Nematic alignment of self-propelled particles in the macroscopic regime

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Section 1

Context and motivation: what is nematic alignment?

Nematic alignment and collective dynamics

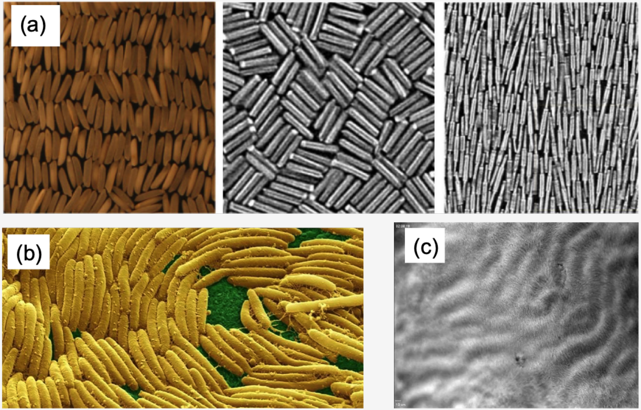

Figure: (a) Inert particles (from left to right: basmati rice, cylinders and rolling pins) from *'Nonequilibrium steady states in a vibrated-rod monolayer: tetratic, nematic, and smectic correlations'* Narayan et. al. (2006). (b) Nematic swarming in myxobacteria (active particles), from G. Velicer (Indiana U. Bloomington) and J. Bergen (Max-Planck I. for Developmental Biology). (c) Macroscopic view of swarming myxobacteria: formation of ripples, `https://youtu.be/0ALM7X1_LqA`

Some previous results on nematic phenomena

Vast literature, specially in the Physics community (mostly particle simulations):

- **Inert matter:** many interest in liquid crystals (excluded volume interaction of rod-like polymers).
  $\rightarrow$ See, e.g., John Ball for mathematical works.

Great interest in the Physics community, but rigorous derivation missing.

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- **Inert matter:** many interest in liquid crystals (excluded volume interaction of rod-like polymers).
  $\rightarrow$ See, e.g., John Ball for mathematical works.
- **Active matter** (a selection):
  - Chaté et. al. *Modeling collective motion: variations on the vicsek model*. Eur. Phys. J. B (2008) **and** Ginelli et. al. *Large-scale collective properties of self-propelled rods* Phys. Rev. Lett., (2010) **and** Chaté et. al. *Simple model for active nematics: quasi-long-range order and giant fluctuations*. Phys. Rev. Lett. 2006).
    $\rightarrow$ **particle simulation/pattern formation.**
  - F. Peruani, A. Deutsch, and M. Bär. *A mean-field theory for self-propelled particles interacting by velocity alignment mechanisms.* Eur. Phys. J-spec. top., 157(1), 2008.
  - Bertin et. al. *Boltzmann and hydrodynamic description for self-propelled particles*. Phys. Rev. E, 74(2), 2006.
    $\rightarrow$ **Heuristic derivation.**

Great interest in the Physics community, but rigorous derivation missing.

Maths literature (in relation to collective dynamics):

- P. Degond, A. Manhart, and H. Yu. *A continuum model for nematic alignment of self-propelled particles.* Discrete Contin. Dyn. Syst. Ser. B, 22(4), 2017.

  A complexified model for myxobacteria:

  - P. Degond, A. Manhart, and H. Yu. *An age-structured continuum model for myxobacteria.* Math. Models Methods Appl. Sci., 28(09), 2018.

  $\rightarrow$ **hyperbolic limit.**

- P. Degond, A. Frouvelle, S. Merino-Aceituno, and A. Trescases. *Quaternions in collective dynamics.* Multiscale Model. Simul., 16(1), 2018.

  $\rightarrow$ **nematic alignment in dim. 4; quadratic transport term; hyperbolic limit.**

In this work we consider the diffusive limit.

Degond-Manhart-Yu model (without time reversal)

$(X_i, \theta_i)_{i=1,\dots,N}$, $X_i \in R^2$, $\theta_i \in [-\pi, \pi)$:

$$\begin{cases} dX_i = a\, v(\theta_i) dt, \qquad v(\theta_i) = (\cos\theta_i(t), \sin\theta_i(t)),\ a > 0, & \text{(1a)} \\ d\theta_i = \nu\, \partial_{\theta_i} |\cos(\theta_i - \bar\theta_i)| dt + \sqrt{2d\cos^2(\theta_i - \bar\theta_i)}\, dB_t^i, & \text{(1b)} \end{cases}$$

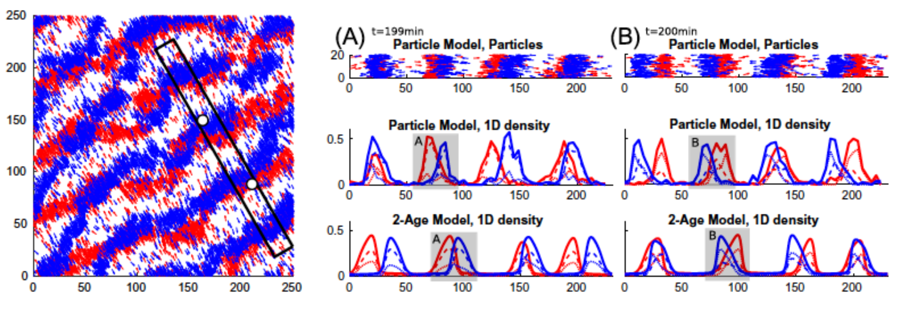
where $\bar\theta_i$ is the 'mean nematic angle' of the neighbours. Noise cancels when $\theta_i = \bar\theta_i + \pi/2$. This affects the shape of the equilibria

$$f_{\rho_+,\rho_-,\bar\theta}(\theta) = \begin{cases} \rho_+ M_{\bar\theta}(\theta),\ \cos(\theta - \bar\theta) > 0, \\ \rho_- M_{\bar\theta}(\theta),\ \cos(\theta - \bar\theta) < 0. \end{cases} \quad \Bigg| \quad M_{\bar\theta}(\theta) \sim \exp\left(\frac{-\nu}{d|\cos(\theta - \bar\theta)|} \right)$$

Macroscopic equations:

$$\begin{cases} \partial_t \rho_+ + d_1 \nabla_x \cdot \left(\rho_+ v(\bar\theta)\right), & \text{(2a)} \\ \partial_t \rho_- - d_1 \nabla_x \cdot \left(\rho_- v(\bar\theta)\right) = 0, & \text{(2b)} \\ (\rho_+ + \rho_-)\partial_t \bar\theta + d_2(\rho_+ - \rho_-)(v(\bar\theta)\cdot\nabla_x)\bar\theta + \mu v(\bar\theta)^\perp \cdot \nabla_x(\rho_+ - \rho_-) = 0. \end{cases}$$

Adding 'reversal motion': ripple (wave) formation.



(a) Ripples in the IBM

(b) Comparison to 2-Age model

Figure: From: P. Degond, A. Manhart, and H. Yu. *An age-structured continuum model for myxobacteria.* Math. Models Methods Appl. Sci., 28(09), 2018.

The Vicsek-nematic: discrete dynamics

$(X_i, \omega_i) \in \mathbb{R}^d \times \mathbb{S}^{d-1}$, $i = 1, \ldots, N$, $d \geq 2$. SDE in Stratonovich sense:

$$\begin{cases} dX_i = \omega_i dt, & (3a) \\ d\omega_i = \frac{\nu}{2} \nabla_{\omega_i} (\omega_i \cdot \bar{\omega}_i)^2 dt + P_{\omega_i^\perp} \circ \sqrt{2D} dB_t^i, & (3b) \end{cases}$$

$\nu, D > 0$; $(B_t^i)_{i=1,\ldots,N}$ independent Brownian motions; $P_{\omega^\perp}$ orthogonal projection onto the orthogonal space to ω denoted by $\{\omega\}^\perp$.

Finally, $\bar{\omega}_i$ denotes any of the two unitary leading eigenvectors of

$$Q_i = \frac{1}{N} \sum_{j=1}^{N} \frac{1}{R^d} K\left(\frac{|X_i - X_j|}{R} \right) \left(\omega_j \otimes \omega_j - \frac{1}{d} \mathrm{Id} \right), \qquad (4)$$

where the function K corresponds to a sensing kernel and $R > 0$ is the typical radius of the sensing region. We assume that $K \geq 0$ and

$$\int_{\mathbb{R}^d} \frac{1}{R^d} K\left(\frac{|x|}{R} \right) dx = 1.$$

We assume that the leading eigenvalue of Q_i is simple.

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Compare with Vicsek: different potential and average orientation.

Vicsek model

$$
\begin{cases}
dX_i = \omega_i dt, & \text{(5a)} \\
d\omega_i = \frac{\nu}{2}\nabla_{\omega_i}(\omega_i \cdot \bar{\omega}_i)dt + P_{\omega_i^\perp} \circ \sqrt{2D}dB_t^i, & \text{(5b)}
\end{cases}
$$

$$
\bar{\omega}_i = \frac{J_i}{|J_i|}, \quad J_i = \frac{1}{N}\sum_{j=1}^{N} \frac{1}{R^d} K\left(\frac{|X_i - X_j|}{R}\right)\omega_j.
$$

Kinetic equation

Proposition (Formal mean-field limit)

The empirical distribution converges to a function $f = f(t, x, \omega)$ which satisfies the following kinetic equation:

$$\partial_t f + \nabla_x \cdot (\omega f) = \nabla_\omega \cdot \left[-\frac{\nu}{2} \nabla_\omega (\omega \cdot \bar{\omega}_{R,f})^2 f + D \nabla_\omega f \right] := C_R(f), \quad (6)$$

where ∇_ω and $\nabla_\omega \cdot$ denote the gradient and divergence operators on $\mathbb{S}^{d-1}$, respectively, and where $\bar{\omega}_{R,f}$ is the unitary leading eigenvector (up to a sign) of

$$Q_{R,f}(t,x) := \int_{\mathbb{R}^d} \int_{\mathbb{S}^{d-1}} \frac{1}{R^d} K\left(\frac{|x-y|}{R}\right) \left(\omega \otimes \omega - \frac{1}{d} Id\right) f \, d\omega \, dy. \quad (7)$$

The initial condition to (6) is $f(0, x, \omega) = f_0(x, \omega)$.

Section 2

Main result and interpretation

Parabolic rescaling

- Hyperbolic rescaling:

$$t' = \varepsilon t, \quad x' = \varepsilon x,$$

leads to

$$\partial_t \rho = 0,$$

because

$$\int v M(v) dv = 0 \quad \text{(no net motion of the particles)}$$

where M is the equilibria of C_R (as we will see next).

- Parabolic rescaling:

$$t' = \varepsilon^2 t, \quad x' = \varepsilon x$$

$$f^\varepsilon(t', x', \omega) = \frac{1}{\varepsilon^d} f(\frac{t'}{\varepsilon^2}, \frac{x'}{\varepsilon}, \omega),$$

and $R = \varepsilon R'$ (localized interactions).

Fokker-Planck formulation

After some expansions:

$$\varepsilon^2 \partial_t f^\varepsilon + \varepsilon \nabla_x \cdot (\omega f^\varepsilon) = \Gamma(f^\varepsilon) + \mathcal{O}(\varepsilon^4). \tag{8}$$

where

$$\Gamma(f) = D\nabla_\omega \cdot \left[M_{u_f} \nabla_\omega \left(\frac{f}{M_{u_f}} \right) \right], \tag{9}$$

and u_f is one of the two normalized leading eigenvector of Q_f

$$Q_f := \int_{\mathbb{S}^{d-1}} \left(\omega \otimes \omega - \frac{1}{d} \mathrm{Id} \right) f \, d\omega, \tag{10}$$

and

$$M_u(\omega) \sim \exp\left(\frac{\kappa}{2} (\omega \cdot u)^2 \right), \int_{\mathbb{S}^{d-1}} M_u(\omega) \, d\omega = 1, \qquad \kappa := \frac{\nu}{D}. \tag{11}$$

We will see that, formally, $f^\varepsilon \to \rho M_u$ as $\varepsilon \to 0$, for that we will use:

Proposition (Properties of the operator Γ)

We have the following properties:

(i) *Entropy dissipation: the following inequality holds:*

$$H(f) := \int_{\mathbb{S}^{d-1}} \Gamma(f)\, \frac{f}{M_{u_f}}\, d\omega = -D \int_{\mathbb{S}^{d-1}} \left|\nabla_\omega\left(\frac{f}{M_{u_f}}\right)\right|^2 M_{u_f}\, d\omega \leq 0. \tag{12}$$

(ii) *Consistency relation: u is the leading eigenvector (up to a sign) of*

$$Q_{M_u} = \int_{\mathbb{S}^{d-1}} M_u(\omega) \left(\omega \otimes \omega - \frac{1}{d} Id\right) d\omega.$$

(iii) *Equilibria: the set $\mathcal{E}$ of functions $f = f(\omega) \geq 0$ such that $\Gamma(f) = 0$ are given by*

$$\mathcal{E} = \{\rho M_u \,|\, \rho \in [0, \infty),\ u \in \mathbb{S}^{d-1}\}. \tag{13}$$

Main result

Suppose that f^ε converges to f as $\varepsilon \to 0$. Then, it holds that

$$f^\varepsilon \to \rho M_u, \quad \text{with} \quad \rho = \rho(t,x) \in [0,\infty), \quad u = u(t,x) \in \mathbb{S}^{d-1}.$$

If the convergence is strong enough and ρ, u are smooth enough, then:

$$\begin{cases} \partial_t \rho + \nabla_x \cdot \big(C_1\,(u\cdot\nabla_x\rho)\,u + C_2\,P_{u^\perp}\nabla_x\rho + C_3\,\rho\,(u\cdot\nabla_x)u \\ \qquad + C_4\,(\nabla_x\cdot u)\,\rho u \big) = 0, \\ \rho\partial_t u + E_1\,P_{u^\perp}\nabla_x\big((u\cdot\nabla_x)\rho\big) \\ \qquad + F_1\,\rho\,P_{u^\perp}\big[(u\cdot\nabla_x)\big((u\cdot\nabla_x)u\big)\big] + F_2\,\rho\,P_{u^\perp}\big(\nabla_x\cdot(P_{u^\perp}\nabla_x u)\big) \\ \qquad + F_3\,\rho\,P_{u^\perp}\nabla_x(\nabla_x\cdot u) \\ \qquad + G_1\,(u\cdot\nabla_x\rho)\,(u\cdot\nabla_x)u + G_2\,(P_{u^\perp}\nabla_x u)(P_{u^\perp}\nabla_x\rho) \\ \qquad + G_3\,\big((P_{u^\perp}\nabla_x\rho)\cdot P_{u^\perp}\nabla_x\big)u + G_4\,(\nabla_x\cdot u)\,P_{u^\perp}\nabla_x\rho \\ \qquad + H_1\,(u\cdot\nabla_x\log\rho)\,(P_{u^\perp}\nabla_x\rho) + H_2\,\rho\,(P_{u^\perp}\nabla_x u)\big((u\cdot\nabla_x)u\big) \\ \qquad + H_3\,\rho\big[\big((u\cdot\nabla_x)u\big)\cdot P_{u^\perp}\nabla_x\big]u + H_4\,\rho\,(\nabla_x\cdot u)\,(u\cdot\nabla_x)u = 0, \\ |u| = 1. \end{cases}$$

Meaning of the equations

- System of cross diffusion equations. Well-posedness? Is the system of higher order derivatives elliptic?... Future work. Coefficients are positive.

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- If $|u(t=0)| = 1$, then $|u(t)| = 1$.
- The equations are invariant under the change $u \to -u$.
  - Nematic symmetry 'wins over' polar transport operator,
  - but, it generates an anisotropy in the direction of u. This leads to an anisotropic diffusion for the mass density ρ and mean direction u.

Anisotropic dynamics

Decompose the differential operators into their component parallel to u: $(u \cdot \nabla_x \ldots)$; and the one on $\{u\}^\perp$: $P_{u^\perp}\nabla_x \ldots$. For second order operators:

$$(u\cdot\nabla_x)\big((u\cdot\nabla_x\ldots)\big), \quad P_{u^\perp}\nabla_x(u\cdot\nabla_x\ldots), \quad P_{u^\perp}\nabla_x(P_{u^\perp}\nabla_x\ldots), \tag{15}$$

In the equation of ρ:

$$\begin{aligned}\partial_t\rho \quad &+ \quad \nabla_x\cdot\big(C_1\,(u\cdot\nabla_x\rho)\,u + C_2\,P_{u^\perp}\nabla_x\rho\big)\\ &+ \quad \nabla_x\cdot\big(C_3\,\rho\,(u\cdot\nabla_x)u + C_4\,(\nabla_x\cdot u)\,\rho u\big) = 0,\end{aligned}$$

Diffusive terms (C_1, C_2)
Cross-diffusion terms (C_3, C_4)

If we had that $C_1 = C_2$ and $C_3 = C_4$, then

$$\begin{aligned}\partial_t\rho \quad &+ \quad C_1\Delta_x\rho\\ &+ \quad C_3\nabla_x\cdot(\rho\,\nabla_x\cdot(u\otimes u)) = 0\end{aligned}$$

but $C_1 \neq C_2$ and $C_3 \neq C_4$ (anisotropy).

The equation for u

$$
\begin{aligned}
\rho\partial_t u &+ E_1\, P_{u^\perp}\nabla_x\big((u\cdot\nabla_x)\rho\big)\\
&+ F_1\,\rho\, P_{u^\perp}\big[(u\cdot\nabla_x)\big((u\cdot\nabla_x)u\big)\big] + F_2\,\rho\, P_{u^\perp}\big(\nabla_x\cdot(P_{u^\perp}\nabla_x u)\big)\\
&+ F_3\,\rho\, P_{u^\perp}\nabla_x(\nabla_x\cdot u)\\
&+ G_1\,(u\cdot\nabla_x\rho)\,(u\cdot\nabla_x)u + G_2\,(P_{u^\perp}\nabla_x u)(P_{u^\perp}\nabla_x\rho)\\
&+ G_3\,\big((P_{u^\perp}\nabla_x\rho)\cdot P_{u^\perp}\nabla_x\big)u + G_4\,(\nabla_x\cdot u)\,P_{u^\perp}\nabla_x\rho\\
&+ H_1\,(u\cdot\nabla_x\log\rho)\,(P_{u^\perp}\nabla_x\rho) + H_2\,\rho\,(P_{u^\perp}\nabla_x u)\big((u\cdot\nabla_x)u\big)\\
&+ H_3\,\rho\big[\big((u\cdot\nabla_x)u\big)\cdot P_{u^\perp}\nabla_x\big]u + H_4\,\rho\,(\nabla_x\cdot u)\,(u\cdot\nabla_x)u = 0,
\end{aligned}
$$

Diffusive terms (E). Cross-diffusion terms (F). Convection terms (G, H)
All the possible quadratic terms:

$$
\begin{aligned}
&\big((u\cdot\nabla_x)\rho\big)\,\big((u\cdot\nabla_x)u\big)\,[G_1], \quad \big((u\cdot\nabla_x)\rho\big)\,(P_{u^\perp}\nabla_x u),\\
&\big((u\cdot\nabla_x)u\big)\,(P_{u^\perp}\nabla_x\rho), \quad (P_{u^\perp}\nabla_x\rho)\,(P_{u^\perp}\nabla_x u)\,[G_2, G_3, G_4],\\
&\big((u\cdot\nabla_x)\rho\big)^2, \quad \big((u\cdot\nabla_x)\rho\big)\,(P_{u^\perp}\nabla_x\rho)\,[H_1], \quad (P_{u^\perp}\nabla_x\rho)^2,\\
&\big((u\cdot\nabla_x)u\big)^2[H_2, H_3, H_4], \quad \big((u\cdot\nabla_x)u\big)\,(P_{u^\perp}\nabla_x u), \quad (P_{u^\perp}\nabla_x u)^2.
\end{aligned}
$$

Back to the original Vicsek model

Something similar seen before:
P. Degond and T. Yang. *Diffusion in a continuum model of self-propelled particles with alignment interaction*. Math. Models Methods Appl. Sci., 20(supp01):14591490,

$$\partial_t \rho + \nabla_x \cdot (c_1 \rho \Omega) = \varepsilon R_1,$$
$$\rho(\partial_t \Omega + c_2(\Omega \cdot \nabla_x)\Omega) + P_{\Omega^\perp} \nabla_x \rho = \varepsilon R_2.$$

Section 3

Comments on the proof: Hilbert expansion and GCI

Generalised Collision Invariant

Momentum is not conserved.

Definition

Let $u \in \mathbb{S}^{d-1}$ be given. The operator $\bar{\Gamma}(f, u)$ is defined by

$$\bar{\Gamma}(f, u) := D\, \nabla_\omega \cdot \left[M_u \nabla_\omega \left(\frac{f}{M_u} \right) \right]. \tag{16}$$

With this definition, we have

$$\Gamma(f) = \bar{\Gamma}(f, u_f). \tag{17}$$

Definition (GCI)

Let $u \in \mathbb{S}^{d-1}$ be given. A function ψ: $\mathbb{S}^{d-1} \to \mathbb{R}$ is called a 'Generalised Collision Invariant (GCI)' associated to u if and only if

$$\int_{\mathbb{S}^{d-1}} \bar{\Gamma}(f, u)\, \psi \, d\omega = 0, \quad \text{for all } f \text{ such that } P_{u^\perp}(Q_f\, u) = 0. \tag{18}$$

It holds

$$\int_{\mathbb{S}^{d-1}} \Gamma(f)\,\psi\,d\omega = \int_{\mathbb{S}^{d-1}} \bar{\Gamma}(f,u_f)\,\psi\,d\omega = 0. \qquad (19)$$

Proposition (Characterisation of the GCI)

The set $\mathcal{G}_u$ of GCIs associated to u is given by

$$\mathcal{G}_u = \left\{ B\cdot\vec{\psi}_u + C \mid B\in\{u\}^{\perp},\ C\in\mathbb{R}\right\}. \qquad (20)$$

where the vector GCI $\vec{\psi}_u$ is written:

$$\vec{\psi}_u(\omega) = P_{u^\perp}\omega\ h(\omega\cdot u), \qquad (21)$$

where the function h is the unique solution to an explicit differential equation; h is an odd function of r and $h(r)\leq 0$ for $r\geq 0$.

Hilbert expansion

Hilbert expansion for f^ε: $\varepsilon^2\partial_t f^\varepsilon + \varepsilon(v\cdot\nabla_x)f^\varepsilon = L(f^\varepsilon)$

$$f^\varepsilon = f_0 + \varepsilon f_1 + \varepsilon^2 f_2 + \mathcal{O}(\varepsilon^3), \tag{22}$$

Taylor expanding Γ about f_0:

$$\begin{aligned}&\Gamma(f_0 + \varepsilon f_1 + \varepsilon^2 f_2 + \mathcal{O}(\varepsilon^3))\\ &\quad = \Gamma(f_0) + \varepsilon D_{f_0}\Gamma(f_1) + \varepsilon^2\left(D_{f_0}\Gamma(f_2) + \frac{1}{2}D^2_{f_0}\Gamma(f_1, f_1)\right) + \mathcal{O}(\varepsilon^3),\end{aligned} \tag{23}$$

At each order we get

$$\mathcal{O}(\varepsilon^0): \qquad \Gamma(f_0) = 0, \tag{24}$$

$$\mathcal{O}(\varepsilon^1): \qquad D_{f_0}\Gamma(f_1) = (\omega\cdot\nabla_x)f_0, \tag{25}$$

$$\mathcal{O}(\varepsilon^2): \qquad D_{f_0}\Gamma(f_2) = \partial_t f_0 + (\omega\cdot\nabla_x)f_1 - \frac{1}{2}D^2_{f_0}\Gamma(f_1, f_1). \tag{26}$$

We have that

$$f_0(t,x,\omega) = \rho_0(t,x)M_{u_0(t,x)}(\omega), \quad \forall (t,x,\omega)\in[0,\infty)\times\mathbb{R}^d\times\mathbb{S}^{d-1}. \tag{27}$$

Theorem (Inversion of the linearized operator $D_{\rho_0 M_{u_0}}\Gamma$)

(i) *Let* $(\rho_0, u_0) \in [0,\infty) \times \mathbb{S}^{d-1}$ *and* $g \in L^2(\mathbb{S}^{d-1})$. *There exists* $f \in H^1(\mathbb{S}^{d-1})$ *such that*

$$D_{\rho_0 M_{u_0}}\Gamma(f) = g \tag{28}$$

holds if and only if g *satisfies the solvability conditions:*

$$\int_{\mathbb{S}^{d-1}} g(\omega)\, d\omega = 0, \quad \int_{\mathbb{S}^{d-1}} g(\omega)\, \vec{\psi}_{u_0}(\omega)\, d\omega = 0. \tag{29}$$

(ii) *If condition* (29) *is satisfied, Eq.* (28) *has a unique solution* f *satisfying the two properties*

$$f \in \dot{H}^1_0(\mathbb{S}^{d-1}) \quad \text{and} \quad P_{u_0^\perp}(Q_f u_0) = 0, \tag{30}$$

where $\dot{H}^1_0(\mathbb{S}^{d-1}) = \left\{\varphi \in H^1(\mathbb{S}^{d-1}) \,\middle|\, \int_{\mathbb{S}^{d-1}} \frac{\varphi}{M_{u_0}}\, d\omega = 0\right\}$. *This solution is also the unique solution to the problem*

$$\bar{\Gamma}(f, u_0) = g, \tag{31}$$

in $\dot{H}^1_0(\mathbb{S}^{d-1})$ *(where* $\bar{\Gamma}$ *is defined in* (16)*) and conversely, the unique solution to* (31) *in* $\dot{H}^1_0(\mathbb{S}^{d-1})$ *is also the unique solution to* (28) *satisfying the two conditions* (30).

Theorem (Continuation)

(iii) *If f is the above solution, the set $\mathcal{S}_{u_0}$ of all solutions of (28) in $H^1(\mathbb{S}^{d-1})$ is given by*

$$\mathcal{S}_{u_0} = \left\{ f + M_{u_0} \left(\hat{\rho} + (\omega \cdot u_0)(\omega \cdot \hat{u}) \right) \mid \hat{\rho} \in \mathbb{R},\ \hat{u} \in \{u_0\}^\perp \right\}. \tag{32}$$

Lemma (Linearised operator)

Let $f_0 = \rho_0 M_{u_0}$ with $\rho_0 > 0$ and $u_0 \in \mathbb{S}^{d-1}$. For all functions $f_1 = f_1(\omega)$ it holds that

$$D_{f_0}\Gamma(f_1) = \bar{\Gamma}(f_1, u_0) - \kappa \nabla_\omega \cdot \left[f_0 \nabla_\omega \big((\omega \cdot u_0)\,(\omega \cdot u_1) \big) \right], \tag{33}$$

where u_1 is related to f_1 through

$$u_1 = \frac{d-1}{d\,\lambda_{\|}\rho_0}\, P_{u_0^\perp}(Q_{f_1} u_0), \tag{34}$$

where $\lambda_{\|}$ is the leading eigenvalue of Q_{f_0}.

Outlook

1. The Generalised Collision Invariant plays a key role in the inversion of the linearized collision operator.
2. Well-posedness?
3. Understand the terms in the equations.
4. Numerical simulations? Develop a method that preserves the symmetries of the system (in particular the rotational invariance).
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Outlook

1. The Generalised Collision Invariant plays a key role in the inversion of the linearized collision operator.
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3. Understand the terms in the equations.
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THANK YOU!