\},$$

and stability of Σ is equivalent to $\lambda_1(L, \Sigma) \geq 0$.

Theorem 16.10 *Let (Σ^n, g) be a Riemannian manifold, $q \in C^\infty(\Sigma)$ and $L = \Delta + q$. If there exists $u \in C^\infty(\Sigma)$ such that $Lu = 0$ and $u > 0$ in Σ, then $-L \geq 0$ in Σ.*

Proof. It suffices to prove that $\lambda_1(L, \Omega) = 0$ for every relatively compact open set Ω of Σ. This equality is a consequence of applying Lemma 16.6 to $u|_\Omega$. □

Remark 16.11 The converse of Theorem 16.10 holds.

A way of having the hypotheses of Theorem 16.10 in the geometric case of a minimal surface $\Sigma \subset (M^3, g)$ is the following one.

Theorem 16.12 *Let Y be a nowhere vanishing Killing field in a Riemannian manifold (M^{n+1}, g), and let $\Sigma \subset M$ be a two-sided minimal hypersurface such that Y is transversal to Σ (i.e., $\langle Y, N\rangle$ has no zeros in Σ). Then, Σ is stable.*

Proof. Let N be a unit normal field to Σ. Given $f \in C_0^\infty(\Sigma)$, the first variation formula for the mean curvature for a variation ψ_t of Σ with normal part of its variational field fN, is given by

$$2 \left.\frac{d}{dt}\right|_0 H_t = Lf.$$

Since Y is a Killing field on M, the 1-parameter group $\{\psi_t\}_t$ of Y consists of isometries of (M, g). Thus, the mean curvature H_t of $\psi_t(\Sigma)$ is zero. Thus, $L\langle Y, N\rangle = 0$, i.e., $\langle Y, N\rangle$ is a Jacobi function on Σ. Now the theorem holds by applying Theorem 16.10 to this Jacobi function. $\square$

For example, the axis of a helicoid divides the helicoid into two (congruent) stable minimal surfaces, since the rotations around that axis generate a Killing field on $\mathbb{R}^3$ that satisfies the hypotheses of Theorem 16.12 on each of these two halves of the helicoid. Another example is a minimal graph over the vertical projection in a product manifold $M^2 \times \mathbb{R}$: just take $Y = \partial_t$, which is a Killing field in the product metric product on $M \times \mathbb{R}$, and is transverse to all such graphs.

Lemma 16.13 (Harnack inequality for $\Delta + q$)
If $f \in C^\infty(\Sigma)$ satisfies $Lf = 0$ and $f > 0$ in Σ, then given a compact subset $K \subset \Sigma$ there exists $C = C(K) > 0$ such that $\sup_K f \leq C \cdot \inf_K f$.

Proof. See Theorem 8.20 in [8] for a version of the Harnack's inequality for an open subset of $\mathbb{R}^n$ and a uniformly elliptic operator with measurable coefficients. In order to pass this result to a manifold Σ, we first cover the compact set K by a finite number of charts, and then apply the above result in $\mathbb{R}^n$. This gives a finite number of positive constants $C > 0$. Taking the minimum of all these constants we will get a Harnack inequality for the compact subset K. $\square$

Theorem 16.14 (Maximum principle for $\Delta + q$)
Let (Σ^n, g) be a Riemannian manifold, $q \in C^\infty(\Sigma)$ and $L = \Delta + q$. If $f \in C^\infty(\Sigma)$ satisfies $Lf = 0$ and $f \geq 0$ in Σ, then $f = 0$ or $f \geq 0$ in Σ.

Proof. Suppose that f vanishes at some point $p_0 \in \Sigma$. Take a compact subset $K \subset \Sigma$ that $p_0 \in K$. Then, Harnack's inequality implies $0 \leq \sup_K f \leq C \cdot \inf_K f = f(p_0) = 0$, hence $f \equiv 0$ in K. Since K is any compact subset of Σ that contains p_0, we conclude that $f \equiv 0$ in Σ. $\square$

We conclude this section with the following comment about stability and covering spaces. If (Σ^n, g) is a Riemannian manifold, $q \in C^\infty(\Sigma)$ and $L = \Delta + q$ is an operator such that $-L \geq 0$, then after lifting L through a covering map $\pi\colon \widetilde{\Sigma} \to \Sigma$ (i.e., $\widetilde{L} = \widetilde{\Delta} + \widetilde{q}$ with $\widetilde{g}$ being the covering metric and $\widetilde{q} = q \circ \pi$), then $-\widetilde{L} \geq 0$. In other words, *stability is preserved by lifting to a cover space.*

Nevertheless, the converse fails in general (the next counterexample is due to Schoen): take a compact Riemannian surface (Σ, g) with Gauss curvature $K \equiv -1$. Take a function $f\colon \mathbb{R} \to (0, 1]$ such that $f(0) = 1$ and $-\frac{1}{8} < f''(0) < 0$. The first eigenvalue of the operator $L := \Delta_\Sigma - 2f''(0)$ in Σ is $2f''(0) < 0$, hence $-L$ does not satisfy $-L \geq 0$ (L is not stable). But the universal cover of Σ is isometric to the hyperbolic plane, that has $\lambda_1(\Delta, \mathbb{H}^2) = 1/4$. Thus, the first eigenvalue of $\widetilde{L}$ is $\frac{1}{4} + 2f''(0) > 0$, i.e., $-\widetilde{L} \geq 0$ in $\mathbb{H}^2$. This operator L can be seen as the Jacobi operator of a minimal surface in a Riemannian manifold (consider the *warped product* $(\Sigma \times \mathbb{R}, f^2 g + dt^2)$, that has $\Sigma \times \{0\}$ as a totally geodesic surface with Jacobi operator $L = \Delta + \mathrm{Ric}(\partial_t) = \Delta - 2f''(0)$).

17 Parabolicity and area growth

Definition 17.1 A Riemannian manifold without boundary (Σ^n, g) ia called *parabolic* if it does not admit non-constant, non-negative superharmonic functions:

If $u \in C^\infty(\Sigma)$ satisfies $u \geq 0$ and $\Delta u \leq 0$, then u is constant.

Remark 17.2 (1) The above definition is equivalent to the non-existence of non-constant positive superharmonic functions, by the classical minimum principle for subharmonic functions.

(2) The notion of parabolicity is independent of the metric g in Σ, it only depends on the conformal class of g.

Definition 17.3 A Riemannian surface without boundary (Σ, g) is said to have *quadratic area growth* if there exists[24] $x_0 \in \Sigma$ and $C = C(x_0) > 0$ such that $A(B_\Sigma(x_0, r)) \leq Cr^2$ for all $r > 0$.

Theorem 17.4 *If a connected Riemannian surface without boundary (Σ, g) has quadratic area growth, then Σ is parabolic.*

Proof. By item (1) of Remark 17.2, it suffices to prove that if $u \in C^\infty(\Sigma)$ satisfies $u > 0$ and $\Delta u \leq 0$, then u is constant.

Let $w = \log u \in C^\infty(\Omega)$. Thus, $\nabla w = \frac{\nabla u}{u}$ and

$$\Delta w = \frac{\Delta u}{u} - \frac{\|\nabla u\|^2}{u^2} = \frac{\Delta u}{u} - \|\nabla w\|^2 \leq -\|\nabla w\|^2. \tag{122}$$

[24] It is not difficult to prove that if $\exists x_0 \in \Sigma$, $C > 0$ such that $A(B_\Sigma(x_0, r)) \leq Cr^2$ for all $r > 0$, then for each $x \in \Sigma$ the function $r > 0 \mapsto A(B_\Sigma(x_0, r))/r^2$ is bounded.

Given $x_0 \in \Sigma$ and $R > 0$, consider the *logarithmic cut-off function* $\psi_R \colon \Sigma \to [0, \infty)$ given by:

$$\psi_R(r) = \begin{cases} 1 & \text{if } 0 \le r \le \sqrt{R}, \\ 2 - \frac{\log(r^2)}{\log R} & \text{if } \sqrt{R} \le r \le R, \\ 0 & \text{if } r \ge R, \end{cases} \tag{123}$$

where $r = \mathrm{dist}_\Sigma(x_0, \cdot)$.

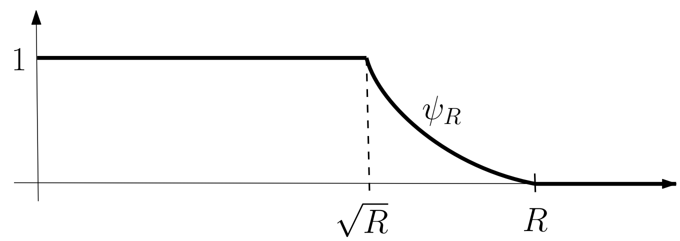


Figure 36: The logarithmic cut-off function ψ_R.

Thus, $\psi_R \in H^1_0(\Sigma)$ and its weak gradient is

$$(\nabla \psi_R)(r) = \begin{cases} 0 & \text{if } 0 < r < \sqrt{R} \text{ or } r > R, \\ -\frac{2}{\log R}\frac{\nabla r}{r} & \text{if } \sqrt{R} < r < R. \end{cases}$$

Hence, $\|\nabla \psi_R\| = \frac{2}{\log R}\frac{1}{r}$ in $\{\sqrt{R} < r < R\}$ and $\|\nabla \psi_R\| = 0$ away from this last set.

Let us denote $B(r) = B_\Sigma(x_0, r)$ for each $r > 0$. Using (122), we have

$$\int_{B(R)} \psi_R^2 \|\nabla w\|^2 \le -\int_{B(R)} \psi_R^2 \Delta w \overset{(*)}{=} 2\int_{B(R)} \psi_R \langle \nabla \psi_R, \nabla w\rangle$$

$$\le 2\int_{B(R)} \psi_R \|\nabla \psi_R\| \|\nabla w\| \overset{(**)}{\le} \frac{1}{2}\int_{B(R)} \psi_R^2 \|\nabla w\|^2 + 2\int_{B(R)} \|\nabla \psi_R\|^2,$$

where in $(*)$ we have applied the Divergence Theorem to the vector field $\psi_R^2 \nabla w$, and in $(**)$ we have used that $2ab \le \frac{1}{2}a^2 + 2b^2$ where $a = \psi_R \|\nabla w\|$ and $b = \|\nabla \psi_R\|$. Therefore,

$$\frac{1}{2}\int_{B(R)} \psi_R^2 \|\nabla w\|^2 \le 2\int_{B(R)} \|\nabla \psi_R\|^2. \tag{124}$$

On the other hand,

$$\int_{B(\sqrt{R})} \|\nabla w\|^2 = \int_{B(\sqrt{R})} \psi_R^2 \|\nabla w\|^2 \le \int_{B(R)} \psi_R^2 \|\nabla w\|^2 \overset{(124)}{\le} 4\int_{B(R)} \|\nabla \psi_R\|^2 \le \frac{16}{(\log R)^2}\int_{B(R)} \frac{1}{r^2}, \tag{125}$$

If we check that $\int_{B(R)} \frac{1}{r^2} \leq C_1 \log R$ for some $C_1 > 0$ independent of R, then we will get

$$\int_{B(\sqrt{R})} \|\nabla w\|^2 \overset{(125)}{\leq} \frac{16C_1}{\log R} \overset{(R\to\infty)}{\longrightarrow} 0$$

hence w will be constant in Σ and so u. We will split the integral $\int_{B(R)} \frac{1}{r^2}$ in a sum of integrals in annular regions:

$$\int_{B(R)} \frac{1}{r^2} = \sum_{j=0}^{m-1} \int_{B(e^{j+1}\sqrt{R})\setminus B(e^j\sqrt{R})} \frac{1}{r^2}$$

where $e^m\sqrt{R} = R$ or equivalently, $m = \log\sqrt{R}$. Since m is a positive integer, R must take values in a sequence (tending to ∞), which does not affect the above reasoning. Since $r \geq e^j\sqrt{R}$ in each one of the above annuli, we can estimate the last right-hand-side from above by

$$\sum_{j=0}^{m-1} \int_{B(e^{j+1}\sqrt{R})\setminus B(e^j\sqrt{R})} \frac{1}{Re^{2j}} \leq \sum_{j=0}^{m-1} \int_{B(e^{j+1}\sqrt{R})} \frac{1}{Re^{2j}} = \sum_{j=0}^{m-1} \frac{1}{Re^{2j}} A(B(e^{j+1}\sqrt{R})). \tag{126}$$

Since Σ has quadratic area growth, there exists $C > 0$ such that $A(B(e^{j+1}\sqrt{R})) \leq Ce^{2(j+1)}R$, and the right-hand-side of (126) can be bound from above by

$$\sum_{j=0}^{m-1} \frac{Ce^{2(j+1)}R}{Re^{2j}} = \sum_{j=0}^{m-1} Ce^2 = Ce^2 m = Ce^2 \log\sqrt{R},$$

hence taking $C_1 = \frac{C}{2}e^2$ we have completed the proof. $\square$

Remark 17.5 The converse of Theorem 17.4 is false: the helicoid in $\mathbb{R}^3$ is a surface with cubic area growth[25] and its conformal structure is parabolic. However, in a certain sense the theorem cannot be improved: there are examples of rotationally symmetric complete metrics in a non-parabolic cylinder with $A(B_\Sigma(x_0, r)) \sim Cr^{2+\varepsilon}$ for all $\varepsilon > 0$ and for some $C = C(\varepsilon) > 0$.

Theorem 17.6 (Bernstein) *The only entire*[26] *minimal graphs in* $\mathbb{R}^3$ *are planes.*

Proof. Let $\Sigma = \mathrm{Gr}(f) \subset \mathbb{R}^3$ be the graph of a solution $f\colon \mathbb{R}^2 \to \mathbb{R}$ of the PDE (6). Given $x_0 \in \Sigma$ and $r > 0$, the triangle inequality gives $B_\Sigma(x_0, r) \subset \Sigma \cap \mathbb{B}(x_0, r)$, hence Theorem 3.4 implies that

$$A\left(B_\Sigma(x_0, r)\right) \leq A\left(\Sigma \cap \mathbb{B}(x_0, r)\right) \leq 2\pi r^2,$$

and so, Σ has quadratic area growth. By Theorem 17.4, Σ is parabolic, and since it is simply connected (because it is a graph over $\mathbb{R}^2$), Σ is conformally equivalent to $\mathbb{C}$. Now we can finish in two ways:

[25] That is, $A(B_\Sigma(x_0, r)) \sim Cr^3$ for some $C > 0$ and for all r sufficiently large.
[26] Entire means defined in the whole $\mathbb{R}^2$.

(a) Consider the Jacobi function $u = N_3 = \langle N, E_3 \rangle$ (observe that E_3 is a Killing field in $\mathbb{R}^3$). Orient Σ so that $u > 0$. The Jacobi equation gives $\Delta u = 2Ku \leq 0$, hence u is superharmonic in Σ and thus, u is superharmonic in $\mathbb{C}$. Since u is bounded, Liouville's Theorem implies that u is constant $c > 0$. Thus, $2cK = \Delta c = 0$ which implies $K \equiv 0$, i.e., Σ is flat hence a plane.

(b) Let $N \colon \Sigma \to \mathbb{S}^2$ be the Gauss map of Σ, and $g \colon \Sigma \equiv \mathbb{C} \to \mathbb{C} \cup \{\infty\}$ its stereographic projection from the North pole. We know that g is a meromorphic function as Σ is minimal. Orient Σ so that $N_3 < 0$, which implies $|g| < 1$. As g is a bounded holomorphic function on $\mathbb{C}$, then g is constant by Liouville's Theorem. Thus, Σ is a plane. □

We next give another proof of Bernstein's Theorem, based on the maximum principle: Let $\Sigma = \mathrm{Gr}(f) \subset \mathbb{R}^3$ be the graph of a solution $f \colon \mathbb{R}^2 \to \mathbb{R}$ of the PDE (6). If Σ is flat, there is nothing to prove. So assume that there exists $p_0 \in \Sigma$ such that $K(p_0) < 0$ (Gauss curvature) and we will arrive to a contradiction.

Recall the doubly periodic Scherk surface S (example 5 in Section 2). Since the Gauss map of S in each square C over which S is a graph with boundary values $\pm\infty$ covers a half-sphere, we can adjust S in $\mathbb{R}^3$ by applying the composition of a rigid motion and homothety so that

(a) S and Σ are tangent at $p_0 \in \Sigma \cap \mathrm{Int}(S)$.

(b) The principal directions of S and Σ at p_0 coincide.

(c) The Gauss curvatures of S and Σ at p_0 coincide.

Conditions (a), (b), (c) above tell us that S and Σ have a contact of order ≥ 2 at p_0. Both S and Σ are minimal graphs of functions f_1, f defined in a common open neighborhood of the origin in the common tangent plane at p_0 to both surfaces. Since the second fundamental forms of S and Σ at p_0 are given respectively by the hessians of f_1 and f at p_0, and both second fundamental forms at p_0 match because the order of contact of S and Σ at p_0 is ≥ 2, we deduce that the function $h := f_1 - f$ vanishes at the origin of $T_{p_0}\Sigma$ with vanishing order ≥ 3. By Theorem 10.3, the zeros of h form an equiangular system of $k \geq 3$ curves that intersect at the origin of $T_{p_0}\Sigma$, and on each component of the complement of the union of these curves in a neighborhood of the origin, the graphs corresponding to S, Σ are one at one side of the other, alternating this relative position when changing consecutive sectors. Consider the vertical projection Γ of the previously described equiangular system on the square C; this system of projected curves is no longer necessarily equiangular at the projection q_0 of p_0 on C (because the projection $(x, y, z) \mapsto (x, y)$ does not preserve angles between pairs of non-vertical vectors), but the curve system Γ still expresses the intersection of S and Σ as graphs over an open neighborhood of q_0 in the (x, y)-plane. In addition, Γ can be extended to a system of curves that start at q_0 and which consists of the points of C that have the same image by the graphs of S and Σ. We will continue denoting by Γ this extended system of curves in C. Note that the ends of curves in Γ are contained in the set of vertices of C (because f is a function defined in all $\mathbb{R}^2$), and there are at least six of these ends of curves in Γ (because $k \geq 3$). This creates a closed loop γ in $\Gamma \cup \{\text{vertices de } C\}$. g bounds an open topological disk in $C \setminus \Gamma$, in which is possible to find a contradiction with the maximum principle for minimal surfaces by moving one of the two surfaces vertically with respect to the other until finding a first contact point between the two surfaces. This contradiction proves Bernstein's Theorem.

18 The Kawai technique

The technique that we will see next allows to control the growth of area and total curvature under hypotheses that include stability. Although we call it 'Kawai technique', many researchers have contributed to it. The first one to do so was Pogorelov [16] in the simply connected case with $K \leq 0$ and $a = 2$. Kawai [10] improved these results for $a > \frac{1}{4}$. The general case for topology and curvature, with $a > \frac{1}{2}$ was considered by Colding-Minicozzi [2], Rosenberg [18] and Castillon [4].

Theorem 18.1 (Kawai, Colding-Minicozzi) *Let (Σ, g) be a complete Riemannian surface, possibly with boundary. Let $L = \Delta + V - cK$ be a Schrödinger operator on Σ, where $V \in C^\infty(\Sigma)$ is non-negative, K denotes the Gauss curvature of (Σ, g) and $c > \frac{1}{2}$. If $-L \geq 0$ in Σ, then given $x_0 \in \Sigma$ and $R > 0$ such that $B_\Sigma(x_0, R) \cap \partial\Sigma = \varnothing$, we have:*

$$\frac{A(B(R))}{R^2} + \frac{1}{2c-1} \int_{B(R)} V \left(1 - \frac{r}{R}\right)^2 \leq \frac{2\pi c}{2c-1}.$$

Proof. Let r be the distance function in (Σ, g) to x_0. Reasoning as in the proof of (88), we have that the length function $l(r)$ of $\partial B_\Sigma(x_0, r)$ (recall that $l(r)$ is C^1 a.e. in r by Theorem 13.2) satisfies

$$l'(r) \overset{(87)}{=} \int_{\partial B(r)} \kappa_g(s)\, ds + 2\sum_{i=1}^{n_r} \tan(\theta_i/2) \leq \int_{\partial B(r)} \kappa_g(s)\, ds + \sum_{i=1}^{n_r} \theta_i \overset{\text{(G-B)}}{=} 2\pi\chi(B(r)) - \widetilde{K}(r),$$

where we have abbreviated $B(r) = B_\Sigma(x_0, r)$, κ_g is the geodesic curvature of $\partial B(r)$, s is its arclength parameter, $\theta_1, \ldots, \theta_{n_r}$ are the external angles of $\partial B(r)$ (in particular, $-\pi < \theta_i < 0$ for each i), in (G-B) we have used Gauss-Bonnet formula, $\chi(D_M(p_0, R))$ stands for the Euler characteristic of $B(r)$ and we have defined the function

$$\widetilde{K}(r) = \int_{B(r)} K\, dA.$$

Consider a smooth function $\eta\colon [0, R] \to (0, \infty)$ such that $\eta(0) = 1$, $\eta(R) = 0$, $\eta' \leq 0$ in $[0, R]$, and let $f = \eta \circ r$. Thus, f lies in the Sobolev space $H_0^1(B(R))$ and its weak gradient is $\nabla f = \eta'(r)\nabla r$. In particular, $\|\nabla f\| = \eta'(r)$.

Since $-(\Delta + V - cK) \geq 0$ in $B(R)$, we have

$$\int_{B(R)} Vf^2 \leq \int_{B(R)} \|\nabla f\|^2 + c\int_{B(R)} Kf^2. \tag{127}$$

The two integrals of the right-hand-side of (127) can be computed by means of the co-area formula:

$$\int_{B(R)} \|\nabla f\|^2 = \int_{B(R)} \eta'(r)^2 = \int_0^R \eta'(r)^2 l(r)\, dr, \tag{128}$$

and

$$\int_{B(R)} Kf^2 = \int_0^R \eta(r)^2 \widetilde{K}'(r)\, dr \overset{(*)}{=} -\int_0^R (\eta^2)'(r)\widetilde{K}(r)\, dr, \tag{129}$$

where in $(*)$ we have integrated by parts and used that $\eta(R) = \widetilde{K}(0) = 0$.

Since $(\eta^2)' = 2\eta\eta' \leq 0$ and $l'(r) \leq 2\pi\chi(B(r)) - \widetilde{K}(r) \leq 2\pi - \widetilde{K}(r)$, (129) implies that

$$\int_{B(R)} Kf^2 \leq \int_0^R (\eta^2)'(r)\left[l'(r) - 2\pi\right] dr = \int_0^R (\eta^2)'(r) l'(r)\, dr + 2\pi,$$

where in the last equality we have used that $\eta(0) = 1$, $\eta(R) = 0$. Joining this last inequality with (127) and (128), we obtain

$$\int_{B(R)} Vf^2 \leq \int_0^R \eta'(r)^2 l(r)\, dr + c\int_0^R (\eta^2)'(r) l'(r)\, dr + 2\pi c.$$

Now replace $\eta(r) = 1 - \frac{r}{R}$ in the last expression:

$$\begin{aligned}\int_{B(R)} V\left(1 - \frac{r}{R}\right)^2 &\leq \frac{1}{R^2}\int_0^R l(r)\, dr - \frac{2c}{R}\int_0^R \left(1 - \frac{r}{R}\right) l'(r)\, dr + 2\pi c \\ &= \frac{A(B(R))}{R^2} - \frac{2c}{R}\int_0^R \left(1 - \frac{r}{R}\right) l'(r)\, dr + 2\pi c.\end{aligned}$$

We want to integrate by parts in the last right-hand-side. But the function $r \mapsto l(r)$ might be discontinuous, although it is differentiable a.e. in r. Indeed, $l(r)$ can be decomposed as

$$l(r) = H(r) - J(r)$$

where H is absolutely continuous[27] in $[0, \infty)$ and J is non-decreasing and continuous except in a closed countable set, where J has finite jump discontinuities (see Shiohama and Tanaka [22]).

[27] This means that $\forall \varepsilon > 0$, $\exists \delta = \delta(\varepsilon)$ such that if $\{(x_k, y_k)\}_k$ is a sequence of pairwise disjoint intervals inside $[0, \infty)$ satisfying $\sum_k (y_k - x_k) < \delta$, then $\sum_k |f(y_k) - f(x_k)| < \varepsilon$.

Therefore, for each non-negative smooth function $\psi\colon [0,R]\to\mathbb{R}$ we have

$$\int_0^R \left[\psi(r)J'(r)+\psi'(r)J(r)\right] dr \leq \psi(R)J(R)-\psi(0)J(0), \tag{130}$$

and if we replace $J(r)$ by $H(r)$ in (130) we obtain an equality. Hence,

$$\int_0^R \left[\psi(r)l'(r)+\psi'(r)l(r)\right] dr \geq \psi(R)l(R)-\psi(0)l(0)=\psi(R)l(R). \tag{131}$$

Applying (131) to $\psi(r)=1-\frac{r}{R}$ and using that $\psi(R)=0$, we get

$$\int_0^R \left(1-\frac{r}{R}\right) l'(r)\, dr \geq \frac{1}{R}\int_0^R l(r)\, dr = \frac{1}{R}A(B(R)).$$

Thus,

$$\int_{B(R)} V\left(1-\frac{r}{R}\right)^2 \leq \frac{1-2c}{R^2}A(B(R))+2\pi c. \tag{132}$$

Finally, (132) finishes the proof of the theorem because $1-2c<0$. □

Corollary 18.2 *Let Σ be a complete, two-sided, stable minimal surface without boundary in a Riemannian manifold (M^3,g) with scalar curvature $S\geq 0$. Then, Σ has quadratic area growth and $\int_\Sigma |A_\Sigma|^2<\infty$.*

Proof. Since the Jacobi operator of Σ is $L=\Delta_\Sigma+|A_\Sigma|^2+\mathrm{Ric}_M(N)=\Delta+V-K_\Sigma$ where $V=\frac{1}{2}\left(S+|A_\Sigma|^2\right)\geq 0$ (see equation (120)), we can apply Theorem 18.1 with $c=1$, obtaining

$$\frac{A(B(R))}{R^2}+\int_{B(R)} V\left(1-\frac{r}{R}\right)^2 \leq 2\pi, \tag{133}$$

hence $A(B(R))\leq 2\pi R^2$ (here $B(R)=B_\Sigma(x_0,R)$ and $x_0\in\Sigma$ is any point in Σ).

Let us see that the 'total curvature' is finite. Since $S\geq 0$,

$$\int_\Sigma |A_\Sigma|^2 \leq \int_\Sigma \left(S+|A_\Sigma|^2\right) = 2\int_\Sigma V, \tag{134}$$

hence it suffices to prove that V is integrable in Σ. Take $R>0$. In $B(R/2)$ we have $1-\frac{r}{R}\geq\frac{1}{2}$ hence

$$\frac{1}{4}\int_{B(R/2)} V \leq \int_{B(R/2)} V\left(1-\frac{r}{R}\right)^2 \overset{(*)}{\leq} \int_{B(R)} V\left(1-\frac{r}{R}\right)^2 \overset{(133)}{\leq} 2\pi. \tag{135}$$

where in $(*)$ we have used that $V\geq 0$. □

The next result generalizes Bernstein's Theorem, and is central in the classical theory of minimal surfaces:

Corollary 18.3 (Schoen, do Carmo-Peng) *The only orientable complete stable minimal surfaces without boundary in $\mathbb{R}^3$ are planes.*

Proof. By Corollary 18.2, if $\Sigma \subset \mathbb{R}^3$ satisfies the hypotheses of Corollary 18.3 then Σ has quadratic area growth and finite total curvature. As Σ is stable, we can apply the inequality (117) to conclude that

$$\int_\Sigma |A_\Sigma|^2 f^2 \leq \int_\Sigma \|\nabla f\|^2, \quad \forall H_0^1(\Sigma).$$

Take as f the logarithmic cut-off function ψ_R defined in (123) in terms of $r = \mathrm{dist}_\Sigma(x_0, \cdot)$ (here x_0 is a previously chosen point in Σ). Abbreviating $B(R) = B_\Sigma(x_0, R)$, we have

$$\int_{B(\sqrt{R})} |A_\Sigma|^2 = \int_{B(\sqrt{R})} |A_\Sigma|^2 \psi_R^2 \leq \int_{B(R)} |A_\Sigma|^2 \psi_R^2 \leq \int_{B(R)} \|\nabla \psi_R^2\|$$

$$= \frac{4}{(\log R)^2} \int_{B(R)\setminus B(\sqrt{R})} \frac{1}{r^2} \leq \frac{4}{(\log R)^2} \int_{B(R)} \frac{1}{r^2} \overset{(*)}{\leq} \frac{4}{(\log R)^2} Ce^2 \log \sqrt{R} = \frac{2Ce^2}{\log R}.$$

where in $(*)$ we have used the last part of the proof of Theorem 17.4, which was valid under the hypothesis $A(B(r)) \leq Cr^2$ $\forall r > 0$. Fixing $R_0 > 0$ and taking $R \geq R_0$, the above reasoning proves that

$$\int_{B(\sqrt{R_0})} |A_\Sigma|^2 \leq \int_{B(\sqrt{R})} |A_\Sigma|^2 \leq \frac{2Ce^2}{\log R} \overset{(R\to\infty)}{\longrightarrow} 0,$$

hence $A_\Sigma \equiv 0$ in $B(\sqrt{R_0})$. As $R > 0$ is arbitrary, we conclude the proof. $\square$

19 Curvature estimates for stable minimal surfaces

Theorem 19.1 (Schoen) *There exists $C > 0$ such that for every orientable stable minimal surface Σ in $\mathbb{R}^3$ possibly with boundary, we have*

$$|A_\Sigma| \ \mathrm{dist}_\Sigma(\cdot, \partial\Sigma) \leq C.$$

Remark 19.2 (1) If under the hypotheses of the theorem, Σ has no boundary, then Σ is a plane by Corollary 18.3.

(2) We cannot apply Theorem 15.1, because in our current situation

$$\int_\Sigma |A_\Sigma|^2 \overset{(134)}{=} 2\int_\Sigma V = 8 \cdot \frac{1}{4}\int_\Sigma V \overset{(135)}{\leq} 8 \cdot 2\pi,$$

but we need $\int_\Sigma |A_\Sigma|^2 \leq C < 8\pi$ in order to apply Theorem 15.1. However, we will use some of the ideas in its proof.

Proof. Arguing by contradiction, suppose for each $n \in \mathbb{N}$ there exists an orientable stable minimal surface Σ_n in $\mathbb{R}^3$, possibly with boundary, and a point $p_n \in \Sigma_n$ such that

$$|A_{\Sigma_n}|(p_n)\ \mathrm{dist}_{\Sigma_n}(p_n, \partial\Sigma_n) \to \infty.$$

(By Corollary 18.3, $\partial\Sigma_n \neq \varnothing$). Replacing Σ_n by $B_{\Sigma_n}(p_n, \mathrm{dist}_{\Sigma_n}(p_n, \partial\Sigma_n))$, we can assume that Σ_n is compact. We can also replace p_n by a maximum of the function $f_n\colon \Sigma_n \to [0,\infty)$ defined by (97). Traslate Σ_n in $\mathbb{R}^3$ so that $p_n = \vec{0}$ and define

$$\Sigma_n' = |A_{\Sigma_n}|(p_n)|\,\Sigma_n \quad \text{(homothety)}, \ \forall n \in \mathbb{N}.$$

Reasoning as in the proof of Theorem 15.1 we have that $\mathrm{dist}_{\Sigma_n'}(\vec{0}, \partial\Sigma_n') \to \infty$ if $n \to \infty$, $|A_{\Sigma_n'}| \leq 2\ \forall n \in \mathbb{N}$ and after passing to a subsequence, $\{\Sigma_n'\}_n$ converges to a complete minimal surface (without boundary) $\Sigma_\infty' \subset \mathbb{R}^3$, with $\vec{0} \in \Sigma_\infty'$ and $|A_{\Sigma_\infty'}|(\vec{0}) = 1$. Σ_∞' is orientable, because it is the uniform limit on compact subsets of $\mathbb{R}^3$ in the C^k topology for each k of orientable surfaces.

Let us see that Σ_∞' is stable: otherwise, there would exist a relatively open subset $\Omega \subset \Sigma_\infty'$ with $\lambda_1(L, \Omega) < 0$ (here L is the Jacobi operator of Σ_∞'). Since Ω is compact, Ω is the uniform limit in the C^k topology for each k of a sequence of relatively compact open subsets $\Omega_n \subset \Sigma_n'$. Thus, $\lambda_1(L_n, \Omega_n) < 0$ from a certain positive integer, where L_n is the Jacobi operator of Σ_n'. This is impossible, because Σ_n' is stable for each n since Σ_n is. Therefore, Σ_∞' is stable.

Finally, Corollary 18.3 ensures that Σ_∞' is a plane, which contradicts que $|A_{\Sigma_\infty'}|(\vec{0}) = 1$. This finishes the proof. □

The previous proof cannot be generalized by replacing $\mathbb{R}^3$ with a complete and homogeneously regular Riemannian manifold (M^3, g), since we do not have ensured that the limit surface Σ_∞' falls in $\mathbb{R}^3$ (this would be ensured if we knew that $|A_{\Sigma_n}|(p_n) \to 0$ with the previous notation); in fact, in $\mathbb{H}^3$ there exist complete orientable stable minimal surfaces, which are not totally geodesic: just take a rectifiable Jordan curve $\Gamma \subset \partial_\infty\mathbb{H}^3 \equiv \mathbb{S}^2$ not being an equator and construct Σ as the limit when $R \to \infty$ of a sequence of solutions to the Plateau problem for Jordan curves $\Gamma_R \subset \partial B^{\mathbb{H}^3}(x_0, R) \subset \mathbb{H}^3$ converging to Γ in a certain sense (here $B^{\mathbb{H}^3}(x_0, R)$ is the metric ball of radius $R > 0$ centered at a point $x_0 \in \mathbb{H}^3$). This same construction can be made in $\mathbb{H}^2 \times \mathbb{R}$ by taking $\Gamma \subset (\partial_\infty\mathbb{H}^2) \times \mathbb{R} \equiv \mathbb{S}^1 \times \mathbb{R}$ not being a horizontal circle but so that Γ is contained in a vertical strip of width $< \pi$, for which it is known that the corresponding Plateau problem can be solved.

Similarly as the relation between Theorems 15.1 and 15.2, we have the following version of Theorem 19.1 for homogeneously regular three-manifolds.

Theorem 19.3 (Rosenberg, Souam, Toubiana) *Given $\Lambda > 0$, there exists $C = C(\Lambda) > 0$ such that if (M^3, g) is a homogeneously regular Riemannian manifold with $|K_{\mathrm{sec}}| \leq \Lambda$ and Σ is a two-sided*[28]*, complete stable surface with constant mean curvature $H \in \mathbb{R}$ in (M^3, g), possibly*

[28]Every surface with constant mean curvature $H \in \mathbb{R} \setminus \{0\}$ is two-sided.

with boundary, then:

$$|A_\Sigma| \cdot \min\left\{\mathrm{dist}_\Sigma(\cdot,\partial\Sigma), \frac{\pi}{2\sqrt{\Lambda}}\right\} \le C. \tag{136}$$

Remark 19.4 (1) Surfaces of constant mean curvature $H \in \mathbb{R}$ in a Riemannian three-manifold are the critical points of the functional Area$-2H\cdot$Volume for compactly supported normal variations, and the second derivative of this functional at a surface Σ with constant mean curvature constant H (which in the sequel will be called a H-surface) is given by the right-hand-side of (113), where L is the Jacobi operator of Σ, defined as in the minimal case. The notion of stability extends to H-surfaces, imposing that (115) holds. Do not confuse this notion of stability in the CMC case (sometimes called *strong stability*) con the notion of stability associated to the isoperimetric problem, where (115) is only required for functions $f \in H_0^1(\Sigma)$ with zero mean.

(2) What is the meaning of inequality (136)? If $x \in \Sigma$ is very close to $\partial\Sigma$, the minimum that appears in (136) is $\mathrm{dist}_\Sigma(x,\partial\Sigma)$, hence the inequality is the same as the one in Theorem 19.1; i.e., $|A_\Sigma|$ can grow at most inversely proportional to the distance to the boundary for these points. When x is far from $\partial\Sigma$, the minimum in (136) is $\frac{\pi}{2\sqrt{\Lambda}}$, hence $|A_\Sigma|$ is bounded for these points and the bound is inversely proportional to the square root of a bound for the absolute sectional curvature of the ambient space.

Before proving Theorem 19.3, let us see some consequences. The first one is a strong generalization of Corollary 18.3.

Corollary 19.5 *Given $\Lambda > 0$, there exists $C = C(\Lambda) > 0$ such that if (M^3, g) is a homogeneously regular Riemannian manifold with $|K_{\mathrm{sec}}| \le \Lambda$ and Σ is a complete stable surface with mean curvature constant $H \in \mathbb{R}$ (two-sided if $H = 0$) in (M^3, g) without boundary, then*

$$|A_\Sigma| \le \frac{2\sqrt{\Lambda}}{\pi} C.$$

Proof. Apply (136) taking into account that $\mathrm{dist}_\Sigma(\cdot,\partial\Sigma) = +\infty$ in Σ. $\square$

The second consequence of Theorem 19.3 is a version of Corollary 19.5 for compact surfaces.

Corollary 19.6 *Given $\Lambda > 0$, there exists $C_1 = C_1(\Lambda) > 0$ such that if (M^3, g) is a homogeneously regular Riemannian manifold with $|K_{\mathrm{sec}}| \le \Lambda$ and Σ is a compact stable surface with mean curvature constant $H \in \mathbb{R}$ (two-sided if $H = 0$) in (M^3, g) without boundary, then*

$$|A_\Sigma| \le \max\left\{\frac{1}{\mathrm{diam}(\Sigma)}, \frac{\sqrt{\Lambda}}{\pi}\right\} C_1.$$

Proof. Let us call $d = \mathrm{diam}(\Sigma)$. Take $x \in \Sigma$ and apply (136) to the surface $\Sigma_1 = B_\Sigma(x, d/2)$:

$$|A_\Sigma|(x) \cdot \min\left\{\mathrm{dist}_{\Sigma_1}(x, \partial\Sigma_1), \frac{\pi}{2\sqrt{\Lambda}}\right\} \leq C.$$

But $\mathrm{dist}_{\Sigma_1}(x, \partial\Sigma_1) = \frac{d}{2}$, hence the above left-hand-side is $\frac{|A_\Sigma|(x)}{2} \cdot \min\left\{d, \frac{\pi}{\sqrt{\Lambda}}\right\}$, hence it suffices to take $C_1 = 2C$ in order to deduce the corollary. $\square$

We will only give a sketch of the proof of Theorem 19.3, avoiding technicalities. We will need the notion of *blow-up pair* for a compact surface with boundary $(\Sigma, \partial\Sigma) \subset (\mathbb{B}(r_0), \partial\mathbb{B}(r_0))$ (in particular, $\Sigma \subset \mathbb{R}^3$). The idea is that if $|A_\Sigma|$ is large at $x \in \Sigma \setminus \partial\Sigma$, then we can choose $y \in \Sigma$ near x where not only $|A_\Sigma|(y)$ is proportional to $|A_\Sigma|(x)$ but also $|A_\Sigma| \leq 2|A_\Sigma|(y)$ in $B_\Sigma(y, s)$ for some $s > 0$ (that is, $|A_\Sigma||_{B_\Sigma(y,s)}$ essentially achieves its maximum at y):

Definition 19.7 Let $\Sigma \subset \mathbb{R}^3$ be a compact surface with boundary, not necessarily minimal, with $(\Sigma, \partial\Sigma) \subset (\mathbb{B}(r_0), \partial\mathbb{B}(r_0))$. Suppose that

$$\sup_{\Sigma \cap \mathbb{B}(\frac{r_0}{2})} |A_\Sigma| \geq \frac{4C}{r_0}, \tag{137}$$

where $C > 0$. Given $y \in \Sigma \setminus \partial\Sigma$, $s > 0$, we will say that (y, s) is a *blow-up pair* centered at y with scale s if the following conditions hold:

(i) $|A_\Sigma|(z) \cdot \mathrm{dist}_{\mathbb{R}^3}(z, \partial\mathbb{B}(r_0)) \leq |A_\Sigma|(y) \cdot \mathrm{dist}_{\mathbb{R}^3}(y, \partial\mathbb{B}(r_0))$, $\forall z \in \Sigma$.

(ii) $2s \leq r_0 - \|y\| (= \mathrm{dist}_{\mathbb{R}^3}(y, \partial\mathbb{B}(r_0)))$.

(iii) $|A_\Sigma| \leq 2|A_\Sigma|(y)$ in $B_\Sigma(y, s)$.

Lemma 19.8 *Let $\Sigma \subset \mathbb{R}^3$ be compact surface with boundary, not necessarily minimal, with $(\Sigma, \partial\Sigma) \subset (\mathbb{B}(r_0), \partial\mathbb{B}(r_0))$ and $C > 0$ such that (137) holds. Then, there exists a blow-up pair (y, s), where $y \in \Sigma \setminus \partial\Sigma$ and $s > 0$ is given by*

$$|A_\Sigma|(y) = \frac{C}{s}. \tag{138}$$

Proof. Consider the continuous function $f\colon \Sigma \to [0, \infty)$ given by

$$f(z) = |A_\Sigma|(z) \cdot \mathrm{dist}_{\mathbb{R}^3}(z, \partial\mathbb{B}(r_0)).$$

Then, $f|_{\partial\Sigma} = 0$. Let $y \in \Sigma \setminus \partial\Sigma$ be a maximum of f. In particular, $|A_\Sigma|(y) > 0$ and we can define $s > 0$ by (138).

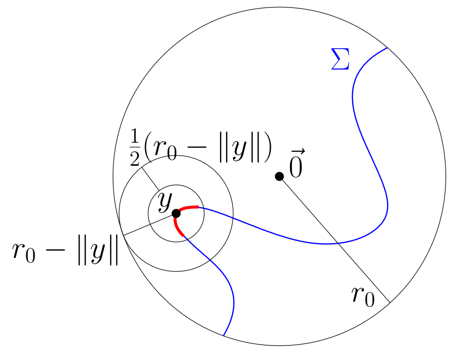


Figure 37: In red, $B_\Sigma(y,s)$ (contained in $\mathbb{B}(y, \frac{1}{2}(r_0 - \|y\|))$, where inequality (iii) of Definition 19.7 holds.

Let us see that inequality (ii) of Definition 19.7 holds: Take $x \in \Sigma \cap \mathbb{B}(r_0/2)$.

$$|A_\Sigma|(x)\frac{r_0}{2} \leq |A_\Sigma|(x)(r_0 - \|x\|) = f(x) \leq f(y) = |A_\Sigma|(y)(r_0 - \|y\|),$$

from where

$$\frac{r_0}{2} \sup_{\Sigma \cap \mathbb{B}(\frac{r_0}{2})} |A_\Sigma| \leq |A_\Sigma|(y)(r_0 - \|y\|).$$

By (137), the left-hand-side is $\geq 2C$, hence

$$2C \leq |A_\Sigma|(y)(r_0 - \|y\|) \stackrel{(138)}{=} \frac{C}{s}(r_0 - \|y\|),$$

which is (ii).

Let us now check inequality (iii) of Definition 19.7: Take $z \in B_\Sigma(y,s)$. Since $B_\Sigma(y,s) \subset \mathbb{B}(y,s)$ (by the triangle inequality) and $\mathbb{B}(y,s) \subset \mathbb{B}(y, \frac{1}{2}(r_0 - \|y\|))$ (by item (ii)), we have $r_0 - \|z\| \geq \frac{1}{2}(r_0 - \|y\|) > 0$ (see Figure 37). On the other hand,

$$|A_\Sigma|(z)(r_0 - \|z\|) = f(z) \leq f(y) = |A_\Sigma|(y)(r_0 - \|y\|),$$

hence

$$\begin{aligned} |A_\Sigma|(z) &\leq \frac{r_0 - \|y\|}{r_0 - \|z\|}|A_\Sigma|(y) = \left(1 + \frac{\|z\| - \|y\|}{r_0 - \|z\|}\right)|A_\Sigma|(y) \\ &\leq \left(1 + \frac{\|z - y\|}{r_0 - \|z\|}\right)|A_\Sigma|(y) < \left(1 + \frac{s}{r_0 - \|z\|}\right)|A_\Sigma|(y) \stackrel{(*)}{\leq} 2|A_\Sigma|(y), \end{aligned}$$

where in $(*)$ we have used that $r_0 - \|z\| \geq \frac{1}{2}(r_0 - \|y\|) \stackrel{\text{(iii)}}{\geq} s$. Therefore, (iii) hods. (i) is trivial, since y maximum of f. $\square$

Let us see what happens after re-scaling by curvature around a blow-up point.

Lemma 19.9 *Let $\Sigma \subset \mathbb{R}^3$ be a surface in the hypotheses of Lemma 19.8. Let (y, s) be a blow-up pair satisfying (138). We denote by $\Sigma(y, s)$ the component of $\Sigma \cap \mathbb{B}(y, s)$ that contains y (observe that $B_\Sigma(y, s) \subset \Sigma(y, s)$ by the triangle inequality). Consider the new surface*

$$\widetilde{\Sigma} := |A_\Sigma|(y)\,[\Sigma(y, s) - y] \subset \mathbb{B}\left(\vec{0}, |A_\Sigma|(y) \cdot s\right) \overset{(138)}{=} \mathbb{B}(\vec{0}, C).$$

(which verifies $\partial \widetilde{\Sigma} \subset \partial \mathbb{B}(\vec{0}, C)$ because $\partial \Sigma(y, s) \subset \partial \Sigma \cap \mathbb{B}(y, s)$). Then, $\vec{0} \in \widetilde{\Sigma}$, $|A_{\widetilde{\Sigma}}|(\vec{0}) = 1$, and $|A_{\widetilde{\Sigma}}| \leq 2$ in $B_{\widetilde{\Sigma}}(\vec{0}, C)$.

Proof. It is clear that $\vec{0} \in \widetilde{\Sigma}$ and $|A_{\widetilde{\Sigma}}|(\vec{0}) = 1$. Let us see that $|A_{\widetilde{\Sigma}}| \leq 2$ in $B_{\widetilde{\Sigma}}(\vec{0}, C)$: Given $\widetilde{z} \in B_{\widetilde{\Sigma}}(\vec{0}, C) = |A_\Sigma|(y)\,[B_\Sigma(y, s) - y]$, there exists $z \in B_\Sigma(y, s)$ con $\widetilde{z} = |A_\Sigma|(y)(z - y)$, hence

$$|A_{\widetilde{\Sigma}}|(\widetilde{z}) = \frac{|A_\Sigma|(z)}{|A_\Sigma|(y)} \overset{\text{(iii)}}{\leq} 2. \qquad \square$$

Next we give a sketch of the proof of Theorem 19.3.

Proof. [of Theorem 19.3]. Arguing by contradiction, suppose that for some $\Lambda > 0$, there exists a sequence Σ_n of stable surfaces with constant mean curvature $H_n \in \mathbb{R}$ (two-sided if $H_n = 0$) in homogeneously regular Riemannian manifolds (M_n^3, g_n) with $|K_{\text{sec}}(M_n)| \leq \Lambda$ for each n, possibly with $\partial \Sigma_n \neq \varnothing$, in such a way that

$$|A_\Sigma|(p_n^*) \cdot \min\left\{\text{dist}_\Sigma(p_n^*, \partial \Sigma_n), \frac{\pi}{2\sqrt{\Lambda}}\right\} > n \quad \forall n \in \mathbb{N}, \tag{139}$$

where p_n^* is a point of Σ_n. We divide the argument into two steps.

(A) Using the uniform bound $|K_{\text{sec}}(M_n)| \leq \Lambda$ for every n, it can be proven that there exists $r_n > 0$ such that the metric ball $B_{\Sigma_n}(p_n^*, r_n)$ is contained in the domain of a chart of M_n, which allows to identify $B_{\Sigma_n}(p_n^*, r_n)$ with a surface is an extrinsic ball of fixed radius in $(\mathbb{R}^3, g_n)$ (this is a slight abuse of notation, since the metric in this extrinsic ball shoud be the pullback of g_n by the chart).

(B) The construction can be made so that $|A_{\Sigma_n}|(p_n^*) r_n \to \infty$. Next re-scale both the ambient metric g_n and the surface Σ_n by the factor $|A_{\Sigma_n}|(p_n^*)$, obtaining a sequence of Riemannian metrics defined on three-dimensional balls whose radii diverge to ∞ (which converge on compact subsets of $\mathbb{R}^3$ to the usual inner product on $\mathbb{R}^3$), and stable surfaces with constant mean curvature, all passing through the origin of $\mathbb{R}^3$, with uniformly bounded geometry and with norm of the second fundamental form equals to 1 origin. Taking $n \to \infty$, a subsequence of these surfaces converges to a non-flat, complete stable with constant mean curvature in $\mathbb{R}^3$, which contradicts Corollary 18.3 (for this, we need an extension of Corollary 18.3 to the case of constant mean curvature).

We now comment on some of the technical difficulties of the previous steps.

The first idea one has to prove (A) is to use exponential coordinates in (M_n, g_n). Finding a uniform geodesic radius in n is not a problem, since the hypothesis $|K_{\rm sec}(M_n)| \leq \Lambda$ guarantees that in $[0, \frac{\pi}{\sqrt{\Lambda}})$ there are no conjugate values along any of the unitary radial geodesics in M_n, and therefore we have a uniform lower bound for the radius of the extrinsic ball $\mathbb{B}(\vec{0}, \frac{\pi}{\sqrt{\Lambda}}) \subset T_x M_n$ where the exponential map $\exp_x^{M_n}$ is a local diffeomorphism (x is any point in M_n). Hence, given $n \in \mathbb{N}$, we can lift $\Sigma_n \cap \exp_{p_n^*}^{M_n}\left(\mathbb{B}(\vec{0}, \frac{\pi}{\sqrt{\Lambda}})\right)$ to an immersed surface $\widetilde{\Sigma}_n$ in $\mathbb{B}(\vec{0}, \frac{\pi}{\sqrt{\Lambda}}) \subset T_{p_n^*} M_n$. If we also lift the ambient metric g_n to $\mathbb{B}(\vec{0}, \frac{\pi}{\sqrt{\Lambda}})$, then $\widetilde{\Sigma}_n$ will have the same constant mean curvature H_n as Σ_n. The problem is that the use of the exponential map does not ensure more than a control C^0 over the metric g_n pullback by the exponential map (there are counterexamples), but we need a control C^1 on the coefficients of g_n to conclude that the limit surface constructed in (B) has constant mean curvature. Therefore, we should not use exponential coordinates. This problem can be solved by using another kind of local coordinates in M_n, called *harmonic coordinates*[29] which produce a control $C^{1,\alpha}$ on the coefficients of g_n and that can be defined at least in metric balls of (M_n, g_n) with radius bounded from below independently of n, in terms of an upper bound of $K_{\rm sec}(M_n)$ (this is the content of Theorem 2.1 in [19]). Once this is done, it can be checked that the radius r_n of step (A) can be taken as

$$r_n = \min\left\{ \mathrm{dist}_\Sigma(p_n^*, \partial\Sigma_n), \frac{\pi}{2\sqrt{\Lambda}} \right\}.$$

In this way, the condition $|A_{\Sigma_n}|(p_n^*) r_n \to \infty$ that appears in step (B) is a consequence of (139). After replacing M_n by $\mathbb{B}(r_n) = \mathbb{B}(\vec{0}, r_n)$ via the harmonic coordinates ($\mathbb{B}(r_n)$ is endowed with the pullback metric, also denoted by g_n), we can also view the surfaces Σ_n inside $\mathbb{B}(r_n)$; in fact, it is possible to prove that there exists $r_0 > 0$ such that $(\Sigma_n, \partial\Sigma_n) \subset (\mathbb{B}(r_0), \partial\mathbb{B}(r_0))$ for each n and that (139) implies

$$\sup_{\Sigma_n \cap \mathbb{B}(\frac{r_0}{2})} |A_{\Sigma_n}| \geq \frac{2n}{r_0}. \tag{140}$$

(Compare to (137) taking $C = n/2$). By Lemma 19.8 (generalized to the metric g_n instead of the usual inner product), there exists a blow-up pair (y_n, s_n) where $y_n \in \Sigma_n \setminus \partial\Sigma_n$ and $s_n > 0$ is given by

$$|A_{\Sigma_n}|(y_n) = \frac{C}{s_n} = \frac{n}{2s_n}. \tag{141}$$

We now re-scale Σ_n as in Lemma 19.9:

$$\widetilde{\Sigma}_n := |A_{\Sigma_n}|(y_n) \left[\Sigma_n(y_n, s_n) - y_n\right] \subset \mathbb{B}(n/2).$$

[29] The coordinate functions associated to each of these charts are harmonics functions over the open subset of M_n in which they are defined.

Lemma 19.9 implies that $\partial\widetilde{\Sigma}_n \subset \partial\mathbb{B}(n/2)$, $\vec{0} \in \widetilde{\Sigma}_n$, $|A_{\widetilde{\Sigma}_n}|(\vec{0}) = 1$, and $|A_{\widetilde{\Sigma}_n}| \leq 2$ in $B_{\widetilde{\Sigma}_n}(n/2)$. Furthermore, $\widetilde{\Sigma}_n$ has constant mean curvature

$$\widetilde{H}_n := \frac{H_n}{|A_{\Sigma_n}|(y_n)} \overset{(141)}{=} \frac{2s_n}{n} H_n.$$

On the other hand, (9) ensures that $2\widetilde{H}_n^2 \leq |A_{\widetilde{\Sigma}_n}|^2 \leq 4$, hence $|\widetilde{H}_n| \leq \sqrt{2}$. A standard diagonal argument allows us to take limits of a subsequence of the $\widetilde{\Sigma}_n$ to a surface $\widetilde{\Sigma}_\infty \subset \mathbb{R}^3$ (now $\mathbb{R}^3$ is endowed with the usual inner product because before passing to the limit we considered harmonic coordinates and the scale factors $|A_{\Sigma_n}|(y_n)$ diverge to ∞ thanks to (140)), with constant mean curvature $\widetilde{H}_\infty := \lim_n \widetilde{H}_n$ (this limit exists after passing to a subsequence), satisfying $\vec{0} \in \widetilde{\Sigma}_\infty$, $|A_{\widetilde{\Sigma}_\infty}|(\vec{0}) = 1$ and $|A_{\widetilde{\Sigma}_\infty}| \leq 2$ in $\widetilde{\Sigma}_\infty$. Furthermore, $\widetilde{\Sigma}_\infty$ is stable because is a smooth limit of stable surfaces, which concludes the sketch of the proof. $\square$

20 The Jenkins-Serrin method

Let $C \subset \mathbb{R}^2$ be a piecewise C^1 Jordan curve with a finite number of vertices. We can write $C = \gamma_1 \cup \ldots \cup \gamma_n$ where each γ_i is a Jordan arc of class C^1. Suppose that C is the boundary of a compact convex domain $D \subset \mathbb{R}^2$. Given a continuous function $f \colon \mathrm{Int}(\gamma_1) \cup \ldots \cup \mathrm{Int}(\gamma_n) \to \mathbb{R}$, When there exists a minimal graph over $\mathrm{int}(D)$ with these boundary values? In other words, we are interested in solutions of the Dirichlet problem

$$\begin{cases} \mathrm{div}\left(\frac{\nabla u}{W}\right) = 0 & \text{in } \mathrm{Int}(D), \\ u = f & \text{in } \partial D, \end{cases} \tag{142}$$

where $W = \sqrt{1 + \|\nabla u\|^2}$.

Theorem 20.1 *If f is bounded, then (142) has a unique solution.*

Proof. Construct a Jordan curve $\Gamma \subset \mathbb{R}^3$ whose vertical projection is $C \subset \mathbb{R}^2 \equiv \{z = 0\}$, in such a way that on each $\mathrm{Int}(\gamma_i)$, Γ is the graph of $f|_{\mathrm{Int}(\gamma_i)}$, and over each vertex of C, Γ consists of vertical segments that joint the extrema of the graphs $f|_{\mathrm{Int}(\gamma_i)}$, $f|_{\mathrm{Int}(\gamma_{i+1})}$ (in a cyclic way).

Douglas-Radó's Theorem (Theorem 8.1) ensures that there exists a solution M of the Plateau problem for the contour Γ. M is unique by the maximum principle (translate vertically), and an approximation argument as the one in Section 12.1 allow us to use Radó's Theorem to ensure that the interior of M is a graph over the interior of D. $\square$

Remark 20.2 The same result holds if we replace $\mathbb{R}^2$ by a Riemannian surface (M^2, g) (for instance, $M = \mathbb{H}^2$) and the equation $\mathrm{div}\left(\frac{\nabla u}{W}\right) = 0$ by $\mathrm{div}_M\left(\frac{\nabla u}{W}\right) = 0$ with $W = \sqrt{1 + \|\nabla_M u\|^2}$. Now, the graph of u is a minimal surface in $(M \times \mathbb{R}, g \times dt^2)$.

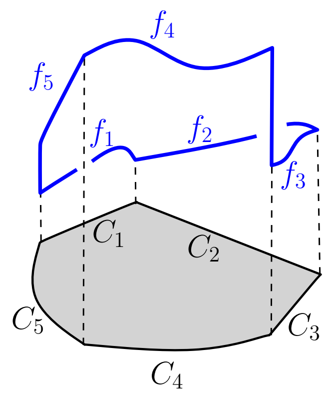


Figure 38: By adding vertical segments at the discontinuities of f, we construct a Jordan curve in $\mathbb{R}^3$.

We are interested in the problem (142) admitting values $\pm\infty$ on a portion of C. Before analyzing the general case, let us study some particular cases. Suppose that D is a rectangle of sides $a, b > 0$. Given $n \in \mathbb{N}$, we set (consecutive) boundary values $0, n, 0, n$ as in Figure 39 left and center. We solve the corresponding Plateau problem, as in the proof of Theorem 20.1.

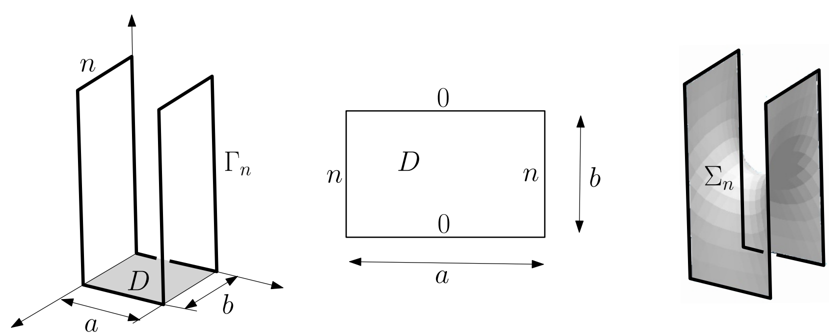


Figure 39: Left: contour Γ_n with height n over a rectangle D of sides a, b. Right: the solution Σ_n to the Plateau problem over D.

Thus, we produce a solution u_n of (142) and a minimal graph $\Sigma_n = \text{Gr}(u_n)$, see Figure 39 right. The maximum principle implies that $u_n \leq u_{n+1}$ for each $n \in \mathbb{N}$. What happens with Σ_n as $n \to \infty$?

Intuitively we see that if $a \ll b$, Σ_n converges to two vertical strips (i.e., there is no limit graph over D, or we could say that it is constant $+\infty$), see Figure 40 left.

If $a \gg b$, Σ_n converges to a well-defined graph on the interior of D, see Figure 40 right. One

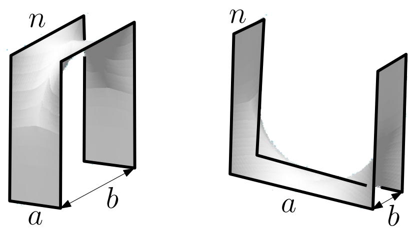


Figure 40: Left: Σ_n converges as $n \to \infty$ to two vertical strips of width a. Right: Σ_n converges to a graph over D, with boundary values $0, +\infty, 0, +\infty$. In fact, both surfaces Σ_n are congruent for each $n \in \mathbb{N}$. What produces different limits is the normalization we are using to compute the limit (for instance, the point that we are assuming to be fixed).

way to justify this is as follows: Since $\{u_n\}_n$ is a monotonous sequence, the limit $\lim_n u_n$ will exist in $\mathrm{int}(D)$ if and only if given $x \in \mathrm{Int}(D)$, there exists $C(x) > 0$ such that $u_n(x) \leq C(x)$ for all $n \in \mathbb{N}$. If $a \gg b$, we can construct a catenoid $\mathcal{C}$ as in Figure 41. Moving vertically

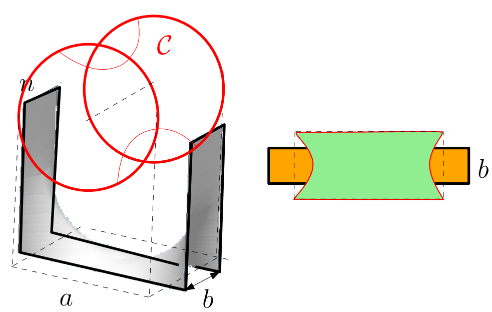


Figure 41: Place a catenoid $\mathcal{C}$ above Σ_n, in such a way that $\mathcal{C}$ projects vertically onto the green region of the figure at the right. Such a catenoid $\mathcal{C}$ exists provided that $a \gg b$.

$\mathcal{C}$ upwards (for n fixed), $\mathcal{C}$ does not touch Σ_n. Now we start dropping $\mathcal{C}$ towards Σ_n. The maximum principle ensures that when the two circles in $\partial\mathcal{C}$ touch for the first time the plane $\{z = 0\}$, $\mathcal{C}$ is still completely above Σ_n. This last catenoid serves as a barrier for Σ_n from above. As this barrier is independent of n, we ensure the existence of the limit of u_n when $n \to \infty$, at least in the portion of D enclosed by the vertical projection of $\mathcal{C}$ (in green in Figure 41). To prove that $\lim_n u_n$ exists in the rest of D (i.e. in the two orange zones in Figure 41) we will use another minimal annulus as a barrier. The idea consists of replacing the pair of circles $\partial\mathcal{C}$ by two vertical rectangles R_1, R_2 at the same height, with the same vertical projection as $\partial\mathcal{C}$. That vertical projection is a rectangle of sides $a-\varepsilon$, $b+\varepsilon$ (being $\varepsilon > 0$ arbitrarily small), see Figure 42.

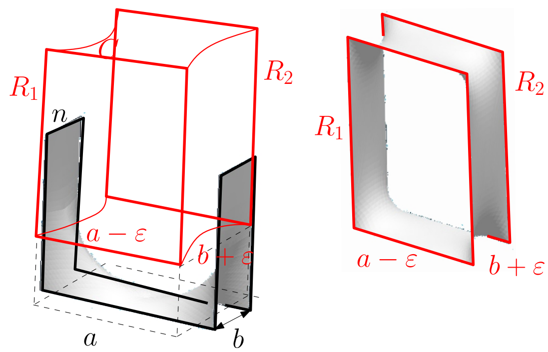


Figure 42: The rectangle in $\{z=0\}$ bounded by the vertical projections of the red rectangles $R_1 \cup R_2$ which form the contour of the barrier, has sides $a-\varepsilon$, $b+\varepsilon$, where $\varepsilon > 0$.

To know if there is a minimal annulus with boundary $R_1 \cup R_2$, we apply the Douglas criterion (Theorem 11.3): it is enough to find a compact annulus Σ with boundary $R_1 \cup R_2$ whose area is strictly less than the sum of the areas of the rectangles R_1 and R_2. The annulus Σ that we get moving R_1 parallel to R_2 has area $2(a-\varepsilon)(b+\varepsilon) + 2n(b+\varepsilon)$ (we are assuming that the height of R_i is n, $i = 1, 2$), while the rectangles $R_1 \cup R_2$ have area $2(a-\varepsilon)n$. So it suffices to check the following inequality:

$$(a-\varepsilon)(b+\varepsilon) + n(b+\varepsilon) < (a-\varepsilon)n.$$

Dividing by $b-\varepsilon > 0$, the above inequality is equivalent to $a-\varepsilon < \left(\frac{a-\varepsilon}{b-\varepsilon} - 1\right) n$. Since $\frac{a-\varepsilon}{b-\varepsilon} - 1 = \frac{a-b}{b-\varepsilon}$, we have transformed our desired inequality in $a-\varepsilon < \frac{a-b}{b-\varepsilon} n$, i.e., $(a-\varepsilon)(b-\varepsilon) < (a-b)n$, which clearly holds for n large enough provided that $a > b$. By the Douglas criterion, if $a > b$ then for n large enough there exists a minimal annulus A_n with boundary $R_1 \cup R_2$, which we use as a barrier for Σ_{n_0} from above (n_0 is fixed). This argument can be made for $\varepsilon > 0$ arbitrarily small, which gives that $\lim_n u_n$ exists in the interior of D provided that $a > b$. Therefore:

> The Dirichlet problem (142) has a (unique) solution over the interior of the rectangle $D = D(a, b)$ with boundary values 0 over the sides with length a and $+\infty$ over the sides with length b, provided that $a > b$.

We cannot prove it yet, but this same Dirichlet problem has no solution if $a \leq b$: this will be a consequence of Theorem 20.4, which will give us necessary and sufficient conditions for the existence of the limit of the graphs Σ_n, conditions that will be translated into $a > b$ (see Note 20.5(3)).

Another interesting Dirichlet problem (142) over the rectangle $D = D(a, b)$ consists of imposing boundary values $-\infty, +\infty, -\infty, +\infty$. If $a = b$ (i.e., D is a square), then we explicitly know a solution: a homothetical image of the doubly periodic Scherk surface (the graph of $z = u(x, y) = \log \frac{\cos x}{\cos y}$). Scherk also proved the existence of a solution of an analogous Dirichlet problem over a rhombus with arbitrary angle, with the same boundary values $-\infty, +\infty, -\infty, +\infty$. Theorem 20.4 will also ensure the existence of this minimal graph over a rhombus.

Before stating Theorem 20.4 and having in mind the previous examples, the following question arises:

> Is it mandatory to take geodesic arcs of the closed curve $C = \partial D \subset \mathbb{R}^2$ if we want to prescribe the boundary values $\pm\infty$ on these arcs?

Lemma 20.3 (Straight line lemma) *Let (M^2, g) be a Riemannian surface, $D \subset M$ a convex domain whose boundary is piecewise C^2 and $c \subset \partial D$ a C^2 arc. If there exists a solution $u\colon \mathrm{Int}(D) \to \mathbb{R}$ of $\mathrm{div}_M \left(\frac{\nabla_M u}{W}\right) = 0$ in $\mathrm{Int}(D)$ and for each $q \in \mathrm{Int}(c)$ we have $\lim\limits_{x \to q} u = +\infty$, then c is a geodesic arc in (M, g) (the same holds if we replace $+\infty$ by $-\infty$).*

Proof. That c is of class C^2 makes possible to talk about the geodesic curvature κ_g of c in $\mathrm{Int}(c)$. To prove the lemma we must see that κ_g vanishes identically in $\mathrm{Int}(c)$. So take a point $q \in \mathrm{Int}(c)$, and we will prove that $\kappa_g(q) = 0$. The first part of the argument will discard $\kappa_g(q) > 0$, for which the argument is relatively simple (based on the maximum principle applied to the graph of u and an auxiliary minimal graph of 'type Scherk'. The second part will discard $\kappa_g(q) < 0$ using a more delicate argument, based on the curvature estimate for stable minimal surfaces and a limit process.

Suppose first that $\kappa_g(q) > 0$ (i.e., the geodesic curvature vector at q points inwards D). In the case of $M = R^2$, let us consider a solution $v_{-\infty}$ to the Dirichlet problem (142) in a triangle $T \subset \mathbb{R}^2$, with boundary values $0, 0, -\infty$ as in Figure 43 (for example, a quarter of a fundamental domain of a doubly periodic Scherk surface on a square); in the general case for M, we will need a solution $v_{-\infty}$ of the same problem in a triangle $T \subset M$, whose sides are three curves of class C^2 with common vertices, so that T is convex, such that $v_{-\infty}$ takes the boundary values $0, 0, -\infty$; this general existence of $v_{-\infty}$ can be justified as follows: let us consider for each $n \in \mathbb{N}$ the Dirichlet problem (142) over T with boundary values $0, 0, -n$, seen as a Plateau problem in $M \times \mathbb{R}$ once we have added suitable vertical segments in $M \times \mathbb{R}$ on the vertices of T to have a closed polygonal curve. Theorem 3.2 (that we can apply even if $M \times R$ is not complete, after an appropriate modification of the product metric outside $T \times \mathbb{R}$) ensures the existence of a least area disk with this boundary; the convexity of T and the generalization of Radó's Theorem to this situation (the arguments for this generalization to hold are analogous to the ones we explained to generalize the proof of Radó's Theorem to spaces $\mathbb{E}(\kappa, \tau)$ in Section 12.1) implies that these minimal disks are minimal graphs over T of functions v_n. Taking $n \to \infty$, a subsequence of v_n converges to the solution $v_{-\infty}$ of the Dirichlet problem in T that we are looking for. Note that $v_{-\infty} \geq 0$ in T (in fact, $v_{-\infty} < 0$ in $\mathrm{int}(T)$). In addition, we make the construction of $v_{-\infty}$ so that the triangle T is placed as in Figure 43 regarding its intersection with D.

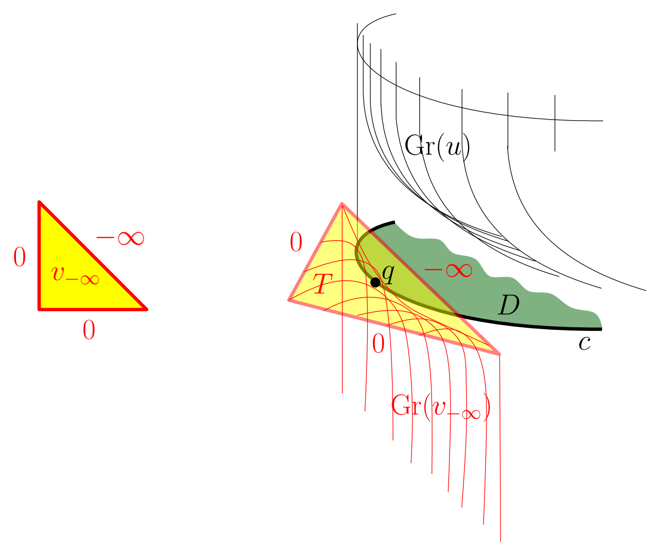


Figure 43: Translating Gr(u) downwards we will contradict the interior maximum principle.

As u tends to $+\infty$ along c, we can move vertically the graph of u up so that Gr(u) lies strictly above Gr(v_{infty}). If we now start moving continuously Gr(u) downwards we will find a first point of interior contact between both surfaces, contradicting the maximum principle. Note that this last argument can be made in $(M \times \mathbb{R}, g \times dt^2)$.

Suppose now that $\kappa_g(q) < 0$ (the geodesic curvature vector at q points outwards D). Since $\Sigma_u := \mathrm{Gr}(u)$ is a stable minimal surface stable (by Theorem 16.12 applied to the Killing field $Y = \frac{\partial}{\partial t}$ in $M \times \mathbb{R}$), then Theorem 19.3 implies that the second fundamental form $|A_{\Sigma_u}|$ is bounded by a constant $C(\varepsilon) > 0$ in $\Sigma_u(\varepsilon) = \{p \in \Sigma_u \mid d_{\Sigma_u}(p, \partial\Sigma_u) \geq \varepsilon\}$, given $\varepsilon > 0$.

Take $q \in \mathrm{Int}(c)$ and $\varepsilon > 0$ sufficiently small so that $d_{\Sigma_u}(p, \partial\Sigma_u) \geq \varepsilon$ for all $p = (x, u(x))$ with x in a neighborhood U_q of q in $\overline{D}$ (this can be done leaving a positive distance from ∂U_q to $\partial D \setminus c$), see Figure 44). This implies that $|A_{\Sigma'_u}|$ is bounded in the portion $\Sigma'_u := \mathrm{Gr}(u|_{U_q}) \subset \Sigma_u$.

Take $p_n = (x_n, u(x_n))$ with $x_n \in U_q$ such that $x_n \to q$ as $n \to \infty$. For each $n \in \mathbb{N}$, let

$$\Sigma'_u(n) := \Sigma'_u - u(x_n)e_3$$

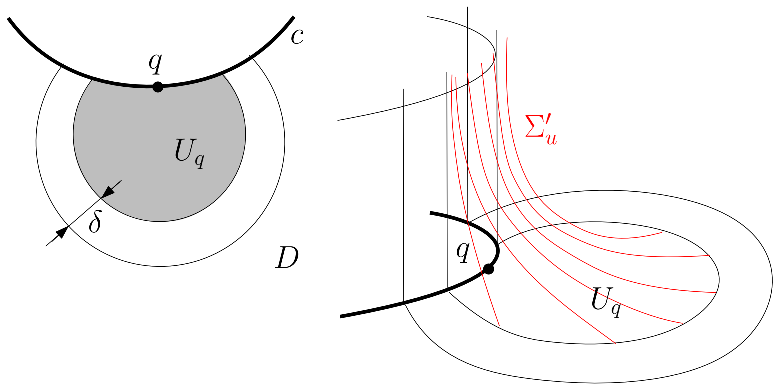


Figure 44: $|A_{\Sigma'_u}|$ is bounded in the sub-graph $\Sigma'_u = \mathrm{Gr}(u|_{U_q}) \subset \Sigma_u$.

be the vertical translation of Σ'_u in $M \times \mathbb{R}$ passing through $(x_n, 0)$. $\Sigma'_u(n)$ is a minimal surface in $(M \times \mathbb{R}, g \times dt^2)$ and $\{\Sigma'_u(n)\}_n$ has uniformly bounded second fundamental form (with the same bound as for $|A_{\Sigma'_u}|$). By the Uniform Graph Lemma (Lemma 12.4, observe that we stated it in $\mathbb{R}^3$ but it can be generalized to $M \times \mathbb{R}$), each $\Sigma'_u(n)$ can be locally written around $(x_n, 0)$ as a graph in exponential coordinates in $M \times \mathbb{R}$ of a function defined on a disk of uniform radius $r_0 > 0$ in the tangent plane $T_{(x_n,0)}\Sigma'_u(n)$ (this radius only depends on the bound for $|A_{\Sigma'_u}|$). Therefore after passing to a subsequence, the $\Sigma'_u(n)$ converge to a minimal surface $\Sigma'_\infty \subset M \times \mathbb{R}$ passing through $(q, 0)$ and that can be written in exponential coordinates in $M \times \mathbb{R}$ as the graph of a function defined over a disk of radius $r_0/2$ in the tangent plane $T_{(q,0)}\Sigma'_\infty$. In particular, $T_{(q,0)}\Sigma'_\infty$ is vertical (if not, Σ'_∞ would project vertically covering an open neighborhood of q in M, and the same would hold for $\Sigma'_u(n)$ for n large enough, which contradicts that $\Sigma'_u(n)$ is a vertical graph over a subset of D).

Consider the Jacobi function $N_3 = \langle N, e_3 \rangle$ over Σ'_∞ (recall that in the Riemannian product $M \times \mathbb{R}$, vertical translations are isometries). N_3 is ≥ 0 in Σ'_∞ (because Σ'_∞ is a vertical graph, since is a limit of vertical graphs). Since $N_3(q) = 0$ and q is an interior point to Σ'_∞, the maximum principle for the Jacobi operator (Theorem 16.14[30]) implies that $N_3 \equiv 0$ in Σ'_∞, i.e., $\Sigma'_\infty = \gamma \times \mathbb{R}$ where γ is a geodesic arc in M passing through q. Moreover, γ has an uniform length ($\geq r_0/2$).

The above argument shows that given $q \in \mathrm{Int}(c)$, there exists a limit of vertical translations

[30]We can avoid using Theorem 16.14 to conclude that $\Sigma'_\infty = \gamma \times \mathbb{R}$ by analyzing the intersection of Σ'_∞ with the minimal surface $\gamma \times \mathbb{R}$, where γ is the geodesic in M that passes through q such that $\gamma \times \mathbb{R}$ is tangent to Σ'_∞ at q: this intersection produces an equiangular system of curves Γ at q and locally around q, Σ'_∞ stays at one side of $\gamma \times \mathbb{R}$ in each component of the complement of Γ; this last property prevents Σ'_∞ from being a vertical graph in any neighborhood of q. This last property is still valid just before the limit of the $\Sigma'_u(n)$ (for n large enough) to Σ'_∞), which contradicts the fact that $\Sigma'_u(n)$ is a vertical graph.

of Σ_u that converges around q to a vertical cylinder $\gamma_q \times \mathbb{R}$ over a geodesic arc γ_q in M that passes through q. Now exchange q by another point $q' \in \mathrm{Int}(c)$ close to q. Since the lengths of $\gamma_q, \gamma_{q'}$ are uniform and c has negative geodesic curvature (smaller than a negative constant that only depends of q for q' close enough to q), we conclude that the geodesic arcs $\gamma_q, \gamma_{q'}$ intersect at a point $x(q, q')$ close to q and q' (Figure 45). This is impossible, as the minimal surfaces $\gamma_q \times \mathbb{R}$, $\gamma_{q'} \times \mathbb{R}$ intersect transversally and are limits of vertical translations of Σ_u (here we are using that Σ_u is a vertical graph). This contradiction finishes the proof of the lemma. $\square$

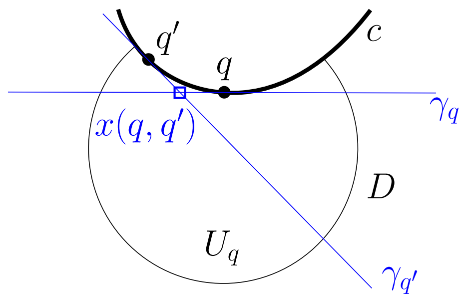


Figure 45: The points q, q' produce geodesics arcs $\gamma_q, \gamma_{q'}$ that intersect at $x(q, q')$.

In order to give a definitive solution to the problem raised at the beginning of this section, we need some notation. Let $C \subset \mathbb{R}^2$ be a closed embedded piecewise C^1 curve, with a finite number of vertices, such that C is the boundary of a compact convex domain $D \subset \mathbb{R}^2$. Decompose C as a finite union of arcs $C = (\cup_i A_i) \cup (\cup_j B_j) \cup (\cup_k C_k)$, where each arc is of class C^1, the A_i, B_j are straight line segments, and there are no two A_i nor two B_j consecutive.

Given a polygon $\mathcal{P}$ inscribed[31] in C (i.e., $\{\text{vertices of } \mathcal{P}\} \subset \{\text{vertices of } \partial C\}$), we will denote

$$a(\mathcal{P}) := \sum_{A_i \subset \mathcal{P}} L(A_i), \qquad b(\mathcal{P}) := \sum_{B_j \subset \mathcal{P}} L(B_j).$$

Hence, $L(\mathcal{P}) = a(\mathcal{P}) + b(\mathcal{P}) + \sum_{C_k \subset \mathcal{P}} L(C_k)$.

Theorem 20.4 (Jenkins-Serrin) *Consider a continuous function $f_k \colon \mathrm{Int}(C_k) \to \mathbb{R}$ for each k. Then, the following conditions are equivalent:*

(A) There exists a solution $u \colon \mathrm{Int}(D) \to \mathbb{R}$ of $\mathrm{div}\left(\frac{\nabla u}{W}\right) = 0$ in $\mathrm{Int}(D)$ with boundary values

$$u = +\infty \text{ in each } A_i, \quad u = -\infty \text{ in each } B_j, \quad u = f_k \text{ in each } C_k.$$

[31] Possibly $\mathcal{P} = C$.

(B) $\underline{\text{If } \cup_k C_k = \varnothing\text{:}}$ $\displaystyle\sum_i L(A_i) = \sum_j L(B_j)$, $2a(\mathcal{P}) < L(\mathcal{P})$ *and* $2b(\mathcal{P}) < L(\mathcal{P})$, *for each polygon* $\mathcal{P} \neq C$ *inscribed in* C.

$\underline{\text{If } \cup_k C_k \neq \varnothing\text{:}}$ $2a(\mathcal{P}) < L(\mathcal{P})$ *and* $2b(\mathcal{P}) < L(\mathcal{P})$, *for each polygon* $\mathcal{P} \neq C$ *inscribed in* C.

Furthermore, if u *exists, then* u *is unique provided that* $\cup_k C_k \neq \varnothing$ *(resp. unique up to vertical translations if* $\cup_k C_k = \varnothing$*).*

Remark 20.5 (1) The theorem holds in $M \times \mathbb{R}$ if we impose that D is geodesically convex and A_i, B_j are geodesic arcs, with the same condition (B) about lengths.

(2) In the case that C is a triangle, the triangle inequality implies that condition (B) is always satisfied. Thus, the Dirichlet problem associated to the boundary values given by Figure 46 has a solution:

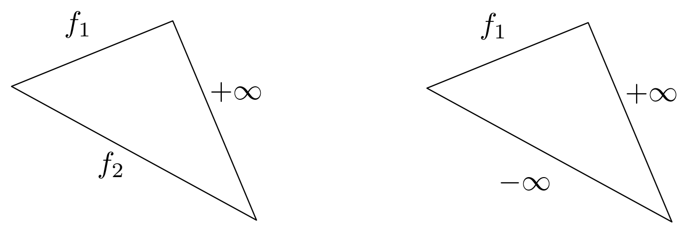


Figure 46: f_1, f_2 are arbitrary continuous functions. We can replace $+\infty$ by $-\infty$ in the triangle on the left.

(3) We can now understand when there exists a solution to the Dirichlet problem associated to the boundary values $0, +\infty, 0, +\infty$ on a rectangle $D(a, b)$ of sides $a, b > 0$ (we impose the zero value on the sides of length a), which we treated after Remark 20.2. In this case, condition (B) of Jenkins-Serrin' Theorem must be checked over any inscribed triangle in the rectangle. On an inscribed triangle, the condition is satisfied by the triangle inequality. On the whole rectangle C, one has $a(C) = 2b$, $b(C) = 0$ and $L(C) = 2a+2b$, hence the condition about lengths is equivalent to $4b < 2a + 2b$, i.e., $b < a$ as we advanced before Lemma 20.3.

Exercise 20.1 *Use Jenkins-Serrin's Theorem to prove that if on a regular polygon with* $2k$ *vertices we impose the boundary values* $+\infty, -\infty$ *cyclically, then there exists a solution of the Dirichlet problem associated to the minimal surface equation with these boundary values, see Figure 47.*

The singly periodic Scherk surface with angle $\pi/2$ (example 6 in Section 2) can also be constructed by Jenkins-Serrin graphs: Consider a rectangle $D = D(a, 1)$ of sides $a > 0$, $b = 1$, and the boundary values given by Figure 48 left.

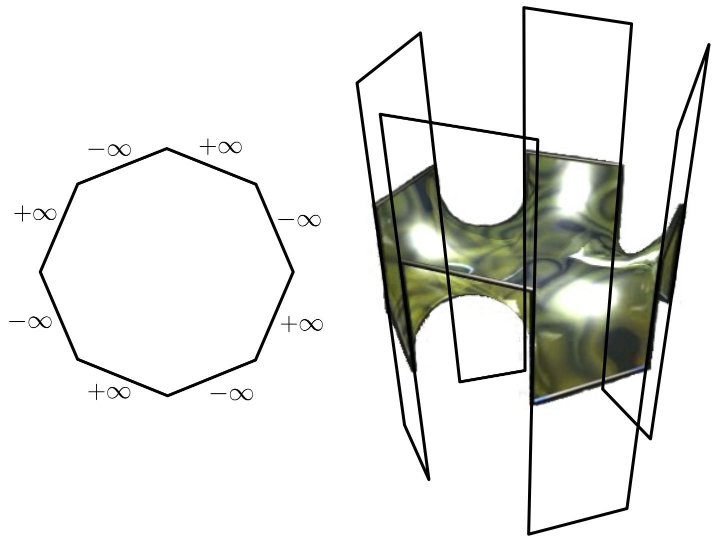


Figure 47: Jenkins-Serrin graph over a regular octagon, with con boundary values $+\infty, -\infty$ (the surface on th right is a compact approximation).

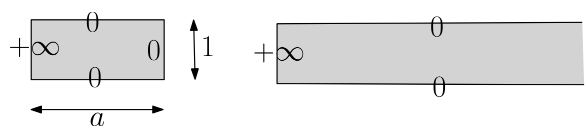


Figure 48: Left: Jenkins-Serrin data on a rectangle of sides $a > 0$ and 1 to produce a singly periodic Scherk surface (after taking $a \to \infty$, right).

It is easy to verify that condition (B) of Jenkins-Serrin's Theorem is satisfied, which produces a minimal graph Σ_a over $D(a, 1)$ with these boundary values (we could have also built Σ_a as a limit when $n \to \infty$ of Douglas-Radó solutions to Plateau problems for Jordan curves Γ_n which project onto $\partial D(a, 1)$, replacing the boundary value $+\infty$ by $n \in \mathbb{N}$). If we now take $a \to \infty$ we will obtain[32] a limit minimal graph $\Sigma_\infty = \lim_{a\to\infty} \Sigma_a$ over the half-strip of width 1 with boundary values $\infty, 0$ (Figures 48 right and 49 left). If we now rotate Σ_∞ an angle of 180^o around each segment or half-line in its boundary, we will generate the singly periodic Scherk surface (Figure 49 right).

[32]We need to use barriers to ensure the existence of this limit; these barriers can be constructed by taking horizontal catenoids suitably arranged above the graphs Σ_a.

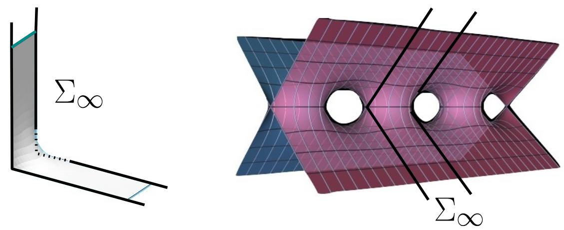


Figure 49: After Schwarz reflection, the surface Σ_∞ generates the singly periodic Scherk surface.

20.1 The Jenkins-Serrin method in $\mathbb{H}^2 \times \mathbb{R}$

The version of Theorem 20.4 in $\mathbb{H}^2 \times \mathbb{R}$ was proven by Nelli and Rosenberg, replacing A_i, B_j by geodesic arcs of $\mathbb{H}^2$ and measuring the lengths of inscribed polygons with respect to the hyperbolic metric. For instance, consider a hyperbolic regular polygon with $2k$ vertices (and geodesic edges). Since we do not have homotheties in $\mathbb{H}^2$ (more precisely, they are not hyperbolic isometries), then we have a 1-parametric family of these non-congruent hyperbolic polygons of $2k$ vertices, which can be parameterized by the inner angle θ of the polygon. θ varies in $\left(0, \pi\left(1-\frac{1}{k}\right)\right)$ ($t \to 0^+$ corresponds to an ideal polygon[33], and $t \to \pi\left(1-\frac{1}{k}\right)^-$ corresponds to the Euclidean case[34], which is the limit of the hyperbolic polygon when it degenerates to a point.

In particular, there is no hyperbolic square ($k = 2$) with internal angle $\pi/2$, but for each $k \in \mathbb{N}$, $k \geq 3$, there exists a unique[35] hyperbolic regular polygon $\mathcal{P}_{2k}$ with $2k$ edges and internal angle $\pi/2$. This polygon satisfies condition (B) of the Jenkins-Serrin Theorem for consecutive boundary values $+\infty, -\infty$. Thus, there exists a minimal graph Σ_u over the interior of this polygon, with these boundary values. Since the angle at each vertex of $\mathcal{P}_{2k}$ is $\pi/2$, we can rotate Σ_u in $\mathbb{H}^2 \times \mathbb{R}$ by angle 180^o with respect to the vertical axes that pass through each vertex of $\mathcal{P}_{2k}$ (these rotations are isometries of $\mathbb{H}^2$). In this way we can extend Σ_u to a minimal graph over the 'black boxes of a chessboard', a tessellation of the hyperbolic plane, in a similar way as with the doubly periodic Scherk surface in $\mathbb{R}^3 = \mathbb{R}^2 \times \mathbb{R}$.

21 The Alexandrov Theorem: the moving plane method

In 1956, Alexandrov applied the maximum principle to surfaces of constant mean curvature and gave one of the most famous characterizations of the sphere:

[33] A hyperbolic polygon is called ideal if its vertices lie on $\partial_\infty \mathbb{H}^2$.

[34] The internal angle of a regular polygon of $2k$ edges in $\mathbb{R}^2$ is $\pi\left(1-\frac{1}{k}\right)$; this can be proved by elementary plane geometry, or in a much simpler way using the Gauss-Bonnet formula.

[35] Up to congruences in $\mathbb{H}^2$.

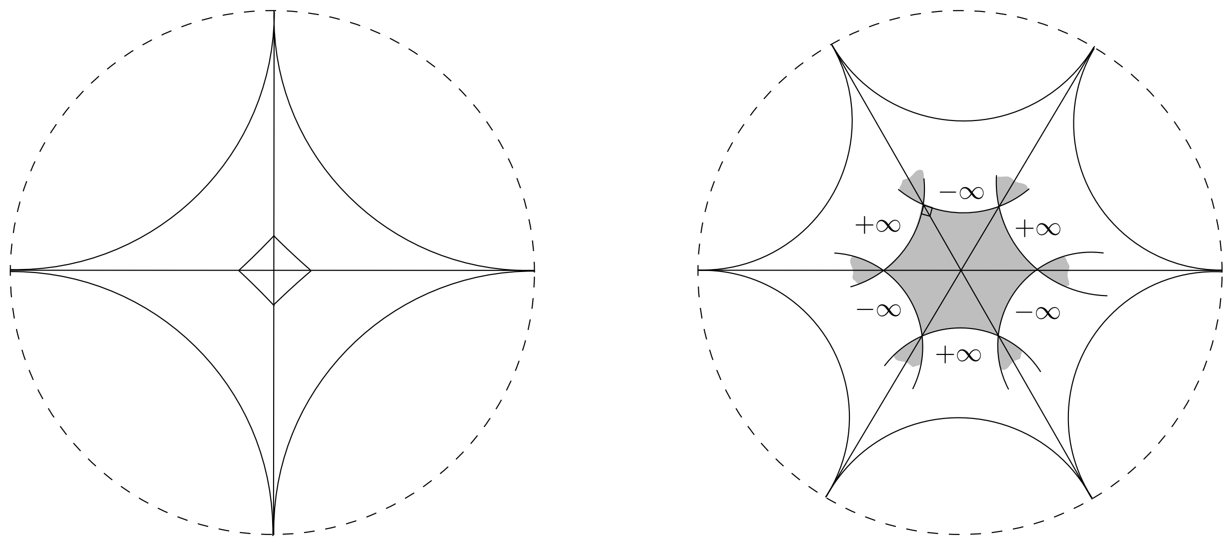


Figure 50: Left: There is no hyperbolic square with internal angle $\pi/2$. Right: A Jenkins-Serrin minimal graph over a regular hexagon in $\mathbb{H}^2$ with internal angle $\pi/2$, which produces by reflection a minimal surface without boundary over the shadowed boxes of the corresponding tessellation of $\mathbb{H}^2$.

Theorem 21.1 (Alexandrov) *Let $\Sigma \subset \mathbb{R}^3$ be a connected, compact, embedded surface with constant mean curvature. Then, Σ is a sphere.*

This theorem is still valid for hypersurfaces of $\mathbb{R}^n$, $n \geq 3$. We now dispose of different proofs from the original, which do not use the maximum principle but are based on integral geometry. In fact, these alternative proofs made possible to generalize the mean curvature in the theorem to any *generalized mean curvature,* meaning any homogeneous symmetric polynomial in the principal curvatures of the hypersurface. However, the proof method employed by Alexandrov has transcended its application to this situation, becoming a powerful and intuitive tool, valid in other contexts to conclude the existence of symmetries of hypersurfaces whose local geometry is related to some PDE maximum principle. This procedure, called the *Alexandrov reflection method,* or *the moving plane method,* will be explained below in its simplest case.

21.1 Characterizations of the sphere

Throughout this section, Σ will denote a connected, compact embedded surface in $\mathbb{R}^3$, Ω the (open) domain enclosed by Σ and N the Gauss map of Σ that points towards Ω, i.e., $\forall p \in \Sigma$ $\exists \varepsilon > 0$ such that $p + tN(p) \in \Omega$ whenever $0 < t < \varepsilon$.

Lemma 21.2 *If all normal lines to Σ pass through a point $q \in \mathbb{R}^3$, then Σ is a sphere centered at q.*

Proof. Suppose there exists $q \in \mathbb{R}^3$ such that $q \in p + \langle N(p) \rangle$, $\forall p \in \Sigma$. Consider the function $\lambda : \Sigma \to \mathbb{R}$ defined by

$$p + \lambda(p)N(p) = q, \quad \forall p \in \Sigma. \tag{143}$$

Since $\lambda(p) = \|p-q\|$, we conclude that λ is smooth in $\Sigma \setminus \{q\}$. Differentiating (143) at $p \in \Sigma \setminus \{q\}$ in the direction of $v \in T_p\Sigma$, we have $[v + \lambda(p)dN_p(v)] + d\lambda_p(v)N(p) = 0$, hence $d\lambda_p(v) = 0$. As v is arbitrary in $T_p\Sigma$ and $\Sigma \setminus \{q\}$ is connected, λ must be constant in $\Sigma \setminus \{q\}$ and thus, also in Σ. Since Σ cannot reduce to $\{q\}$ we have $\lambda > 0$, and the equality $\|p - q\| \equiv \lambda$ ensures that Σ is contained in a sphere $\mathbb{S}$ centered at q of radius λ, hence Σ is an open subset of $\mathbb{S}$. As Σ is compact, Σ must coincide with $\mathbb{S}$. □

Lemma 21.3 *If Σ is symmetric with respect to every plane passing through a point $q \in \mathbb{R}^3$, then Σ is a sphere centered at q.*

Proof. After possibly a translation, we can assume that $q = \vec{0} \in \mathbb{R}^3$.
Take a point $p \in \Sigma \setminus \{\vec{0}\}$. Let $\mathcal{H}_p$ be the set of planes that contains the straight line $\{\lambda p \mid \lambda \in \mathbb{R}\}$. Let us see that every plane in $\mathcal{H}_p$ is orthogonal to $T_p\Sigma$: Given a plane $\Pi \in \mathcal{H}_p$, let S be the symmetry with respect to Π. Since Π passes through $\vec{0}$, Σ must be invariant by S. Differentiating, we have $S(T_p\Sigma) = T_p\Sigma$, hence either $T_p\Sigma = \Pi$ or $T_p\Sigma \perp \Pi$. The first equality cannot occur, since Σ cannot be locally around p a graph over Π and be symmetric with respect to Π, unless $\Sigma \subset \Pi$, which is impossible. Therefore $T_p\Sigma \perp \Pi$. Varying Π in $\mathcal{H}_p$, we have that every plane in $\mathcal{H}_p$ is orthogonal to $T_p\Sigma$ and so, every plane in $\mathcal{H}_p$ contains $N(p)$. This implies that the straight line $\langle N(p) \rangle$ must be equal to $\{\lambda p \mid \lambda \in \mathbb{R}\}$. In other words, the normal line to Σ at p passes through the origin. Since this is valid for every point in $\Sigma \setminus \{0\}$, Lemma 21.2 applies[36] and we conclude that Σ is a sphere. □

Definition 21.4 The *center of mass of* Σ is the vector

$$c(\Sigma) = \frac{1}{A(\Sigma)} \int_\Sigma p \, dA \in \mathbb{R}^3,$$

where $A(\Sigma) = \int_\Sigma dA$ is the area of Σ.

Lemma 21.5 *Let $\phi : \mathbb{R}^3 \to \mathbb{R}^3$ be an isometry of $\mathbb{R}^3$. Then, the center of mass of $\phi(\Sigma)$ is $c(\phi(\Sigma)) = \phi(c(\Sigma))$. In particular, if Σ is invariant by ϕ then its center of mass is a fixed point of ϕ.*

[36] Observe that Lemma 21.2 assumes that ALL normal lines to Σ pass through the same point, but the same proof works if we replace 'all' by 'all except for finitely many'.

Proof. Using the definition of center of mass and the change of variables formula,

$$c(\phi(\Sigma)) = \frac{1}{A(\phi(\Sigma))}\int_{\phi(\Sigma)} q\, dA = \frac{1}{A(\phi(\Sigma))}\int_{\Sigma} \phi(p)|\mathrm{Jac}(\phi)|\, dA,$$

where $|\mathrm{Jac}(\phi)|$ is the absolute value of the Jacobian of ϕ, which equals 1 since ϕ is an isometry. Writing $\phi(x) = Bx + b$ for some $B \in O(3, \mathbb{R})$ and $b \in \mathbb{R}^3$ and using that ϕ preserves area, the last displayed expression can be written as

$$\frac{1}{A(\Sigma)}\int_{\Sigma} \phi(p)\, dA = \frac{1}{A(\Sigma)}\int_{\Sigma} (Bp + b)\, dA = \frac{1}{A(\Sigma)}\int_{\Sigma} Bp\, dA + b$$

$$= B\frac{1}{A(\Sigma)}\int_{\Sigma} p\, dA + b = Bc(\Sigma) + b = \phi(c(\Sigma)). \qquad \square$$

Lemma 21.6 *If for every direction in $\mathbb{R}^3$ there is a plane orthogonal to that direction such that Σ is symmetric with respect to that plane, then Σ is a sphere.*

Proof. By Lemma 21.3, it suffices to prove that Σ is symmetric with respect to all planes passing through some point in $\mathbb{R}^3$, which will be the center of mass c of Σ: Take an affine plane $\Pi \subset \mathbb{R}^3$ that passes through c. By hypothesis, there exists a plane Π' parallel to Π such that Σ is symmetric with respect to Π'. By Lemma 21.5, c is a fixed point of the reflection in Π', hence $c \in \Pi'$ and thus, $\Pi = \Pi'$. In particular, Σ is symmetric with respect to Π. $\square$

Remark 21.7 We could have avoided Lemmas 21.2 and 21.3 and the notion of center of mass, with the following direct proof of Lemma 21.6: Given a direction $a \in \mathbb{S}^2$, take two planes Π_1, Π_2 containing the straight line $\langle a \rangle$ and making an irrational angle θ between them. By hypothesis, there exist planes Π_1', Π_2' parallel to Π_1, Π_2 respectively, that are planes of reflective symmetry of Σ. Therefore, the composition ϕ of the reflections in Π_1' and Π_2' is a rotation in $\mathbb{R}^3$ of angle $2\theta \in \mathbb{R} \setminus \mathbb{Q}$. Thus, Σ is invariant under each of the rotations in the infinite cyclic group G generated by ϕ. As $\mathbb{Q}$ is dense in $\mathbb{R}$, we deduce that G is dense in the group of all rotations about the straight line $\Pi_1' \cap \Pi_2'$. Therefore, Σ is a surface of revolution with axis $\Pi_1' \cap \Pi_2'$, which is a straight line in the direction of a. If we move a in $\mathbb{S}^2$ we will obtain that Σ is a surface of revolution with respect to a straight line in any prescribed direction, hence Σ is a sphere.

21.2 Producing reflective symmetries

In the section we will use the maximum principle for surfaces of constant mean curvature to obtain symmetries of our surface Σ in the hypotheses of Theorem 21.1. Fix a direction $a \in \mathbb{R}^3$, $\|a\| = 1$. Given $t \in \mathbb{R}$, set $\Pi_t = ta + \langle a \rangle^{\perp}$ and let S_t be the symmetry with respect to Π_t, which is given by

$$S_t(x) = x - 2\left(\langle x, a \rangle - t\right) a, \quad \forall x \in \mathbb{R}^3.$$

Consider the sets

$$\Sigma_t^- = \{p \in \Sigma \mid \langle p, a\rangle < t\}, \qquad \Sigma_t^+ = \{p \in \Sigma \mid \langle p, a\rangle > t\}, \qquad \Sigma_t^* = S_t(\Sigma_t^-).$$

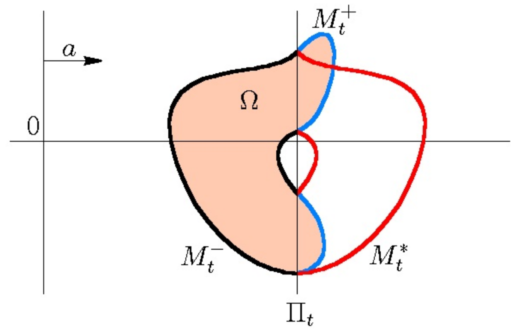


Figure 51: Σ_t^+ (resp. Σ_t^-) is the portion of Σ at the side of Π_t to which a (resp. $-a$) points. Σ_t^* is the reflected image of Σ_t^- by S_t and Ω is the shadowed region.

Since Σ is compact, the set $\{t \in \mathbb{R} \mid \Sigma = \Sigma_t^+\}$ is non-empty and is bounded from above. Thus, there exists

$$t_0 := \sup\{t \in \mathbb{R} \mid \Sigma = \Sigma_t^+\} \in \mathbb{R}. \tag{144}$$

After possibly a translation, we can assume $t_0 = 0$.

Lemma 21.8 *In the above situation,* $\Sigma \cap \Pi_0 \neq \varnothing$ *and* $\forall p \in \Sigma \cap \Pi_0$ *we have* $N(p) = a$.

Proof. Let $h = \langle p, a\rangle$ be the height function with respect to a. For each $t < 0$ we have $h > t$ in Σ, hence passing to the limit we get $h \geq 0$ in Σ. If $h > 0$ in Σ, then by compactness of Σ there exists $t' > 0$ such that $h > t'$ in Σ, hence $\sup\{t \in \mathbb{R} \mid \Sigma = \Sigma_t^+\} \geq t' > 0$, which is a contradiction. Therefore, there exists $p \in \Sigma$ such that $h(p) = 0$, so $\Sigma \cap \Pi_0 \neq \varnothing$.

Take $p \in \Sigma \cap \Pi_0$. Using the notation above, h has a minimum at p, hence $T_p\Sigma$ is parallel to Π_0 or equivalently, $N(p) = \pm a$. Since Σ is contained in the closed half-space $\{x \in \mathbb{R}^3 \mid h(x) \geq 0\}$, this half-space will also contain the domain Ω bounded by Σ. As $N(p)$ points towards Ω and a points inwards this half-space, we must have $N(p) = a$. $\square$

Consider the set

$$\mathcal{A} = \{t > 0 \mid \langle N, a\rangle > 0 \text{ in } \Sigma_t^- \text{ and } \Sigma_s^* \subset \Omega \ \forall s \in (0, t)\}. \tag{145}$$

Lemma 21.9 *There exists* $\varepsilon > 0$ *such that* $(0, \varepsilon) \subset \mathcal{A}$ *and* $\mathcal{A}$ *is an interval.*

Proof. Clearly if $\mathcal{A}$ is non-empty, then it is an interval. Let us see that $\exists\varepsilon_1 > 0$ such that $\langle N, a\rangle > 0$ in $\Sigma^-_{\varepsilon_1}$: otherwise, we can find a sequence $\{t_n\}_n \searrow 0$ such that for each $n \in \mathbb{N}$ there exists $p_n \in \Sigma^-_{t_n}$ such that $\langle N(p_n), a\rangle \leq 0$. Passing to a subsequence, $\{p_n\}_n$ converges to a point $p_\infty \in \Sigma \cap \Pi_0$. Taking limits we have $\langle N(p_\infty), a\rangle \leq 0$, which contradicts Lemma 21.8. Thus, $\exists\varepsilon_1 > 0$ such that $\langle N, a\rangle > 0$ in $\Sigma^-_{\varepsilon_1}$.

Next we check that for $\varepsilon \in (0, \varepsilon_1)$ sufficiently small, we have $\Sigma^*_s \subset \Omega$ whenever $0 < s < \varepsilon$ (which will prove the lemma): arguing by contradiction, suppose that there exists a sequence $\{s_n\}_n \subset (0, \varepsilon_1)$, $s_n \searrow 0$, such that $\Sigma^*_{s_n} \not\subset \Omega$ for all n. In particular, for every $n \in \mathbb{N}$ there exists $p_n \in \Sigma^-_{s_n}$ such that $p^*_n \notin \Omega$, where $p^*_n = S_{s_n}(p_n)$. Consider the segment

$$[p_n, p^*_n] = \{(1-t)p_n + tp^*_n \mid 0 \leq t \leq 1\}.$$

Since $p_n \in \Sigma^-_{s_n}$ and $s_n \in (0, \varepsilon_1)$, we have $\langle N(p_n), a\rangle > 0$ and thus, starting at p_n and moving along $[p_n, p^*_n]$ we find points of Ω arbitrarily close to p_n. Since $p^*_n \notin \Omega$, by connectedness we deduce that the open segment

$$(p_n, p^*_n) = \{(1-t)p_n + tp^*_n \mid 0 < t < 1\}$$

must intersect $\partial\Omega = \Sigma$, hence there exists $x_n \in (p_n, p^*_n) \cap \Sigma$. Passing to a subsequence, $\{p_n\}_n$ converges to a point $p_\infty \in \Sigma \cap \Pi_0$, hence both $\{p^*_n\}_n$ and $\{x_n\}_n$ will have the same limit p_∞ as $n \to \infty$. By Lemma 21.8, $N(p_\infty) = a$. Hence Σ is locally around p_∞ a graph over Π_0, which contradicts that $p_n, x_n \in \Sigma$ have the same projection over Π_0 and both converge to p_∞. This finishes the proof of the lemma. $\square$

$\mathcal{A}$ is bounded from above, because for $s > \inf\{t \in \mathbb{R} \mid \Sigma = \Sigma^-_t\}$ the containment $\Sigma^*_s \subset \Omega$ cannot hold. Thus, there exists $T := \sup \mathcal{A} \in (0, \infty)$.

Lemma 21.10 *In the above situation, $\langle N, a\rangle > 0$ in Σ^-_T and $\Sigma^*_T \subset \overline{\Omega}$.*

Proof. Given $p \in \Sigma^-_T$, there exists $t \in \mathcal{A}$ such that $\langle p, a\rangle < t$ hence $p \in \Sigma^-_t$, and by definition of $\mathcal{A}$, we have $\langle N(p), a\rangle > 0$. For the second part of the lemma, take a point $p \in \Sigma^-_T$ and we will see that $S_T(p)$ lies in $\overline{\Omega}$. Let $\{t_n\}_n \subset \mathcal{A}$ be a sequence with $t_n \nearrow T$. Since $p \in \Sigma^-_T$ and the t_n converge to T, we will have $p \in \Sigma^-_{t_n}$ for n large (except for a finite number of integers, thus we do not loss generality assuming that this occurs $\forall n \in \mathbb{N}$). Since $t_n \in \mathcal{A}$, $S_{t_n}(p)$ lies in Ω for each $n \in \mathbb{N}$. But $S_{t_n}(p) \to S_T(p)$ as $n \to \infty$, hence $S_T(p) \in \overline{\Omega}$. $\square$

From Lemma 21.10, it is clear that $\langle N, a\rangle \geq 0$ in $\Sigma \cap \Pi_T$.

Lemma 21.11 *Suppose that $\langle N, a\rangle > 0$ in $\Sigma \cap \Pi_T$. Then, $\Sigma^*_T \cap \Sigma^+_T \neq \varnothing$.*

Proof. Since $\langle N, a\rangle > 0$ in the compact set $\Sigma \cap \Pi_T$, there exists $\varepsilon > 0$ such that $\langle N, a\rangle > 0$ in $\Sigma \cap \{T - \varepsilon < h < T + \varepsilon\}$. Since $\langle N, a\rangle > 0$ in Σ^-_T, we deduce that $\langle N, a\rangle > 0$ in $\Sigma^-_{T+\varepsilon}$.

Take a sequence $\{t_n\}_n \subset (T, T+\varepsilon)$ with $t_n \searrow T$. In particular, $t_n \notin \mathcal{A}$. Since $\langle N, a \rangle > 0$ in $\Sigma^-_{t_n}$, there exists $\widetilde{t_n} \in (0, t_n)$ such that $S_{\widetilde{t_n}}(\Sigma^-_{\widetilde{t_n}}) \not\subset \Omega$. In fact, $\widetilde{t_n} \in [T, t_n)$ (because $T = \sup \mathcal{A}$ and by definition of $\mathcal{A}$). Therefore, we can find a point $p_n \in \Sigma^-_{\widetilde{t_n}}$ such that $p^*_n = S_{\widetilde{t_n}}(p_n) \notin \Omega$. Consider the segment $[p_n, p^*_n]$. If we start at p_n and move along this segment, we will start by finding points of Ω (because $p_n \in \Sigma$ and $\langle N(p_n), a \rangle > 0$). Since $p^*_n \notin \Omega$, we conclude that there exists $x_n \in (p_n, p^*_n) \cap \Sigma$. Passing to a subsequence, $\{p_n\}_n$ converges to a point $p_\infty \in \Sigma$ and $\lim_n p^*_n = \lim_n S_{\widetilde{t_n}}(p_n) = S_T(p_\infty)$. As $p_n \in \Sigma^-_{\widetilde{t_n}}$, after passing to the limit we will have $p_\infty \in \Sigma^-_T \cup (\Sigma \cap \Pi_T)$. We now discuss two possibilities:

- $\underline{p_\infty \in \Sigma^-_T.}$ In this case, $S_T(p_\infty)$ belongs to Σ^*_T. If we check that $S_T(p_\infty) \in \Sigma^+_T$ then we will have $S_T(p_\infty) \in \Sigma^*_T \cap \Sigma^+_T$, hence the lemma will hold. Since $p_\infty \in \Sigma^-_T$, we have $\langle S_T(p_\infty), a \rangle > T$. By Lemma 21.10, $\Sigma^*_T \subset \overline{\Omega}$ and thus, $S_T(p_\infty) \in \overline{\Omega}$. As $S_T(p_\infty)$ is the limit of $p^*_n \notin \Omega$, we deduce that $S_T(p_\infty) \in \overline{\mathbb{R}^3 \setminus \Omega} = \mathbb{R}^3 \setminus \Omega$. Therefore, $S_T(p_\infty) \in \overline{\Omega} \setminus \Omega = \partial\Omega = \Sigma$ and thus, $S_T(p_\infty) \in \Sigma^+_T$, which finishes this case.

- $\underline{p_\infty \in \Sigma \cap \Pi_T.}$ In this case we have $S_T(p_\infty) = p_\infty$, hence both p_n and $p^*_n = S_{t_n}(p_n)$ converge to the same limit p_∞. In particular, x_n also converges to p_∞. Since by hypothesis $\langle N(p_\infty), a \rangle > 0$, Σ must be locally around p_∞ a graph over Π_T, which contradicts that $p_n, x_n \in \Sigma$ have the same projection over Π_T but both points converge to p_∞. Thus, this second possibility cannot occur and the Lemma is proved. $\square$

Lemma 21.12 *$\Sigma^*_T \cap \Sigma^+_T \neq \varnothing$.*

Proof. Arguing by contradiction, suppose that $\Sigma^*_T \cap \Sigma^+_T = \varnothing$. By Lemma 21.11, there exists $p \in \Sigma \cap \Pi_T$ such that $\langle N(p), a \rangle = 0$. Hence $T_p\Sigma$ is orthogonal to Π_T (we have identified $T_p\Sigma$ with the affine tangent plane to Σ at p), and Σ, Π_T intersect orthogonally at p. Consider Σ^-_T and its reflected image Σ^*_T. Locally around p, both surfaces can be considered to be surfaces with boundary. This boundary is shared by both surfaces, and consists of a regular curve γ passing through p (regularity of γ follows from transversality of $T_p\Sigma$ and Π_T). The Gauss map N^* of Σ^*_T is $N^* = S(N^-)$, where S denotes the linear part of the reflection in Π_T and N^- is the Gauss map of Σ^-_T (i.e., $N^- = N|_{\Sigma^-_T}$). Since $T_p\Sigma$ is orthogonal to Π_T, we have $N(p) \in \Pi_T$, hence $S(N(p)) = N(p)$ and thus, $N^-(p) = N(p) = N^*(p)$. In particular, $T_p(\Sigma^-_T) = T_p(\Sigma^*_T)$. On the other hand, the unit conormal vectors η^-, η^* pointing outwards Σ^-_T and Σ^*_T along $\Sigma \cap \Pi_T$ are clearly opposite and orthogonal to Π_T. In this situation we deduce that

- $T_p(\Sigma^+_T) = T_p\Sigma = T_p(\Sigma^-_T) = T_p(\Sigma^*_T)$,

- $\eta^+(p) = -\eta^-(p) = \eta^*(p) = -a$, Finally, since $\Sigma_T^* \subset \overline{\Omega}$ we have that Σ_T^* lies locally at one side of Σ_T^+ around p, hence we can apply the boundary maximum principle (observe that Σ_T^*, Σ_T^+ have the same constant mean curvature), to conclude that Σ_T^* coincides with Σ_T^+ in a neighborhood of p. This contradicts our hypothesis that $\Sigma_T^* \cap \Sigma_T^+ = \varnothing$. $\square$

Lemma 21.13 *If $p \in \Sigma_T^-$ satisfies $p^* = S_T(p) \in \Sigma_T^+$, then there exists a neighborhood U of p in Σ_T^- such that $U^* = S_T(U) \subseteq \Sigma_T^+$.*

Proof. By Lemma 21.10, $\Sigma_T^* \subset \overline{\Omega}$. As $p^* \in \Sigma_T^* \cap \Sigma_T^+$, we conclude that Σ_T^* and Σ_T^+ intersect tangentially at p^* and one surface lies at one side of the other one around this point. In particular, $N^*(p^*) = \pm N(p^*)$, where $N^* = S(N)$ is the Gauss map of Σ_T^* and S is the linear part of the reflection in Π_T.

Let us see that $N(p^*) = N^*(p^*)$: Recall that N is the Gauss map of Σ pointing towards Ω. If $N(p^*) = -N^*(p^*)$, then after replacing the Gauss map in Σ_T^* by $\nu = -N^*$, and with respect to this common normal vector at p^* we will have $\Sigma_T^* \geq \Sigma_T^+$. Comparing the second fundamental forms of both surfaces we will conclude that the mean curvature ot p of Σ_T^* with respect to ν is not less than that of Σ_T^+ respect to N, which means that $-H \geq H$. But H is a positive constant, which is a contradiction.

Therefore, $N(p^*) = N^*(p^*)$ and we can apply the interior maximum principle to Σ_T^*, Σ_T^+ at p (both surfaces have the same constant mean curvature with respect to the same unit normal vector and one surface lies at one side of the other around p). This proves the lemma. $\square$

Proposition 21.14 *In the above situation, Σ is symmetric with respect to Π_T.*

Proof. By Lemma 21.12, there exists a point $p_0 \in \Sigma_T^-$ such that $p_0^* = S_T(p_0) \in \Sigma_T^+$. Let $\mathcal{C}$ the component of Σ_T^- that contains p_0. Hence, $p_0 \in A := \{p \in \mathcal{C} \mid p^* \in \Sigma_T^+\}$. Furthermore, A is an open subset of $\mathcal{C}$ (by Lemma 21.13). Since A is clearly closed in $\mathcal{C}$ and $\mathcal{C}$ is connected, we have $A = \mathcal{C}$ hence $\mathcal{C}^* = S_T(\mathcal{C}) \subseteq \Sigma_T^+$. In particular, $\mathcal{C} \cup \mathcal{C}^* \subset \Sigma$.

On the other hand, $\mathcal{C}$ is a surface with boundary ($\partial\mathcal{C} \neq \varnothing$ because otherwise $\mathcal{C} = \Sigma$ hance $\Sigma_T^+ = \varnothing$, which contradicts Lemma 21.12), $\mathcal{C} \subset \{x \in \mathbb{R}^3 \mid \langle x, a\rangle < T\}$ and $\partial\mathcal{C} \subset \Pi_T$. Thus, $\mathcal{C}^*$ is also a surface with boundary, $\mathcal{C}^* \subset \{x \in \mathbb{R}^3 \mid \langle x, a\rangle > T\}$ and $\partial\mathcal{C}^* = \partial\mathcal{C}$. From here we conclude that $\mathcal{C} \cup \mathcal{C}^* \cup \partial\mathcal{C}$ is a topological surface, which is of class C^∞ except (possibly) along $\partial\mathcal{C}$.

Let us check that $\mathcal{C} \cup \mathcal{C}^* \cup \partial\mathcal{C}$ is smooth along $\partial\mathcal{C}$: given $q \in \partial\mathcal{C} \subset \Pi_T$, choose a sequence $\{q_n\}_n \subset \mathcal{C}$ such that $q_n \to q$ as $n \to \infty$. Then, $N(q_n)$ converges to $N(q)$ and $q_n^* = S_T(q_n)$ converges to $q^* = q$. As $q_n^* \in \mathcal{C}^* \subset \Sigma$, we have that $N(q_n^*)$ makes sense, which converges to $N(q^*) = N(q)$. But $N(q_n^*) = S(N(q_n))$ (as before, S denotes the linear part of S_T), and taking limits, $N(q) = S(N(q))$. Therefore, $N(q) \in \Pi_T$ and $T_q\Sigma$ is orthogonal to Π_T. This implies that $\mathcal{C}$ and $\mathcal{C}^*$ have the same tangent plane at q as surfaces with boundary. In particular, $\mathcal{C}^*, \Sigma_T^+$ have the same normal and conormal vectors at q, the same (constant) mean curvature and one surface lies at one side of the other around q (because $\Sigma_T^* \subset \overline{\Omega}$), hence $\mathcal{C}^*$ and Σ_T^+ coincide in a neighborhood of q by the boundary maximum principle, and $\mathcal{C} \cup \mathcal{C}^* \cup \partial\mathcal{C}$ is smooth around q.

Finally, as $\mathcal{C} \cup \mathcal{C}^* \cup \partial\mathcal{C}$ is a smooth surface contained in Σ, $\mathcal{C} \cup \mathcal{C}^* \cup \partial\mathcal{C}$ must be an open subset of Σ. Since $\mathcal{C} \cup \mathcal{C}^* \cup \partial\mathcal{C}$ is compact and Σ is connected, it must be $\mathcal{C} \cup \mathcal{C}^* \cup \partial\mathcal{C} = \Sigma$ and thus, Σ is symmetric with respect to Π_T. □

Proof of Theorem 21.1. Given any vector $a \in \mathbb{R}^3$ with $\|a\| = 1$, Proposition 21.14 ensures that there exists a plane orthogonal to a such that Σ is symmetric with respect to this plane. By Lemma 21.6, Σ must be a sphere.

21.3 The moving plane method for compact CMC surfaces with planar boundary

Suppose that $\Sigma \subset \{(x_1, x_2, x_3) \in \mathbb{R}^3 \mid x_3 \geq 0\}$ is a connected, compact, embedded, non-flat surface with non-empty boundary $\partial\Sigma \subset \{x_3 = 0\}$ and constant mean curvature H. We will analyze how to modify the Alexandrov reflection method in this setting.

Let us start by taking $a = (0, 0, -1)$ as the vector that defines the planes Π_t with respect to which we will reflect the surface, given by

$$\Pi_t = \{(x_1, x_2, -t) \mid (x_1, x_2) \in \mathbb{R}^2\}, \qquad \forall t \leq 0.$$

Given $t < 0$, Σ_t^- is the portion of Σ strictly above height $|t|$, Σ_t^+ is the portion of Σ between heights $|t|$ and 0 (Σ_t^+ includes $\partial\Sigma$) and Σ_t^* is the reflected image of Σ_t^- with respect to Π_t, which lies below height $|t|$. We will take as inner domain the open set Ω of $\{x_3 > 0\}$ enclosed by $\Sigma \cup \Pi_0$.

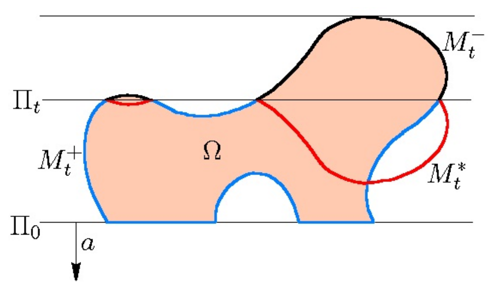


Figure 52: The case with boundary contained in Π_0.

Now the supremum $t_0 = \sup\{t \in \mathbb{R} \mid \Sigma = \Sigma_t^+\}$ is strictly negative. Lemma 21.8 remains valid, and ensures that $\Sigma \cap \Pi_{t_0} \neq \varnothing$ and $N(p) = (0, 0, -1)$ $\forall p \in \Sigma \cap \Pi_{t_0}$. Define

$$\mathcal{A} = \{t > t_0 \mid \langle N, (0, 0, -1)\rangle > 0 \text{ in } \Sigma_t^- \text{ and } \Sigma_s^* \subset \Omega \ \forall s \in (t_0, t)\},$$

Lemma 21.9 is also valid in this context and gives that there exists $\varepsilon > 0$ such that $(t_0, t_0+\varepsilon) \subset \mathcal{A}$. As in the previous case, $\mathcal{A}$ is bounded from above, so it makes sense to define $T = \sup \mathcal{A}$ (and T is clearly negative). Lemmas 21.10, 21.11 and 21.12 (and their proofs) remain valid in our new setting, since the boundary of Σ does not appear in their arguments. However, in Lemma 21.13 we have a new situation: we start with a point $p \in \Sigma_T^-$ such that $p^* = S_T(p) \in \Sigma_T^+$. In this case, either p^* is an interior point to Σ (in this case the previous proof remains valid) or $p^* \in \Sigma_T^* \cap \partial\Sigma$. In this last case, we cannot apply the maximum principle since Σ_T^* and Σ_T^+ are not necessarily tangent at p^*. A way of avoiding this is that the distance between the reflection plane Π_T and the plane Π_0 that contains $\partial\Sigma$ be strictly greater than half of the height of Σ over Π_0, i.e., $|T| < |t_0|/2$ (both T and t_0 are negative). In these conditions, we would conclude that Σ is symmetric with respect to Π_T following the same arguments of Proposition 21.14. But this would lead to $\partial\Sigma = \varnothing$, which is impossible. Therefore, we must have $|T| \geq |t_0|/2$. On the other hand, it is clear that $|T|$ cannot be greater than $|t_0|/2$ (because in that case Σ_T^* would have points below the plane that contains $\partial\Sigma$, in contradiction with $\Sigma_T^* \subset \overline{\Omega}$). Thus, we necessarily have $|T| = |t_0|/2$.

Corollary 21.15 *Let $\Sigma \subset \{(x_1, x_2, x_3) \in \mathbb{R}^3 \mid x_3 \geq 0\}$ be a connected, compact, embedded, non-flat surface with non-empty boundary $\partial\Sigma \subset \{x_3 = 0\}$ and constant mean curvature H. Define $h = \max_\Sigma x_3 > 0$. Then, $\Sigma \cap \{x_3 > h/2\}$ is a graph over its projection to $\{x_3 = 0\}$.*

Proof. By the arguments above, we have $|T| = |t_0|/2 = h/2$. Suppose, arguing by contradiction, that $\Sigma \cap \{x_3 > h/2\}$ is not a graph over its projection to $\Pi_0 = \{x_3 = 0\}$. Thus, there exist $p, q \in \Sigma_T^-$, $p \neq q$, with the same vertical projection over Π_0. We do not loss generality assuming that $x_3(p) < x_3(q)$. Take the horizontal plane that passes by the midpoint of the vertical segment $[p, q]$, which with the notation of the moving plane method is Π_t for $t = -\frac{1}{2}(x_3(p) + x_3(q))$. In particular, Π_t lies strictly above Π_T, hence $\Sigma_t^* \subset \Omega$ (see the definition of $\mathcal{A}$). This is impossible, since $q^* = p \in \Sigma \subset \partial\Omega$. $\square$

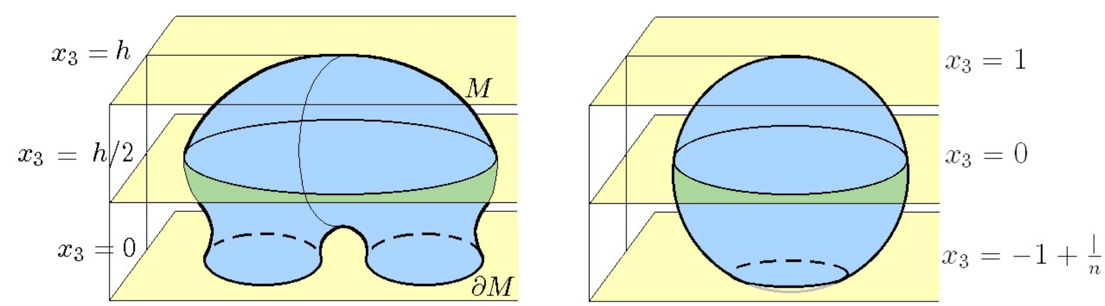


Figure 53: Left: $\Sigma \cap \{x_3 > h/2\}$ is a vertical graph. Right: $\Sigma = \mathbb{S}^2(1) \cap \{x_3 \geq -1 + \frac{1}{n}\}$.

Remark 21.16 *The bound $h/2$ of Corollary 21.15 is sharp: consider the surface $\Sigma_n = \mathbb{S}^2(\vec{0}, 1) \cap \{x_3 \geq -1 + \frac{1}{n}\}$, which satisfies the hypotheses of the corollary. The bound $h/2$ for Σ_n is $1 - \frac{1}{2n}$, which converges to the best possible bound for the sphere.*

21.4 The moving plane method for non-compact CMC surfaces without boundary

Suppose that $\Sigma \subset \mathbb{R}^3$ is a connected, non-compact, embedded surface with constant mean curvature $H > 0$. What can we deduce about Σ by means of the Alexandrov reflection technique?

The basic examples in the above situation are the cylinder of radius $\frac{1}{2H}$ and the Delaunay surfaces. These last ones form a 1-parameter family of singly periodic surfaces of revolution: Given $H > 0$, we can see the parameter of the family of Delaunay surfaces with constant mean curvature H as the neck size d of the generatrix (i.e., the minimum distance from the generatrix curve to the revolution axis). This neck size decreases from $d = \frac{1}{2H}$ (cylinder), passing embedded surfaces called *unduloids,* until an infinite chain of tangent spheres of radius $1/H$, which has $d = 0$. In fact, the family can be extended beyond this configuration, producing non-embedded singly periodic surfaces of revolution and constant mean curvature H called *nodoids.*

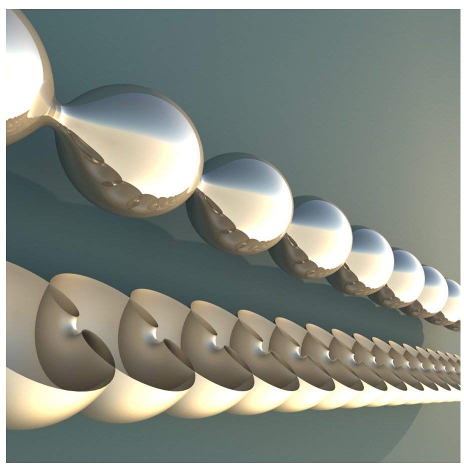


Figure 54: Top: unduloid. Bottom: nodoid.

The family of Delaunay surfaces with a fixed value H for the mean curvature can be studied analytically by writing the first-order ODEs system that a curve in a half-plane has to satisfy in order to generate by revolution around the boundary of the half-plane a surface with constant mean curvature H. It is possible to find a first integral of this system, which can be seen as the parameter for the family (the neck size is a function of this parameter). The classical uniqueness of solutions of an ODE system in terms of initial conditions gives that the unique surfaces of revolution with constant mean curvature $H > 0$ in $\mathbb{R}^3$ are cylinders, Delaunay surfaces and spheres.

Theorem 21.17 (Korevaar, Kusner, Meeks, Solomon) *The unique non-compact, properly embedded annuli with constant mean curvature $H > 0$ in $\mathbb{R}^3$ are cylinders and unduloids.*

We will not give the complete proof, because it exceeds the level of difficulty of these notes. Instead, we will explain a part of the proof that is based on the moving plane method.

Proposition 21.18 *If a surface Σ in the hypothesis of Theorem 21.17 is contained in a solid cylinder of $\mathbb{R}^3$, then the theorem holds.*

Proof. As Σ is properly embedded in $\mathbb{R}^3$, then Σ separates $\mathbb{R}^3$ into two connected components. Let Ω be the component of $\mathbb{R}^3 \setminus \Sigma$ towards which the mean curvature vector of Σ points. Let C be a solid cylinder of $\mathbb{R}^3$ such that $\Sigma \subset \mathrm{Int}(C)$ (thus $\Omega \subset \mathrm{Int}(C)$). Clearly we can assume that C is vertical. To prove the proposition, it suffices to demonstrate that Σ is of revolution around a straight line contained in C (hence vertical). Using the argument in Remark 21.7, it suffices to prove that given a vertical plane P, there exists a plane P' parallel to P such that Σ is symmetric by the reflection in P'. So let us fix a vertical plane P and we will analyze if the moving plane method can be applied to Σ with respect to parallel planes to P.

Note that the first contact point between Σ and P_{t_0} (this t_0 is defined as in (144)) could occur in infinity, which would force the method to stop working. To solve this problem, take a horizontal plane Q that intersects Σ transversally in a compact set (Σ is properly embedded). Q divides C into two halves, C^+ and C^-, both open sets. Let $\overline{\Sigma^+}$ be the portion of Σ in the closed half-space above Q (thus $\overline{\Sigma^+}$ contains its boundary inside Q). We can assume that P does not intersect the cylinder C. Let us call $P(\varepsilon)$ to the plane obtained by rotating P an angle $\varepsilon > 0$ around the straight line $P \cap Q$ as in Figure 55 left. In particular, $P(\varepsilon)$ does not intersect $\overline{\Sigma^+}$.

Start applying the Alexandrov's reflection technique to the surface with boundary $\overline{\Sigma^+}$ with respect to parallel planes to $P(\varepsilon)$; we will denote these planes by $P(\varepsilon)_t$, $t > 0$ (the sign for t is taken so that $P(\varepsilon)_t$ intersects C^+ $\forall t > 0$). Given $t > 0$, we will call

- $P(\varepsilon)_t^+$ to the open half-space with boundary $P(\varepsilon)_t$, which intersects C^+ in an unbounded set, and $P(\varepsilon)_t^-$ to the opposite open half-space.
- $(\overline{\Sigma^+})_t^- = \overline{\Sigma^+} \cap P(\varepsilon)_t^-$, and $(\overline{\Sigma^+})_t^+ = \overline{\Sigma^+} \cap P(\varepsilon)_t^+$.
- $(\overline{\Sigma^+})_t^*$ to the reflected image of $(\overline{\Sigma^+})_t^-$ with respect to $P(\varepsilon)_t$.

If $t > 0$ satisfies $P(\varepsilon)_t \cap \Sigma^+ \neq \varnothing$, then $P(\varepsilon)_t \cap \Sigma^+$ is compact (because $P(\varepsilon)_t \cap C$ is compact and Σ is proper), and $\overline{(\overline{\Sigma^+})_t^-} = (\overline{\Sigma^+})_t^- \cup \partial(\overline{\Sigma^+})_t^-$ is also compact.

Let $t_0 > 0$ be the first time in which $P(\varepsilon)_{t_0}$ intersects $\overline{\Sigma^+}$, which exists by the arguments in the last paragraph. We have two possibilities for t_0:

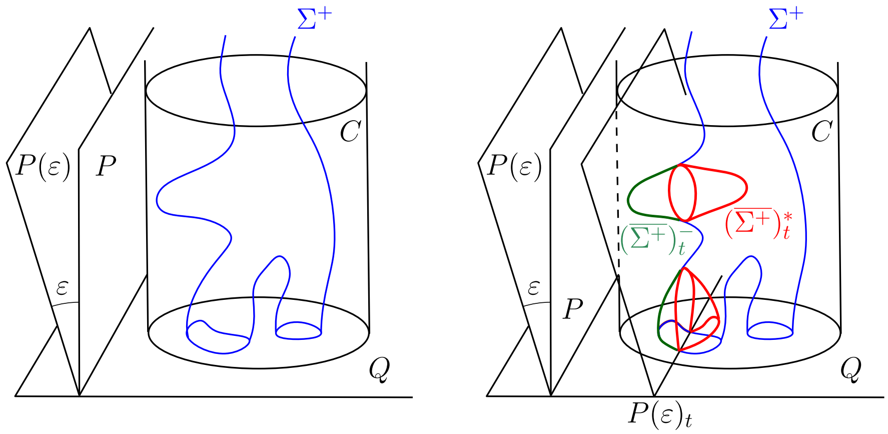


Figure 55: We apply the Alexandrov's reflection technique with respect to the tilted plane $P(\varepsilon)$, reflecting the portion of $\overline{\Sigma^+}$ that $P(\varepsilon)_t$ leaves behind when moving to the right.

(P1) $P(\varepsilon)_{t_0}$ intersects $\overline{\Sigma^+}$ only at interior points to $\overline{\Sigma^+}$. In this case, the normal vector N to Σ for which the mean curvature is $H > 0$, points towards $P(\varepsilon)^+$ at every point of $P(\varepsilon)_{t_0} \cap \overline{\Sigma^+}$.

(P2) $P(\varepsilon)_{t_0}$ intersects $\overline{\Sigma^+}$ in at least one point of $\partial\Sigma^+ \subset Q$.

Let A be the set of real values $t > t_0$ such that $(\overline{\Sigma^+})^*_t \subset \Omega$ and the angle between $(\overline{\Sigma^+})^*_t$ and $P(\varepsilon)_t$ is $< \frac{\pi}{2}$ along $\partial(\overline{\Sigma^+})^*_t$ (this is the analogous definition to (145) for this case with boundary).

If (P1) holds, then for all $t > t_0$ close enough to t_0, we have $t \in \mathcal{A}$ (argue as in the case without boundary). If (P2) holds the same conclusion is true, because we can assume that $\varepsilon > 0$ was taken small enough so that the angle between $(\overline{\Sigma^+})^*_{t_0}$ and $P(\varepsilon)_{t_0}$ is $< \frac{\pi}{2}$ along $\partial(\overline{\Sigma^+})^*_{t_0}$ (recall that Q is transversal to Σ and $Q \cap \Sigma$ is compact).

On the other hand, $\mathcal{A}$ is bounded from above, because $(\overline{\Sigma^+})^*_t$ cannot be contained in C if $t \gg 1$ (this follows from the fact that the distance from $Q \cap C$ to $P(\varepsilon)_t$ tends to $+\infty$ as $t \to +\infty$).

Therefore, there exists $T := \sup \mathcal{A} \in (t_0, \infty)$. If $(\overline{\Sigma^+})^*_T$ (first accident).

Ii $(\overline{\Sigma^+})^*_T$ and $\overline{\Sigma^+}$ are tangent at an interior point to both surfaces, we contradict the interior maximum principle (note that $(\overline{\Sigma^+})^*_T$ cannot be contained in $\overline{\Sigma^+}$ because $P(\varepsilon)_T$ cannot be a symmetry plane of Σ, as $\Sigma \subset C$ and $\varepsilon > 0$).

If the angle between $(\overline{\Sigma^+})^*_T$ and $P(\varepsilon)_T$ equals $\pi/2$ at some point of $\partial(\overline{\Sigma^+})^*_T$, then we contradict the boundary maximum principle.

Therefore, the first accident must occur at a point $q = q(\varepsilon) \in (\partial\Sigma^+)^*_T \cap \Sigma^+$, as in the next figure:

Now move $\varepsilon > 0$ towards zero: observe that if $\varepsilon' \in (0, \varepsilon)$, then $x_3(q(\varepsilon')) < x_3(q(\varepsilon))$. This implies that taking $\varepsilon_n \searrow 0$, we obtain points $q_n = q(\varepsilon_n) \in (\partial\Sigma^+)^{*,\varepsilon_n}_{T(n)} \cap \Sigma^+$ where a first accident

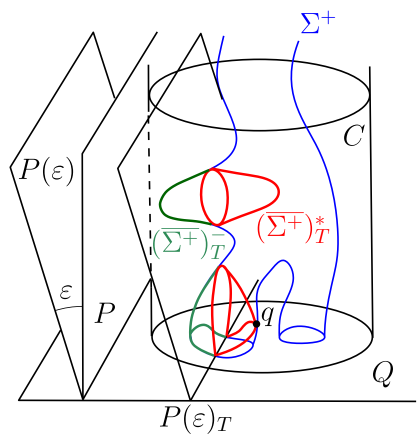


Figure 56: The first accident occurs at q, where the reflected boundary of $(\overline{\Sigma^+})^-_T$ touches $(\Sigma^+)^+_T$ by first time.

occurs when reflecting $\overline{\Sigma^+}$ with respect to planes parallel to $P(\varepsilon_n)$. This first accident points q_n converge as $n \to \infty$ (after passing to a subsequence) to a point $q_\infty \in \partial\Sigma^+$, and the values $T(n)$ also converge to some $T(\infty) > 0$.

As $(\overline{\Sigma^+})^{*,\varepsilon_n}_{T(n)} \subset \overline{\Omega}$ for each $n \in \mathbb{N}$, taking limits we have $(\overline{\Sigma^+})^{*,\varepsilon=0}_{T(\infty)} \subset \overline{\Omega}$. But $(\overline{\Sigma^+})^{*,\varepsilon=0}_{T(\infty)}$ is the reflection of $\overline{\Sigma^+}$ with respect to a plane parallel to P (at distance $T(\infty)$ from P).

The above arguments show that if we apply the Alexandrov's reflection technique to $\overline{\Sigma^+}$ with respect to planes parallel to P, then the first accident occurs when reflecting with respect to $P_{T(\infty)}$, and we find a point $q(\infty) \in \partial\Sigma^+$ such that $(\overline{\Sigma^+})^{*,\varepsilon=0}_{T(\infty)} \subset \overline{\Omega}$, see Figure 57.

We now apply the same reasoning to Σ^- with tilted planes $P(\varepsilon)$ of the type explained in Figure 58.

Reasoning analogously, we obtain a limit of first accident points, that gives a point $q'(\infty) \in \partial\Sigma^-$ such that $(\overline{\Sigma^-})^{*,\varepsilon=0}_{T'(\infty)} \subset \overline{\Omega}$. Clearly $T(\infty) = T'(\infty)$ and $q(\infty) = q'(\infty)$.

Finally, we deduce that if we apply the moving plane method to the whole surface Σ with respect to planes parallel to P, then the primer accident occurs when reflecting with respect to $P_{T(\infty)}$ where we find the above point $q(\infty)$, which is now interior to Σ and satisfies $\Sigma^{*,\varepsilon=0}_{T(\infty)} \subset \overline{\Omega}$. Since $\Sigma^{*,\varepsilon=0}_{T(\infty)}$ is tangent to Σ and lies at one side of Σ, we deduce that $\Sigma^{*,\varepsilon=0}_{T(\infty)} \subset \Sigma$ by the interior maximum principle. Therefore, $P_{T(\infty)}$ is a symmetry plane of Σ and the proof is complete. □

21.5 The moving plane method in $\mathbb{H}^2 \times \mathbb{R}$

The hyperbolic plane $\mathbb{H}^2$ is a homogeneous manifold, i.e., given $p, q \in \mathbb{H}^2$ there exists $\phi \in \text{Iso}(\mathbb{H}^2)$ such that $\phi(p) = q$. Therefore, $\mathbb{H}^2 \times \mathbb{R}$ (with the metric product) is also a homogeneous manifold.

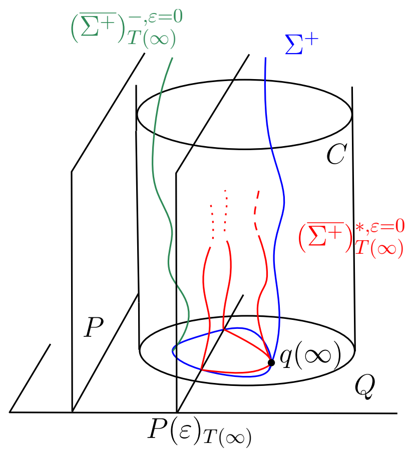


Figure 57: When the reflection plane $P(\varepsilon_n)_{T(n)}$ tends to vertical, the first accident point converges to a point $q(\infty) \in \partial\Sigma^+$.

Using the Poincaré disk model for $\mathbb{H}^2$, the hyperbolic metric is conformal to the usual inner product, with conformal factor depending only on the distance to the origin $\vec{0}$ in the disk. Thus, Euclidean rotations around the origin are isometries of the hyperbolic metric, and Euclidean rotations of $\mathbb{H}^2 \times \mathbb{R}$ around the axis $\{\vec{0}\} \times \mathbb{R}$ are isometries of $\mathbb{H}^2 \times \mathbb{R}$ that fix that axis pointwise. Since $\mathbb{H}^2$ is homogeneous, for each point $p \in \mathbb{H}^2$, the subgroup of $\mathrm{Iso}(\mathbb{H}^2)$ of those isometries that fix p is isomorphic to $\mathbb{S}^1$. Hence, the subgroup of $\mathrm{Iso}(\mathbb{H}^2 \times \mathbb{R})$ given by the isometries that fix $\{p\} \times \mathbb{R}$ pointwise is also isomorphic to $\mathbb{S}^1$.

Take a hyperbolic geodesic $\gamma \subset \mathbb{H}^2$ passing through $\vec{0}$ (γ is an Euclidean diameter of the Poincaré disk). Then, there exists an isometry S_γ of $\mathbb{H}^2$ (a Möbius transformation of the disk) that fixes γ pointwise. We will call S_γ the *reflection with respect to* γ (Figure 59 left).

Taking products with the real line, we have that $\gamma \times \mathbb{R}$ is totally geodesic topological plane in $\mathbb{H}^2 \times \mathbb{R}$ and $S_\gamma \times 1_\mathbb{R}$ is an isometry of $\mathbb{H}^2 \times \mathbb{R}$ that has $\gamma \times \mathbb{R}$ as fixed point set. We will call $S_\gamma \times 1_\mathbb{R}$ the *reflection with respect to* $\gamma \times \mathbb{R}$.

We can apply in $\mathbb{H}^2 \times \mathbb{R}$ the Alexandrov's reflection technique as follows. Take a geodesic $\gamma \subset \mathbb{H}^2$, which we parameterize by its arclength s. For each $s \in \mathbb{R}$, let Γ_s be the geodesic of $\mathbb{H}^2$ that intersects γ orthogonally at $\gamma(s)$. Let $R_s \colon \mathbb{H}^2 \times \mathbb{R} \to \mathbb{H}^2 \times \mathbb{R}$ be the reflection with respect to $\Gamma_s \times \mathbb{R}$. The 1-parametric family $\{R_s \mid s \in \mathbb{R}\}$ can be used to perform the moving plane method. Note that we can do this for any horizontal 'direction', or more precisely, for any horizontal geodesic $\gamma \times \{t_0\}$ of $\mathbb{H}^2 \times \mathbb{R}$. With this idea mind, one can prove the following result:

Theorem 21.19 *Let $\Sigma \subset \mathbb{H}^2 \times \mathbb{R}$ be a connected, compact embedded surface with constant mean curvature. Then, Σ is a rotationally symmetric sphere.*

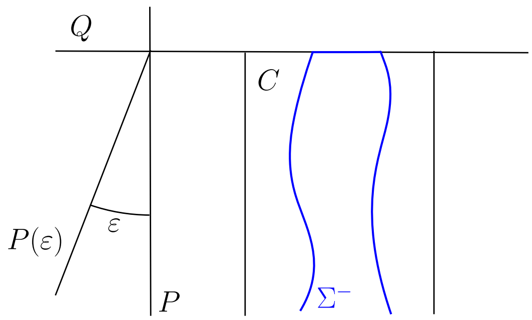


Figure 58: We repeat the process for the bottom part of Σ.

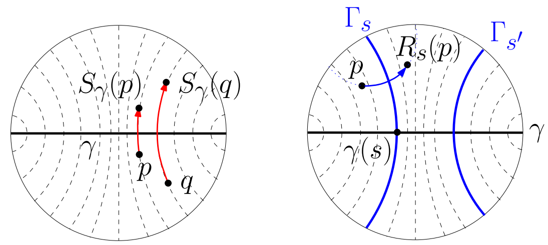


Figure 59: Left: the reflection S_γ with respect to a geodesic γ. Right: R_s is the reflection with with respect to Γ_s, the geodesic orthogonal to γ that passes through $\gamma(s)$.

Unlike what happens in $\mathbb{R}^3$, spheres of mean curvature constant H in $\mathbb{H}^2 \times \mathbb{R}$ do not exist for all positive values of H, but only for $H > 1/2$. This is a consequence of the maximum principle, as we will see below. To do this, we will first construct a foliation of $\mathbb{H}^2 \times \mathbb{R}$ by surfaces of constant mean curvature $1/2$.

In the half-plane model $\{(x, y) \in \mathbb{R}^2 \mid y > 0\}$ of $\mathbb{H}^2$ with the metric $g = \frac{1}{y^2} g_0$, the curve $\alpha(s) = (s, 1)$ is parameterized by arclength and $\nabla_{a'}\alpha' = (0, 1)$, which is also unitary (∇ denotes the Levi-Civita connection of g). This tells us that the absolute geodesic curvature of α equals 1, and the geodesic curvature vector is $\vec{\kappa_g} = (0, 1)$. The trace of α is a horizontal straight line at Euclidean height 1. Going to the Poincaré disk model $\{|z| < 1\}$, α is a *horocycle,* i.e., a circle contained in $\{|z| \leq 1\}$ that touches tangentially $\{|z| = 1\}$ at a single point. All horocycles in $\mathbb{H}^2$ are congruent, since the family of Euclidean homotheties $\{\psi_\lambda(x, y) = (\lambda x, \lambda y)\}_{\lambda>0}$ in the half-plane model map horizontal straight lines into horizontal straight lines. We can also change the tangency point in $\partial_\infty \mathbb{H}^2$ of the horocycle in the Poincaré disk model by considering a Euclidean rotation around $\vec{0}$.

Given a horocycle $\alpha \subset (\mathbb{R}^2)^+ \equiv \mathbb{H}^2$, the topological plane $\alpha \times \mathbb{R} \subset \mathbb{H}^2 \times \mathbb{R}$ is a surface of

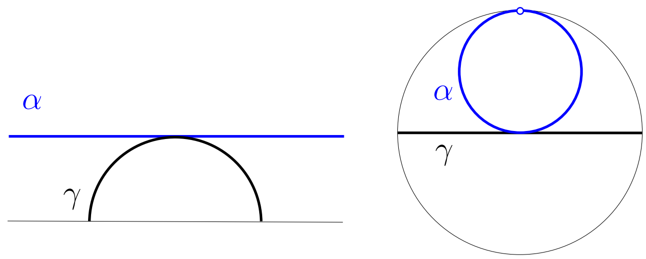


Figure 60: The horocycle α has absolute geodesic curvature 1.

constant mean curvature $H = 1/2$ called *horocylinder*, and the image of $\alpha \times \mathbb{R}$ be the isometry $\phi_\lambda := \psi_\lambda \times 1_{\mathbb{R}}$ of $\mathbb{H}^2 \times \mathbb{R}$ produces a surface congruent to $\alpha \times \mathbb{R}$ and disjoint from this last surface, for each $\lambda > 0$. Therefore, $\mathcal{F} = \{\phi_\lambda(\alpha \times \mathbb{R}) \mid \lambda > 0\}$ is a foliation a $\mathbb{H}^2 \times \mathbb{R}$ by surfaces of constant mean curvature $1/2$.

Once we have the foliation $\mathcal{F}$, we can already prove that there are no spheres of constant mean curvature $H \in (0, 1/2]$ in $\mathbb{H}^2 \times \mathbb{R}$. In fact, we will prove something stronger:

Proposition 21.20 *There are no compact immersed surfaces without boundary in $\mathbb{H}^2 \times \mathbb{R}$ with constant mean curvature $H \in [0, 1/2]$.*

Proof. Suppose that $\Sigma \subset \mathbb{H}^2 \times \mathbb{R}$ is a compact immersed surface without boundary with constant mean curvature $H \geq 0$. H cannot be zero, since otherwise the third component of the immersion of Σ in $\mathbb{H}^2 \times \mathbb{R}$ would be a harmonic function $x_3 \colon \Sigma \to \mathbb{R}$ (use Lemma 4.2 just on the third coordinate function). Since Σ is compact, x_3 would be constant by the maximum principle for harmonic functions. This implies that Σ is contained in $\mathbb{H}^2 \times \{t_0\}$ for some $t_0 \in \mathbb{R}$, which contradicts that Σ is compact without boundary. Thus, $H > 0$.

Now consider the foliation $\mathcal{F} = \{\phi_\lambda(\alpha \times \mathbb{R}) \mid \lambda > 0\}$ constructed above. Observe that given $\lambda > 0$, $\phi_\lambda(\alpha \times \mathbb{R})$ separates $\mathbb{H}^2 \times \mathbb{R}$ into two components (two 'open half-spaces'), namely $(\mathbb{H}^2 \times \mathbb{R}) \setminus \phi_\lambda(\alpha \times \mathbb{R}) = \Omega_\lambda \dot{\cup} [(\mathbb{H}^2 \times \mathbb{R}) \setminus \overline{\Omega_\lambda}]$ where

$$\Omega_\lambda := \bigcup_{\lambda' > \lambda} \phi_{\lambda'}(\alpha \times \mathbb{R}),$$

and that the mean curvature vector of $\phi_\lambda(\alpha_1 \times \mathbb{R})$ points towards Ω_λ. Since Σ is compact, there exists $\lambda_0 > 0$ such that $\Sigma \subset \Omega_{\lambda_0}$. Consider the set $A = \{\lambda > \lambda_0 \mid \Sigma \subset \Omega_\lambda\}$, which is non-empty because Σ is compact. As $\{\phi_\lambda(\alpha \times \mathbb{R}) \mid \lambda > \lambda_0\}$ is a foliation of Ω_{λ_0} and Σ is compact, then A is of the form $A = (\lambda_0, \Lambda)$ for some $\Lambda > \lambda_0$, and the surfaces $\Sigma, \phi_\Lambda(\alpha \times \mathbb{R})$ have a tangential contact point p where Σ lies above $\phi_\Lambda(\alpha \times \mathbb{R})$ (we orient both surfaces by their mean curvature vectors at p, which point to the same direction). By comparison of the second fundamental forms of both surfaces at p we conclude that the mean curvature of Σ is strictly greater than the one of $\phi_\Lambda(\alpha \times \mathbb{R})$, i.e., $H > \frac{1}{2}$. $\Box$

As for the surfaces of revolution with constant mean curvature in $\mathbb{H}^2 \times \mathbb{R}$, let us first analyze vertical cylinders. In the Poincaré disk model for $\mathbb{H}^2$, every circle c contained in the disk enclosed by a horocycle has constant geodesic curvature $\kappa_g = \kappa \in (1, \infty)$ (in absolute value). The cylinder $c \times \mathbb{R} \subset \mathbb{H}^2 \times \mathbb{R}$ has constant mean curvature $H = \frac{\kappa}{2} \in (\frac{1}{2}, \infty)$. Given $H > 1/2$, if we study the system of first-order ODEs that a curve in a vertical half-plane of $\mathbb{H}^2 \times \mathbb{R}$ must satisfy in order to generate by revolution around the boundary of the half-plane a surface of constant mean curvature H, we will find a behavior similar to that of the same problem in $\mathbb{R}^3 = \mathbb{R}^2 \times \mathbb{R}$: it is possible to find a first integral of the system, which can be seen as the parameter of a 1-parameter family of revolution surfaces with constant mean curvature H in $\mathbb{H}^2 \times \mathbb{R}$. These 'Delaunay type' surfaces go from the vertical cylinder with that value of mean curvature, passing through singly periodic embedded surfaces (unduloids) until arriving to an infinite chain of rotationally symmetric spheres that intersect tangentially at an infinite sequence of equally spaced points in the rotation axis. This 1-parameter family of surfaces can be extended analytically beyond the chain of spheres to non-embedded singly periodic surfaces revolution with constant curvature H (nodoids). Again the uniqueness of solutions of this ODE system shows that these are the unique surfaces of revolution with constant mean curvature $H > 1/2$ in $\mathbb{H}^2 \times \mathbb{R}$. (Figure 61).

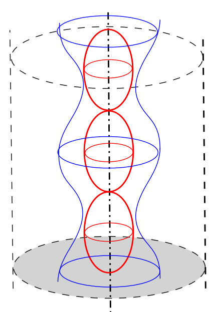


Figure 61: Chain of spheres with constant mean curvature in $\mathbb{H}^2 \times \mathbb{R}$, and an unduloid.

A natural conjecture in this line is the following version of Proposition 21.18 in $\mathbb{H}^2 \times \mathbb{R}$:

Conjecture 21.21 *If $\Sigma \subset \mathbb{H}^2 \times \mathbb{R}$ is a connected, properly embedded surface with constant mean curvature $H > \frac{1}{2}$ and Σ lies inside a solid vertical cylinder in $\mathbb{H}^2 \times \mathbb{R}$, then Σ is a surface of revolution (hence a sphere or unduloid).*

Mazet [12] proved Conjecture 21.21 imposing additionally that Σ has finite topology. We can prove a weaker version of Mazet's result with elementary arguments:

Theorem 21.22 *Let $\Sigma \subset \mathbb{H}^2 \times \mathbb{R}$ be a non-compact, properly embedded annulus with constant mean curvature $H > 0$. If Σ is invariant by a vertical translation $T(p,t) = (p, t+h)$ of $\mathbb{H}^2 \times \mathbb{R}$ for some $h > 0$, then Σ is either a vertical cylinder or an unduloid.*

Proof. Since T is an isometry of $\mathbb{H}^2 \times \mathbb{R}$ that generates a cyclic group $\langle T \rangle$ which acts properly and discontinuously on $\mathbb{H}^2 \times \mathbb{R}$, the quotient set $(\mathbb{H}^2 \times \mathbb{R})/\langle T \rangle$ is a three-dimensional Riemannian manifold isometric to $\mathbb{H}^2 \times \mathbb{S}^1$. The invariance of Σ under T allows us to project Σ to the quotient, thereby producing a compact surface $\Sigma/\langle T \rangle \subset (\mathbb{H}^2 \times \mathbb{R})/\langle T \rangle$ with constant mean curvature H.

The vertical 'planes' $\gamma \times \mathbb{R}$ with γ being a geodesic of $\mathbb{H}^2$ also project to totally geodesic surfaces in $(\mathbb{H}^2 \times \mathbb{R})/\langle T \rangle$ (which are topological annuli, hence we will call them 'vertical annuli'), and there exist reflective symmetries of $(\mathbb{H}^2 \times \mathbb{R})/\langle T \rangle$ with respect to such vertical annuli, each of which is produced by projecting to $(\mathbb{H}^2 \times \mathbb{R})/\langle T \rangle$ the corresponding reflection of $\mathbb{H}^2 \times \mathbb{R}$ with respect to $\gamma \times \mathbb{R}$. Therefore, we can apply the Alexandrov reflection technique to $\Sigma/\langle T \rangle$ in $(\mathbb{H}^2 \times \mathbb{R})/\langle T \rangle$. Since $\Sigma/\langle T \rangle$ is compact, the moving plane method produces for each geodesic $\gamma \subset \mathbb{H}^2$, a vertical annulus in $(\mathbb{H}^2 \times \mathbb{R})/\langle T \rangle$ parallel to $\gamma \times \mathbb{S}^1$ which is of reflective symmetry for $\Sigma/\langle T \rangle$. Lifting this reflective vertical annulus to $\mathbb{H}^2 \times \mathbb{R}$ we get a vertical plane parallel to $\gamma \times \mathbb{R}$, which is of reflective symmetry for Σ. Since γ is any geodesic in $\mathbb{H}^2$, the argument in Remark 21.7 leads us to conclude that Σ is a surface of revolution, which finishes the proof. $\square$

The last proof opens a research line:

What can we say about the theory of minimal or CMC surfaces in $\mathbb{H}^2 \times \mathbb{S}^1$?

As any surface of revolution with constant mean curvature is invariant by a vertical translation, we can view the surfaces of 'Delaunay type" in $\mathbb{H}^2 \times \mathbb{S}^1$, thus producing embedded spheres or tori with constant mean curvature $H > \frac{1}{2}$.

$\mathbb{H}^2 \times \mathbb{R}$ admits other interesting quotients besides $\mathbb{H}^2 \times \mathbb{S}^1$. The role of vertical translation can be played by a 'horizontal translation'. There are two types of translations in $\mathbb{H}^2$ (isometries of $\mathbb{H}^2$ without fixed points): the *hyperbolic translations,* which are translations along a geodesic $\gamma \subset \mathbb{H}^2$, and the *parabolic translations,* which are translations along a horocycle. Next we will see how to generate a properly embedded minimal surface in $\mathbb{H}^2 \times \mathbb{S}^1$ invariant under cyclic group of hyperbolic translations.

After Exercise 20.1 we saw how to generate a singly periodic Scherk surface. We will do now something similar in $\mathbb{H}^2 \times \mathbb{R}$. Take a geodesic $\gamma \subset \mathbb{H}^2$, which we normalize as the diameter of the Poincaré disk. Let us consider all hyperbolic translations ϕ_h along γ, where $h \in \mathbb{R}$ denotes the hyperbolic distance along γ (i.e., if $\gamma = g(s)$ is parameterized by its arclength, then $\phi_h(g(s)) = \gamma(s+h)\ \forall s, h \in \mathbb{R}$). Let $\Gamma = \Gamma(t)$ be the geodesic parameterized by arclength orthogonal to γ such that $\Gamma(0) = \gamma(0) = \vec{0} \in \{|z| < 1\} \equiv \mathbb{H}^2$. Given $t \in \mathbb{R}$, the orbit of $\Gamma(t)$ by $\{\phi_h\}_{h \in \mathbb{R}}$ parameterizes the *equidistant* curve from γ at hyperbolic distance $|t|$ passing through $\Gamma(t)$, which we will call γ_t. The image of Γ by ϕ_h is a geodesic orthogonal to γ, which intersects γ at the point $\gamma(h)$.

Given $h, t > 0$, consider the polygon C of $\mathbb{H}^2$ with vertices $\gamma(-h), \gamma(h), \phi_h(\Gamma(t)), \phi_{-h}(\Gamma(t))$, that has three consecutive geodesic sides, and whose fourth side is an equidistant curve to γ_t joining the vertices $\phi_h(\Gamma(t)), \phi_{-h}(\Gamma(t))$ (see Figure 62 left).

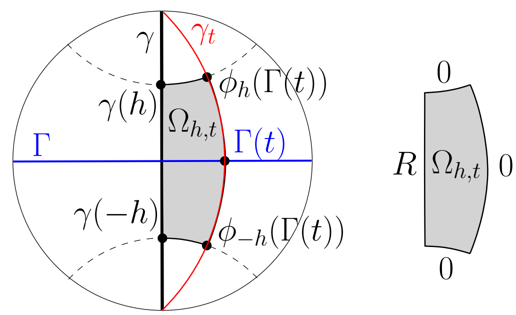


Figure 62: Left: the domain $\Omega_{h,t}$. Right: the boundary data.

Note that the polygon C is convex, because the mean curvature vector of γ_t points towards γ. C is the boundary of an open convex quadrilateral $\Omega_{h,t} \subset \mathbb{H}^2$. Consider the Dirichlet problem (142) over $\Omega_{h,t}$ with boundary conditions $0, 0, 0, R$ (here $R > 0$ is previously chosen), see Figure 62 right.

The version in $\mathbb{H}^2 \times \mathbb{R}$ of Radó's del Theorem implies that there exists a unique minimal surface bounded by the Jordan curve in $\mathbb{H}^2 \times \mathbb{R}$ given by the above boundary values, and this surface is the vertical graph of a function $u_{h,t,R} \colon \Omega_{h,t} \to \mathbb{R}$. We analyze the existence of the limit $\lim_{t\to\infty} u_{h,t,R}$. To do that we will need barriers above the graph of $u_{h,t,R}$, independently of t.

In the half-plane model $\{(x, y) \in \mathbb{R}^2 \mid y > 0\}$, the function[37]

$$v(x, y) = \log\left(\frac{y + \sqrt{x^2 + y^2}}{x}\right)$$

defined in $\{(x, y) \in \mathbb{R}^2 \mid x > 0, y > 0\}$, produces a minimal graph with boundary values 0 over $\{y = 0\}$ and $+\infty$ over $\{x = 0, y > 0\}$. This minimal graph can be passed to the Poincaré disk model of $\mathbb{H}^2$, producing a minimal graph over either of the two disks bounded by an arbitrary geodesic α and $\partial_\infty \mathbb{H}^2$. If we now take $\alpha = \gamma$ in the above construction, the corresponding function v defined over the disk D in $\mathbb{H}^2$ whose boundary is γ and that contains $\Omega_{h,t}$ (D is the shaded region in Figure 63 right) satisfies:

$$v|_{\Omega_{h,t}} > u_{h,t,R},$$

[37]Discovered independently by Abresch and Sa Earp.

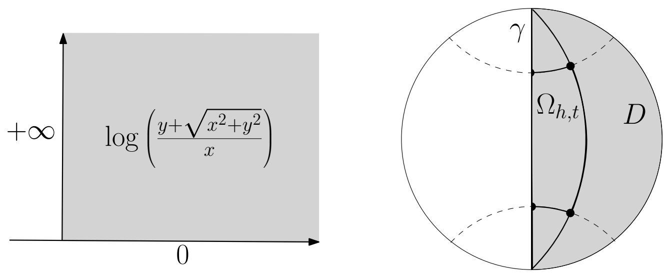


Figure 63: The barrier function v.

by the maximum principle applied to the graphs of both functions. This tells us that the graph of $v|_{\Omega_{h,t}}$ is a barrier from above for the graph of $u_{h,t,R}$, independently of t. Thus, there exists

$$u_{h,R} := \lim_{t\to\infty} u_{h,t,R},$$

defined over the infinite half-strip $\Omega_h := \cup_{t>0}\Omega_{h,t}$. $u_{h,R}$ produces a minimal graph over Ω_h with boundary values $0, R, 0$ (see Figure 64 left).

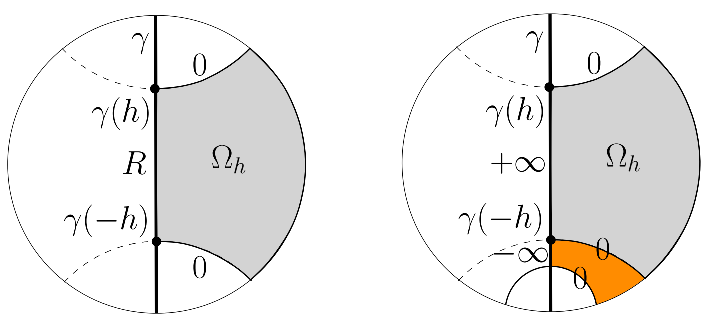


Figure 64: Left: Both the minimal graph $u_{h,R}$ and its limit $u_{h,\infty}$ as $R \to +\infty$ are defined over the infinite half-strip Ω_h of hyperbolic width $2h$. After symmetrization of the graph $\mathrm{Gr}(u_{h,\infty})$ with respect to its boundary, we construct a singly periodic surface.

We now take the limit $u_{h,\infty} := \lim_{R\to\infty} u_{h,R}$, which exists by the same reason as before (use the same barrier v, now restricted to Ω_h). La function $u_{h,\infty}$ is defined over the same infinite half-strip Ω_h, and produces a minimal graph over Ω_h with boundary values $0, +\infty, 0$ (see Figure 64 left). After rotation by angle 180^o of the graph of $u_{h,\infty}$ with respect to each geodesic segment of its boundary, we will find a properly embedded, singly periodic minimal surface $\Sigma \subset \mathbb{H}^2 \times \mathbb{R}$

without boundary, which is invariant by the hyperbolic translation ϕ_{2h}. This surface Σ is the analogous in $\mathbb{H}^2 \times \mathbb{R}$ of the classical singly periodic Scherk minimal surface in $\mathbb{R}^3$.

To finish, we will construct a properly embedded, doubly periodic surface in $\mathbb{H}^2 \times \mathbb{R}$ with positive constant mean curvature. Consider again the geodesic $\gamma = \gamma(s)$ defined above (normalized as a diameter of the disk $\{|z| < 1\}$). Given $t > 0$, let γ_t be an equidistant curve at hyperbolic distance t from γ. Since γ_t has constant geodesic curvature $\kappa(t)$, the topological plane $\Sigma_t := \gamma_t \times \mathbb{R}$ has constant mean curvature $H(t) = \frac{\kappa(t)}{2}$ in $\mathbb{H}^2 \times \mathbb{R}$, with $H(t) \in [0, 1/2)$ provided that we orient Σ_t with the unit normal vector that points towards Σ_0 (Figure 65).

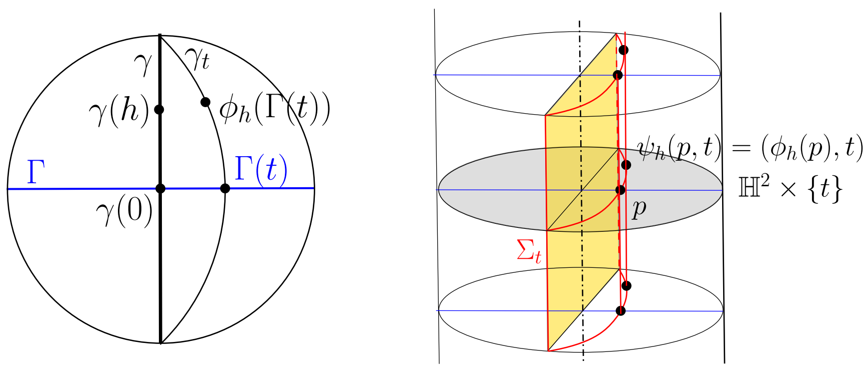


Figure 65: The topological plane Σ_t has constant mean curvature.

Take a hyperbolic translation ϕ_h along γ. ϕ_h leaves γ_t invariant for all t, since ϕ_h is an isometry of $\mathbb{H}^2$ and γ_t an equidistant curve to γ. Thus, the hyperbolic translation $\psi_h(p,t) = (\phi_h(p), t)$ is an isometry of $\mathbb{H}^2 \times \mathbb{R}$ that leaves Σ_t invariant. Since the cyclic group $\langle \psi_h \rangle$ generated by ψ_h acts properly and discontinuously on $\mathbb{H}^2 \times \mathbb{R}$, we deduce that $\Sigma_t / \langle \psi_h \rangle$ is a properly embedded surface, topologically a cylinder, with constant mean curvature in $(\mathbb{H}^2 \times \mathbb{R})/\langle \psi_h \rangle$. If we choose a vertical translation $T_a(p,t) = (p, t+a)$ with $a > 0$, then T_a commutes with ψ_h and the group generated by ψ_h, T_a is isomorphic to $\mathbb{Z} \oplus \mathbb{Z}$. This group acts properly and discontinuously on $\mathbb{H}^2 \times \mathbb{R}$ by isometries, hence the quotient manifold

$$M^3 = M^3(h,a) := (\mathbb{H}^2 \times \mathbb{R})/\langle \psi_h, T_a \rangle$$

is a homogeneous manifold, diffeomorphic to the cartesian product of a two-dimensional torus times and the real line (but the quotient metric is not a product metric). As $T_1(\Sigma_t) = \Sigma_t$, we deduce that

$$\mathcal{T}(t) = \mathcal{T}(t,h,a) := \Sigma_t / \langle \psi_h, T_a \rangle \subset M^3(h,a)$$

is a properly embedded torus in $M^3(h,a)$ with constant mean curvature $H(t)$. In other words, Σ_t can be viewed as a doubly periodic, properly embedded surface in $\mathbb{H}^2 \times \mathbb{R}$ with constant mean curvature. Furthermore,

$$\mathcal{F} := \{\mathcal{T}(t) \mid t \in \mathbb{R}\}$$

is a foliation of $M^3(h,a)$ by tori with constant mean curvature $H(t) \in (-1/2, 1/2)$ (note that the surfaces Σ_t with $t \geq 0$ are oriented as before, while the surfaces Σ_t with $t < 0$ are oriented with the opposite normal vector, not the one pointing towards Σ_0). This produces an analytic choice of the unit normal vector to the foliation $\mathcal{F}$, and the mean curvature value $H(t)$ of the leaves parameterizes transversally the foliation in the interval $(-1/2, 1/2)$. Furthermore, the torus $\mathcal{T}(0)$ is totally geodesic in M^3.

References

[1] S. Brendle, *The isoperimetric inequality for a minimal hypersurface in Euclidean space.* Preprint at `https://128.84.21.199/pdf/1907.09446v1.pdf`.

[2] T. H. Colding and W. P. Minicozzi II. Estimates for parametric elliptic integrands. *Int. Math. Res. Not.*, (6):291–297, 2002.

[3] T. H. Colding and W. P. Minicozzi II. An excursion into geometric analysis. *Surveys of Differential Geometry IX - Eigenvalues of Laplacian and other geometric operators*, International Press, edited by Alexander Grigor'yan and Shing Tung Yau, 83–146, 2004.

[4] P. Castillon. An inverse spectral problem on surfaces. *Comment. Math. Helv.*, 81(2):271–286, 2006.

[5] J. Douglas, *Solutions of the problem of Plateau.* Bull. Am. Math. Soc. **35** (1929), 292.

[6] J. Douglas, *The problem of Plateau.* Bull. Am. Math. Soc. **39** (1933), 227-251.

[7] L. Euler, *Methodus inveniendi lineas curvas maximi minimive propietate gaudeates sive solutio problematis isoperimetrici latissimo sensu accepti.* Opera omnia (1) **24**, Füssli, Turici, 1952. Traducido al inglés parcialmente en D. J. Struik. *A source book in mathematics,* Harvard Univ. Press, Cambridge, Mass., 1969, 399-406.

[8] D. Gilbarg and N. S. Trudinger, *Elliptic partial differential equations of second order.* Springer-Verlag, New York, 2nd edition, 1983.

[9] S. Hildebrandt, *Maximum principles for minimal surfaces and for surfaces of continuous mean curvature.* Math. Z. **128** (1972), 253–269.

[10] S. Kawai. Operator $\Delta - aK$ on surfaces. *Hokkaido Math. J.*, 17(2):147–150, 1988.

[11] J. L. Lagrange, *Essai d'une nouvelle méthode pour déterminer les maxima et minima des formules intégrales indefinies.* Miscellanea Taurinensia **2** (1760-1), 173-95; *Oeuvres de Lagrange.* Vol. **1**, Gauthiers-Villars, Paris, 1867.

[12] L. Mazet, *Cylindrically bounded constant mean curvature surfaces in* $\mathbb{H}^2 \times \mathbb{R}$*.* Trans. Amer. Math. Soc. **367** (2015), 5329–5354.

[13] F. Morgan, *Geometric Measure Theory. A Beginner's Guide, Fourth Edition* Elsevier/Academic Press **1** (2009).

[14] J. C. C. Nitsche, *Lectures on Minimal Surfaces.* Cambridge University Press **1** (1989).

[15] R. Osserman, *A survey of Minimal Surfaces.* Vol. 1, Cambridge Univ. Press, New York (1989).

[16] A. V. Pogorelov. On the stability of minimal surfaces. *Soviet Math. Dokl.*, 24:274–276, 1981.

[17] T. Radó, *The problem of least area and the problem of Plateau.* Math. Z. **32** (1930) 763–796.

[18] H. Rosenberg. Some recent developments in the theory of minimal surfaces in 3-manifolds. In *24th Brazilian Mathematics Colloquium*, Instituto de Matematica Pura e Aplicada (IMPA), Rio de Janeiro, 2003. IMPA Mathematical Publications.

[19] H. Rosenberg and R. Souam and E. Toubiana, *General curvature estimates for stable* H*-surfaces in* 3*-manifolds and applications.* J. Differential Geom. **84(3)** (2010) 623–648.

[20] H. F. Scherk, *De propietatibus superficiei, quae hac continetur aequatione* $(1+q^2)r-2pqs+(1+p^2)t=0$ *disquisitiones analyticae.* Prize-essay. Actis Soc. Jablon. nova **4** (1832), 204-80.

[21] H. F. Scherk, *Bemerkungen über die kleinste Fläche innerhalb gegebener Grenzen.* J. r. angew. Math. **13** (1835), 185-208.

[22] K. Shiohama and M. Tanaka. The length function of geodesic parallel circles. *Progress in Differential Geometry, Adv. Stud. Pure Math.*, 22:299–308, 1993.

[23] S.T. Yau, *Isoperimetric constants and the first eigenvalue of a compact Riemannian manifold.* Ann. Sci. Ecole Norm. Sup. **8** (1975) 487–507.