Global existence of classical solutions for reaction-diffusion systems with mass dissipation

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Reaction-diffusion systems with mass dissipation

Let $\Omega \subset \mathbb{R}^n$ be bounded and $u_i(x,t)$ be the $i-$th concentration (chemical, population, etc.) for $i = 1, \ldots, m$. We consider the following reaction-diffusion system

$$\begin{cases} \partial_t u_i - d_i \Delta u_i = f_i(u), & x \in \Omega, \\ \nabla u_i \cdot \nu = 0, & x \in \partial\Omega, \\ u_i(x,0) = u_{i,0}(x), & x \in \Omega, \end{cases} \tag{1}$$

where $d_i > 0$ are diffusion coefficients and the nonlinearities f_i are *locally Lipschitz continuous* and satisfy

- quasi-positivity condition

$$f_i(u) \geq 0 \quad \text{for all} \quad u \in [0,\infty)^m \quad \text{with} \quad u_i = 0. \tag{\textbf{P}}$$

- mass dissipation condition

$$\sum_{i=1}^{m} f_i(u) \leq 0 \quad \text{for all} \quad u \in [0,\infty)^m. \tag{\textbf{M}}$$

Reversible reactions

$$CH_3COOH + C_2H_5OH \leftrightharpoons CH_3COOC_2H_5 + H_2O \quad \text{or} \quad S_1 + S_2 \leftrightharpoons S_3 + S_4$$

The **Fickian law** and the **law of mass action** lead to

$$\begin{aligned}
\partial_t u_1 - d_1\Delta u_1 &= f_1(u) := -u_1u_2 + u_3u_4, \quad x \in \Omega,\\
\partial_t u_2 - d_2\Delta u_2 &= f_2(u) := -u_1u_2 + u_3u_4, \quad x \in \Omega,\\
\partial_t u_3 - d_3\Delta u_3 &= f_3(u) := +u_1u_2 - u_3u_4, \quad x \in \Omega,\\
\partial_t u_4 - d_4\Delta u_4 &= f_4(u) := +u_1u_2 - u_3u_4, \quad x \in \Omega.
\end{aligned} \tag{2}$$

- It is clear that $f_i(u) \geq 0$ if $u_i = 0$, so (**P**) is satisfied.
- For (**M**) we have a stronger property, namely the *mass conservation*, i.e.

$$\sum_{i=1}^{4} f_i(u) = 0. \tag{M'}$$

- (2) has additionally an *entropy inequality*, i.e.

$$\sum_{i=1}^{4} f_i(u)\log u_i = -(u_1u_2 - u_3u_4)\log\frac{u_1u_2}{u_3u_4} \leq 0. \tag{E}$$

A remark about (**M**) and (**M′**)

$$(\mathbf{M}) \quad \sum_{i=1}^{m} f_i(u) \le 0 \qquad \text{and} \qquad (\mathbf{M}') \quad \sum_{i=1}^{m} f_i(u) = 0.$$

Obviously $(\mathbf{M}') \Rightarrow (\mathbf{M})$.
From (**M**) one can create a new system with (**M′**).

$$\partial_t u_i - d_i \Delta u_i = f_i(u), \quad i = 1, \ldots, m \qquad \text{with} \qquad \sum_{i=1}^{m} f_i(u) \le 0.$$

Add to the system the $(m+1)-$th equation,

$$\partial_t u_{m+1} - \Delta u_{m+1} = -\sum_{i=1}^{m} f_i(u) \ge 0,$$

then the new system (with $m+1$ unknowns) satisfies (**M′**) (and also (**P**)).

Global existence with (**P**) and (**M′**)?

$$\partial_t u_i - d_i \Delta u_i = f_i(u) \quad \text{with} \quad (\mathbf{P}) \quad \text{and} \quad (\mathbf{M}'): \quad \sum_{i=1}^m f_i(u) = 0.$$

Thanks to (**P**), if the initial data is non-negative, the solution remains non-negative as long as it exists.

Local existence on $(0, T_{\max})$**:** Classical result since f_i are locally Lipschitz continuous.

Global existence of classical solutions:

$$\lim_{t \to T_{\max}} \|u_i(t)\|_{L^\infty(\Omega)} < +\infty \text{ for } i = 1, \ldots, m \qquad \Longrightarrow \qquad T_{\max} = +\infty.$$

When $d_i = d$ for all $i = 1, \ldots, m$ one has

$$\partial_t \sum u_i - d\Delta \sum u_i = 0 \xrightarrow{\text{Maximal principle}} \left\|\sum u_i(t)\right\|_{L^\infty(\Omega)} \leq \left\|\sum u_{i,0}\right\|_{L^\infty(\Omega)}.$$

When d_i are different from each other $\longrightarrow$ It is a challenging question!

Available estimates

$$\lim_{t\to T_{\max}} \|u_i(t)\|_{L^\infty(\Omega)} < +\infty \text{ for } i = 1,\ldots,m \quad \Longrightarrow \quad T_{\max} = +\infty.$$

- From $(\mathbf{M}')$,

$$\partial_t \sum_{i=1}^m u_i - \Delta \sum_{i=1}^m d_i u_i = \sum_{i=1}^m f_i(u) = 0, \tag{3}$$

  hence

$$\partial_t \sum_{i=1}^m \int_\Omega u_i(x,t)dx = 0,$$

  and therefore

$$u_i \in L^\infty(0,T;L^1(\Omega)).$$

- From (3), one can apply an improved duality method[1]

$$u_i \in L^{2+\varepsilon}(0,T;L^{2+\varepsilon}(\Omega)).$$

[1] D. Schmidt, M. Pierre, SIAM Review 2000; Cañizo, Desvillettes, Fellner, CPDE 2014

An "almost" counterexample[2]

There exists $f(u,v)$ and $g(u,v)$ and functions $h_1, h_2 : \mathbb{R} \to \mathbb{R}$ such that

$$\textbf{(P)} \quad f(0,v) \geq 0, \quad g(u,0) \geq 0 \quad \text{and} \quad f(u,v) + g(u,v) \leq 0$$

and the solution to

$$\begin{aligned} \partial_t u - d_1 \Delta u &= f(u,v), & x &\in \Omega, \\ \partial_t v - d_2 \Delta v &= g(u,v), & x &\in \Omega, \\ u(x,t) &= h_1(t), & x &\in \partial\Omega, \\ v(x,t) &= h_2(t), & x &\in \partial\Omega \end{aligned}$$

blows up in finite time.

Remark that we have in this case inhomogeneous Dirichlet boundary conditions.

An example with homogeneous Neumann boundary conditions is unknown!

[2]D. Schmidt, M. Pierre, SIAM Review 2000

Polynomial nonlinearities

$$\partial_t u_i - d_i \Delta u_i = f_i(u) \quad \text{with} \quad (\mathbf{P}) \quad \text{and} \quad (\mathbf{M}') : \quad \sum_{i=1}^{m} f_i(u) = 0.$$

Polynomial growth of nonlinearities is pretty common, i.e. there exists $\mu > 1$ such that

$$|f_i(u)| \leq C(1 + |u|^{\mu}) \quad \text{for} \quad i = 1, \ldots, m. \qquad (\mathbf{G})$$

In particular the case of quadratic nonlinearities, i.e. $\mu = 2$. For instance

$$\begin{aligned}
\partial_t u_1 - d_1 \Delta u_1 &= f_1(u) := -u_1 u_2 + u_3 u_4, \quad x \in \Omega,\\
\partial_t u_2 - d_2 \Delta u_2 &= f_2(u) := -u_1 u_2 + u_3 u_4, \quad x \in \Omega,\\
\partial_t u_3 - d_3 \Delta u_3 &= f_3(u) := +u_1 u_2 - u_3 u_4, \quad x \in \Omega,\\
\partial_t u_4 - d_4 \Delta u_4 &= f_4(u) := +u_1 u_2 - u_3 u_4, \quad x \in \Omega.
\end{aligned}$$

or antisymmetric Lotka-Volterra system with $A = (a_{ij}) \in \mathbb{R}^{m \times m}$,

$$\partial_t u_i - d_i \Delta u_i = -\tau_i u_i + u_i \sum_{j=1}^{m} a_{ij} u_j \quad \text{where} \quad A + A^{\top} = 0.$$

The literature

- Goudon, Vasseur (2010): when $n = 1$ and $\mu = 3$ or $n = 2$ and $\mu = 2$ (assuming additionally the entropy dissipation (**E**)). T. (2018): solutions grow at most polynomially in time.
- Caputo, Vasseur (2009): for arbitrary $n \geq 1$ with *strictly subquadratic* nonlinearities, i.e. $\mu < 2$ (still assuming (**E**)).
- Cañizo, Desvillettes, Fellner (2014): for $n \leq 2$ and $\mu = 2$ without assuming (**E**). The solution grows *at most polynomially in time*.
  Pierre, Suzuki, Yamada (2019): The solution is bounded uniformly in time for $n \leq 2$ and $\mu = 2$.
- When $\mu > 2$, Cañizo, Desvillettes, Fellner (2014) or Fellner, Latos, Suzuki (2016) showed global strong solution when

$$\sup_{i,j} |d_i - d_j| \leq \delta \quad \text{for } \delta \text{ small enough.}$$

- Close-to-equilibrium: Cáceres, Cañizo (2017) $n \leq 4$ & $\mu = 2$; T. (2018) $n \leq 4$ and $\mu = 1 + \frac{4}{n}$.
- (Weaker solutions) Pierre (2003), $f_i(u) \in L^1(0, T; L^1(\Omega))$ implies global weak solutions; Fischer (2015), global *renormalized solutions* under (**E**) without any restriction on n and μ.

A forgotten Russian paper

Ya. I. Kanel', Solvability in the large of a system of reaction- diffusion equations with the balance condition, Differ. Uravn., 1990, Volume 26, Number 3, 448–458

РАЗРЕШИМОСТЬ В ЦЕЛОМ СИСТЕМЫ УРАВНЕНИЙ РЕАКЦИИ-ДИФФУЗИИ С БАЛАНСНЫМ УСЛОВИЕМ

В биохимии и химической кинетике [1, 2] встречаются системы уравнений вида

$$L_i u_i \equiv \frac{\partial u_i}{\partial t} - \lambda_i \Delta u_i = f_i(u), \quad i = 1, \ldots, n, \quad \lambda_i = \text{const} > 0,$$
$$u = (u_1, \ldots, u_n), \quad x = (x_1, \ldots, x_m). \tag{1}$$

Поставим задачу Коши для системы (1) в полупространстве $t>0$, $x \in R^m$ при начальных условиях

$$u_i(x, 0) = u_{i0}(x), \quad x \in R^m, \tag{2}$$

где $u_{i0}(x)$ кусочно-непрерывны в R^m,

$$0 \leqslant u_{i0}(x) \leqslant c_i = \text{const}, \quad i = 1, \ldots, n. \tag{3}$$

Предположим, что в (1) функции $f_i(u)$ удовлетворяют условию Липшица по u в любой конечной области из R^n и условиям

$$f_i(u) \geqslant 0 \quad \text{при } u_i = 0, \ u_j \geqslant 0, \ j \neq i, \tag{4}$$

$$f_1(u) + \ldots + f_n(u) = 0. \tag{5}$$

Последнее условие называется балансным.

Теорема 2. *Пусть выполняются условия (3), (4), (5) и условия*

$$|f_i(u)| \leqslant K(1 + |u|^{r+\varepsilon}), \quad i = 1, \ldots, n, \quad K = \text{const} > 0, \tag{9}$$

где $r=2$ при $m>1$; $r=3$ при $m=1$; $\varepsilon>0$ достаточно мало. Тогда существует единственное решение $u(x, t)$ задачи (1), (2) в полупространстве $t>0$, $x \in R^m$.

Three recent works

- Caputo, Goudon, Vasseur. Analysis and PDE (2019): De Giorgi's method. $\Omega = \mathbb{R}^n$, assume (**M**') and (**E**), i.e.
$$\sum_{i=1}^m f_i(u) = 0 \quad \text{and} \quad \sum_{i=1}^m f_i(u)\log u_i \le 0.$$
- Souplet. JEE (2018): Kanel's approach. Ω is bounded, assume (**M**) and (**E**), i.e.
$$\sum_{i=1}^m f_i(u) \le 0 \quad \text{and} \quad \sum_{i=1}^m f_i(u)\log u_i \le 0.$$
- Fellner, Morgan, T. Annales IHP & DCDS-S (in press): Kanel's approach. Ω is bounded, assume only (**M**) (and even weaker), i.e.
$$\sum_{i=1}^m f_i(u) \le L_0 + L_1 \sum_{i=1}^m u_i.$$
Moreover, if $L_0 = L_1 = 0$, i.e. (**M**), then the solution is bounded uniformly in time
$$\sup_{t\ge 0} \|u_i(t)\|_{L^\infty(\Omega)} \le M.$$

Sketch of the proof

A key lemma

Lemma

Consider $\partial_t u - d\Delta u = f$, $\nabla u \cdot \nu = 0$ *and* $u(0) = u_0$. *Assume that*

- $\|f\|_{L^\infty(\Omega\times(0,T))} \le F$; *and*
- *there exists* $\gamma \in [0,1)$ *such that*

$$|u(x,t) - u(x',t)| \le H|x - x'|^\gamma, \quad \text{for all} \quad (x,t),(x',t) \in \Omega \times (0,T).$$

Then,

$$\sup_{\Omega\times(0,T)} |\nabla u| \le \sup_{\Omega} |\nabla u_0| + BH^{\frac{1}{2-\gamma}} F^{\frac{1-\gamma}{2-\gamma}}.$$

When u is not Hölder continuous, we can take $\gamma = 0$ and $H = 2\|u\|_{L^\infty(\Omega\times(0,T))}$.

Sketch of the proof

From

$$\partial_t \sum_{i=1}^m u_i - \Delta \sum_{i=1}^m d_i u_i = \sum_{i=1}^m f_i(u) = 0,$$

follows

$$\sum_{i=1}^m u_i(x,t) = \Delta \underbrace{\left(\int_0^t \sum_{i=1}^m d_i u_i(x,s) ds\right)}_{=:v(x,t)} + \sum_{i=1}^m u_{i,0}(x) = \Delta v(x,t) + \sum_{i=1}^m u_{i,0}(x).$$

Aim: To estimate $\|\Delta v\|_{L^\infty}$ in terms of $U := \sum u_i$.

Estimate of Δv with $v(x,t) = \int_0^t \sum d_i u_i(x,s)ds$

- (1) $b(x,t)\partial_t v - \Delta v = 0$, $0 < m \le b(x,t) \le M$ gives
$$|v(x,t) - v(x',t')| \le H\left(|x-x'|^\delta + |t-t'|^{\delta/2}\right).$$
- (2) $\partial_t v - \Delta v = \sum (d_i - 1)u_i$. Application of key lemma yields
$$|\nabla v| \le C_T\left(1 + |U|^{\frac{1-\delta}{2-\delta}}\right).$$
- Also from (2), $|\Delta v| \le C_T(1 + |\nabla v|^{1/2}|\nabla U|^{1/2})$.
- From $\partial_t u_i - d_i\Delta u_i = f_i(u)$ and $|f_i(u)| \le C_T(1 + |u|^{2+\varepsilon})$ one has from the key lemma
$$|\nabla U| \le C_T\left(1 + |U|^{\frac{3+\varepsilon}{2}}\right).$$
- Therefore, from $\sum u_i = \Delta v + \sum u_{i,0}$,
$$|U| \le C_T(1 + |\Delta v|) \le C_T\left(1 + |U|^{\frac{3+\varepsilon}{4} + \frac{1-\delta}{2(2-\delta)}}\right)$$
with $\frac{3+\varepsilon}{4} + \frac{1-\delta}{2(2-\delta)} < 1$ when ε is small enough, and therefore
$$|U| \le C_T.$$

Theorem (Fellner, Morgan, T. (2019))

Assume (**P**),

$$\sum_{i=1}^{m} f_i(u) \leq L_0 + L_1 \sum_{i=1}^{m} u_i \quad \text{for all} \quad u \in [0,\infty)^m$$

and

$$|f_i(u)| \leq C\left(1 + |u|^{2+\varepsilon}\right) \quad \text{for all} \quad i = 1, \ldots, m.$$

Then reaction-diffusion system (1) *has a unique global classical solution. Moreover,*

- *if* $L_1 < 0$ *then*

$$\|u_i(t)\|_{L^\infty(\Omega)} \leq Ce^{-\lambda t};$$

- *if* $L_0 = L_1 = 0$, *that means assuming* (**M**), *then*

$$\sup_{t\geq 0} \|u_i(t)\|_{L^\infty(\Omega)} \leq M.$$

- *if* $L_1 = 0$ *and* $L_0 > 0$ *then*

$$\|u_i(t)\|_{L^\infty(\Omega)} \leq C(1 + t^p);$$

Super-quadratic nonlinearities

Theorem (Cupps, Morgan, T. (2019))

Assume (**P**), (**M**) *and* $f_i(u) \le C(1 + |u|^\mu)$ *for all* $i = 1, \ldots, m$. *If*

$$\sup_{i,j} |d_i - d_j| \le \delta(\mu, n) \quad \text{or} \quad d_i \ge D(\mu, n) \text{ for all } i = 1, \ldots, m,$$

then

$$\sup_{t \ge 0} \|u_i(t)\|_{L^\infty(\Omega)} \le C.$$

Corollary (Close-to-equilibrium)

Let $u_\infty \in (0, \infty)^m$ *be an equilibrium, i.e.*

$$f_i(u_\infty) = 0 \quad \text{for all} \quad i = 1, \ldots, m.$$

If $\|u_{i,0} - u_\infty\|_{L^\infty(\Omega)} \le \varepsilon$ *then*

$$\sup_{t \ge 0} \|u_i(t)\|_{L^\infty(\Omega)} \le C.$$

Super-quadratic nonlinearities

Conjecture (Global Attractor Conjecture for ODE)

If a chemical reaction network is complex balanced, then the positive complex balanced equilibrium is the global attractor of the dynamics of the differential system.

Corollary (GAC with large diffusion)

Let (1) *represents a complex balanced reaction network. Assume that* $|f_i(u)| \leq C(1 + |u|^{\mu})$ *for all* $i = 1, \ldots, m$. *If*

$$d_i > D(\mu, n) \quad \text{for all} \quad i = 1, \ldots, m,$$

then the GAC holds for the PDE system as long as it holds for the corresponding ODE system.

Quadratic Intermediate Sum Conditions

Theorem (Morgan, T. (2019))

Let $n \le 2$. *Assume* (**P**), (**M**) *and*

$$\begin{cases} a_{11}f_1(u) & \le C(1+|u|^2),\\ a_{21}f_1(u) + a_{22}f_2(u) & \le C(1+|u|^2),\\ \cdots, & \\ a_{m1}f_1(u) + a_{m2}f_2(u) + \cdots + a_{mm}f_m(u) & \le C(1+|u|^2). \end{cases}$$

Then

$$\sup_{t\ge 0} \|u_i(t)\|_{L^\infty(\Omega)} \le C.$$

Quadratic Intermediate Sum Conditions

Consider the reversible reaction $S_1 + pS_2 \leftrightharpoons S_2 + S_3$, for $p \geq 1$ arbitrary.

$$\begin{cases} \partial_t u_1 - d_1 \Delta u_1 &= -u_1 u_2^p + u_2 u_3, \\ \partial_t u_2 - d_2 \Delta u_2 &= -u_1 u_2^p + u_2 u_3, \\ \partial_t u_3 - d_3 \Delta u_3 &= +u_1 u_2^p - u_2 u_3, \end{cases} \quad \text{then} \quad \begin{cases} f_1(u) & \leq u_2 u_3 \\ f_2(u) & \leq u_2 u_3 \\ f_2(u) + f_3(u) & \leq 0. \end{cases}$$

Theorem (Morgan, T. (2019))

Let $n \leq 2$. Then the above system has a unique bounded solution, i.e.

$$\sup_{t \geq 0} \|u_i(t)\|_{L^\infty(\Omega)} \leq C.$$

Moreover, the solution converges exponentially to equilibrium with explicit rates and constants,

$$\sum_{i=1}^{3} \|u_i(t) - u_{i,\infty}\|_{L^\infty(\Omega)} \leq Ce^{-\lambda t} \quad \text{for all} \quad t \geq 0.$$

Conclusion and Outlook

Conclusion: Global existence and uniform-in-time bound of solutions with (**P**) and (**M**) for

- systems with quadratic nonlinearities;
- systems with large enough diffusion coefficients;
- systems with quadratic intermediate sum conditions (when $n = 2$).

Outlook

- Quadratic intermediate sum conditions in all dimensions?
- Blow-up examples with homogeneous Neumann boundary conditions?
- Threshold of growth order for global existence?

Thank you for your attention!

Gracias por su atención!