# Long time dynamics for the Landau-Fermi-Dirac equation with hard potentials

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# The Boltzmann equation with quantum effects

To take into account quantum effects (like Pauli exclusion principle), classical Boltzmann equation is modified as

$$\partial_t f + v \cdot \nabla_x f = \iint_{\mathbb{R}^3 \times \mathbb{S}^2} B(v - v_*, \sigma) \Big\{ f' f'_* (1 - \varepsilon f)(1 - \varepsilon f_*) - f f_* (1 - \varepsilon f')(1 - \varepsilon f'_*) \Big\} \, d\sigma \, dv_* .$$

where $\varepsilon$ is proportional to the Planck constant $\hbar$ and

- $\varepsilon = 0$: the Boltzmann equation;
- $\varepsilon > 0$: the Boltzmann-Fermi-Dirac equation;
- $\varepsilon < 0$: the Boltzmann-Bose-Einstein equation.

with usual post-collisional velocities $v' = \frac{v+v_*}{2} + \frac{|v-v_*|}{2}\sigma$, $v'_* = \frac{v+v_*}{2} - \frac{|v-v_*|}{2}\sigma$.

**Ref.** : Chapman & Cowling (1970), Danielewicz (1980), Lifshitz & Pitaevskivı (1981)

## Case of inverse-power law potentials

$$B(z,\sigma) = |z|^{\gamma}\, b(\theta), \qquad \text{where} \qquad \cos\theta = \left(\frac{v - v_*}{|v - v_*|}\right)\cdot\sigma.$$

The function $b$ is only implicitly defined, locally smooth with a nonintegrable singularity at $\theta = 0$

$$\sin(\theta) b(\theta) \overset{\theta\to 0}{\sim} C\,\theta^{(\gamma-3)/2}$$

- $0 < \gamma \leqslant 1$: hard potentials;
- $\gamma = 0$: the Maxwellian potential;
- $-3 < \gamma < 0$: soft potentials;
- $\gamma = -3$: the Coulomb potential.

# The Boltzmann Fermi Dirac equation

$$\partial_t f + v \cdot \nabla_x f = \iint_{\mathbb{R}^3 \times \mathbb{S}^2} B(v - v_*, \sigma) \Big\{ f' f'_* (1 - \varepsilon f)(1 - \varepsilon f_*) - f f_* (1 - \varepsilon f')(1 - \varepsilon f'_*) \Big\} \, d\sigma \, dv_* .$$

For $\varepsilon > 0$, there are two kinds of equilibrium states :

- Fermi-Dirac distributions

$$\mathcal{M}_\varepsilon(v) = \frac{M_\varepsilon(v)}{1 + \varepsilon\, M_\varepsilon(v)} = \frac{a e^{-b|v - v_0|^2}}{1 + a\varepsilon\, e^{-b|v - v_0|^2}}, \qquad a, b > 0,\ v_0 \in \mathbb{R}^3$$

- characteristic functions of balls

$$F_\varepsilon(v) = \frac{1}{\varepsilon} \mathbf{1}_{|v - v_0| \leqslant R_\varepsilon}$$

Spatially homogeneous case

Lu (2001), Escobedo, Mischler & Valle (2003), Lu & Wennberg (2003)

Spatially inhomogeneous case

Dolbeault (1994), Lions (1994), Alexandre (2000), Lu (2006 and 2008).

## Grazing collisions asymptotic

$$b(\theta)\sin(\theta) \text{ is replaced by } \frac{1}{\delta^3}\, b\left(\frac{\theta}{\delta}\right) \sin\left(\frac{\theta}{\delta}\right)$$

and, letting $\delta \to 0$, one obtains

## The Landau operator with quantum effects

$$\mathcal{Q}_{LFD}(f)(v) = \nabla_v \cdot \int_{\mathbb{R}^3} |v - v_*|^{\gamma+2}\, \Pi(v - v_*) \Big\{ f_*(1 - \varepsilon f_*)\nabla f - f(1 - \varepsilon f)\nabla f_* \Big\}\, dv_*$$

with

$$\Pi(z) = (\Pi_{i,j}(z))_{i,j} \quad \text{and} \quad \Pi_{i,j}(z) = \delta_{i,j} - \frac{z_i z_j}{|z|^2}.$$

**Ref.:** Degond & Lucquin-Desreux (1992), Desvillettes (1992) for $\varepsilon = 0$
Danielewicz (1980) in the general case.

# The classical Landau equation ($\varepsilon = 0$)

$$\partial_t f + v \cdot \nabla_x f = \nabla_v \cdot \int_{\mathbb{R}^3} |v - v_*|^{\gamma+2}\, \Pi(v - v_*) \Big\{ f_* \nabla f - f \nabla f_* \Big\}\, dv_*$$

The equilibrium states are Maxwellian distributions :

$$M(v) = a e^{-b|v-v_0|^2}, \qquad a, b > 0, v_0 \in \mathbb{R}^3$$

Spatially homogeneous case

Arsen'ev & Peskov (1977), Arsen'ev & Buryak (1991), Villani (1998), Desvillettes & Villani (2000), El Safadi (2007), Fournier & Guerin (2009), Chen, Li & Xu (2009 and 2010), Fournier (2010), Morimoto, Pravda-Starov & Xu (2013), Wu (2014), Carrapatoso (2015), Desvillettes (2015), Alexandre, Lia & Lin (2015), Desvillettes (2016), Carrapatoso, Desvillettes & He (2017)

Spatially inhomogeneous case

Villani (1996), Guo (2002), Yu (2006), Guo & Strain (2006 and 2008), Chen, Desvillettes & He (2009), Carrapatoso, Tristani & Wu (2016), Carrapatoso & Mischler (2017)

# The Landau-Fermi-Dirac (LFD) equation

$$\partial_t f = \mathcal{Q}_{LFD}(f)$$
$$= \nabla_v \cdot \int_{\mathbb{R}^3} |v - v_*|^{\gamma+2}\, \Pi(v - v_*) \Big\{ f_*(1 - \varepsilon f_*)\nabla f - f(1 - \varepsilon f)\nabla f_* \Big\}\, dv_*.$$

- Such an equation also arises in the modelling of self-gravitating particles. Kadomtsev & Pogutse (1970), Chavanis (1998)
- (Non quantitative) spectral analysis for the linearization has been obtained by Lemou (2000)
- There are some results in the spatially inhomogeneous case. Liu, Ma & Yu (2012), Liu (2012)

# The Landau-Fermi-Dirac (LFD) equation

$$\partial_t f = \mathcal{Q}_{LFD}(f)$$
$$= \nabla_v \cdot \int_{\mathbb{R}^3} |v - v_*|^{\gamma+2}\, \Pi(v - v_*) \Big\{ f_*(1 - \varepsilon f_*) \nabla f - f(1 - \varepsilon f) \nabla f_* \Big\}\, dv_*.$$

Equilibrium solutions are the same as for Boltzmann-Fermi-Dirac:

- Fermi-Dirac distributions

$$\mathcal{M}_\varepsilon(v) = \frac{M_\varepsilon(v)}{1 + \varepsilon\, M_\varepsilon(v)} = \frac{a e^{-b|v-v_0|^2}}{1 + a\varepsilon\, e^{-b|v-v_0|^2}}, \qquad a, b > 0,\, v_0 \in \mathbb{R}^3$$

- characteristic functions of balls

$$F_\varepsilon(v) = \frac{1}{\varepsilon} \mathbf{1}_{|v-v_0| \leqslant R_\varepsilon}$$

**Equilibrium distributions cannot have arbitrary mass *and* energy**
If $0 \leqslant g \leqslant \varepsilon^{-1}$ is such that

$$\int_{\mathbb{R}^3} g(v) \begin{pmatrix} 1 \\ v \\ |v|^2 \end{pmatrix} dv = \begin{pmatrix} \varrho \\ 0 \\ 3\varrho\, E \end{pmatrix}$$

then

$$\varepsilon \leqslant \frac{4\pi}{3\rho}(5\,E)^{\frac{3}{2}}$$

and equality only occurs only if $g = F_\varepsilon$ is a degenerate equilibrium (X. Lu (2001)).
Fermi-Dirac equilibrium exists only for

$$\varepsilon < \varepsilon_{\rm deg} := \frac{4\pi}{3\rho}\big(5\,E\big)^{3/2}.$$

# *A priori* estimates

$$\partial_t f(t,v) = \nabla_v \cdot \int_{\mathbb{R}^3} a(v-v_*)\Big\{f_*(1-\varepsilon f_*)\nabla f - f(1-\varepsilon f)\nabla f_*\Big\}\,dv_*.$$

where

$$a_{i,j}(z) = |z|^{\gamma+2}\left(\delta_{i,j} - \frac{z_i z_j}{|z|^2}\right).$$

$L^\infty$-bound:

$$0 \leqslant f(0,\cdot) \leqslant \varepsilon^{-1} \Longrightarrow 0 \leqslant f(t,\cdot) \leqslant \varepsilon^{-1} \qquad \forall t \geqslant 0.$$

Weak formulation:

$$\int \mathcal{Q}_{LFD}(f)(v)\,\varphi(v)\,dv$$
$$= -\frac{1}{2}\iint a(v-v_*)\Big\{f_*(1-\varepsilon f_*)\nabla f - f(1-\varepsilon f)\nabla f_*\Big\}\Big\{\nabla\varphi - \nabla\varphi_*\Big\}\,dv_*\,dv$$

Conservation laws

Mass, momentum and energy are preserved, i.e.

$$\frac{d}{dt}\int f(t,v)\,dv = 0 \qquad \frac{d}{dt}\int f(t,v)\,v\,dv = 0 \qquad \frac{d}{dt}\int f(t,v)\,|v|^2 dv = 0$$

## Fermi-Dirac Entropy

For any $\varepsilon > 0$ and $0 \leqslant f \leqslant \varepsilon^{-1}$ we introduce the Fermi-Dirac entropy as

$$\mathcal{S}_\varepsilon(f) = -\frac{1}{\varepsilon}\int_{\mathbb{R}^3}\Big[\varepsilon f \log(\varepsilon f) + (1-\varepsilon f)\log(1-\varepsilon f)\Big]\,dv\,. \tag{1}$$

Then, along solutions to the LFD eq., one has

$$\frac{d}{dt}\mathcal{S}_\varepsilon(f(t)) = -\mathscr{D}_{\varepsilon,\gamma}(f(t)),$$

where the dissipation term reads

$$\begin{aligned}\mathscr{D}_{\varepsilon,\gamma}(f) &= \int_{\mathbb{R}^3}\mathcal{Q}_{LFD}(f)\Big(\log(\varepsilon f(v)) - \log(1-\varepsilon f(v))\Big)dv\\ &= \frac{1}{2}\int_{\mathbb{R}^3\times\mathbb{R}^3} dv dv_*|v-v_*|^{\gamma+2} ff_*(1-\varepsilon f)(1-\varepsilon f_*)\times\\ &\qquad\qquad \times\left|\Pi(v-v_*)\left(\frac{\nabla f}{f(1-\varepsilon f)} - \frac{\nabla f_*}{f_*(1-\varepsilon f_*)}\right)\right|^2 \geqslant 0.\end{aligned}$$

Hence,

$$t \geqslant 0 \longmapsto \mathcal{S}_\varepsilon(f)(t) \text{ is a non-decreasing function.}$$

# Cauchy Theory

$$\partial_t f(t,v) = \nabla_v \cdot \int_{\mathbb{R}^3} a(v-v_*)\Big\{ f_*(1-\varepsilon f_*)\nabla f - f(1-\varepsilon f)\nabla f_* \Big\}\, dv_* \quad (t \geqslant 0) \quad (2)$$

with $f(t=0,v) = f_0(v)$.
The equation can be reformulated as a nonlinear parabolic equation

$$\partial_t f = \nabla \cdot \Big( \boldsymbol{\Sigma}_\varepsilon[f]\, \nabla f - \boldsymbol{b}[f]\, f(1-\varepsilon f) \Big),$$

with

$$\boldsymbol{\Sigma}_\varepsilon[f] = \left(a_{i,j} * f(1-\varepsilon f)\right)_{i,j},$$

and

$$\boldsymbol{b}[f] = \left(b_i * f\right)_i, \qquad b_i(z) = \sum_k \partial_k a_{i,k}(z) = -2\,|z|^\gamma\, z_i.$$

## Cauchy Theory

**Assumption on the initial datum** There is $\varepsilon_0 \in (0,1)$ such that the initial datum

$$0 < \|f_0\|_\infty =: \varepsilon_0^{-1} < \infty \qquad \text{and} \qquad S_0 := \mathcal{S}_{\varepsilon_0}(f_0) > 0, \tag{3}$$

and $f_0 \in L^1_{s_0}(\mathbb{R}^3)$ for some $s_0 > 2$.

**Theorem (Bagland 2004 – for $\varepsilon = 1$.)**

*Under such an assumption, for any $\varepsilon \in (0, \varepsilon_0]$, there exists a weak solution $f$ to (2) satisfying the conservation laws and*

$$f(1-\varepsilon f) \in L^1_{\mathrm{loc}}\left(\mathbb{R}_+; L^1_{s_0+\gamma}\left(\mathbb{R}^3\right)\right); \qquad f \in L^\infty_{\mathrm{loc}}\left(\mathbb{R}_+; L^1_{s_0}\left(\mathbb{R}^3\right)\right) \cap L^2_{\mathrm{loc}}\left(\mathbb{R}_+; H^1_{s_0}\left(\mathbb{R}^3\right)\right).$$

*If $s_0 \geqslant 2+\gamma$, then the entropy is a non-decreasing function while, for $s_0 > 4\gamma + 11$, such a solution is unique.*

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For $s \in \mathbb{R}$, $p \geqslant 1$, $k \in \mathbb{N}$,

$$\|f\|^p_{L^p_s} = \int_{\mathbb{R}^3} |f(v)|^p \langle v\rangle^s \, dv, \qquad \|f\|^2_{H^k_s} = \sum_{0 \leqslant |\beta| \leqslant k} \int_{\mathbb{R}^3} |\partial_\beta f(v)|^2 \langle v \rangle^s \, dv,$$

where $\langle v\rangle = (1+|v|^2)^{1/2}$, $\beta = (i_1, i_2, i_3) \in \mathbb{N}^3$, $|\beta| = i_1 + i_2 + i_3$ and $\partial_\beta f = \partial_1^{i_1}\partial_2^{i_2}\partial_3^{i_3} f$.

The equation can be reformulated as a nonlinear parabolic equation

$$\partial_t f = \nabla \cdot \Big( \boldsymbol{\Sigma}_\varepsilon[f] \nabla f - \boldsymbol{b}[f]\, f(1-\varepsilon f) \Big),$$

with

$$\boldsymbol{\Sigma}_\varepsilon[f] = \left(a_{i,j} * f(1-\varepsilon f)\right)_{i,j},$$

and

$$\boldsymbol{b}[f] = \left(b_i * f\right)_i, \qquad b_i(z) = \sum_k \partial_k a_{i,k}(z) = -2\,|z|^\gamma\, z_i.$$

Crucial estimate (uniform ellipticity): Let $\varepsilon \in (0, \varepsilon_0]$ then

$$\boldsymbol{\Sigma}_\varepsilon[f](v) \geqslant K_0 \langle v \rangle^\gamma \boldsymbol{I}_{3\times 3}, \qquad \forall v \in \mathbb{R}^3$$

for all $f \in L^1_2(\mathbb{R}^3) \cap L^\infty(\mathbb{R}^3)$ satisfying (3) and

$$\int_{\mathbb{R}^3} f|v|^2 dv \leqslant E_0, \quad \mathcal{S}_\varepsilon(f) \geqslant S_0$$

and $K_0 > 0$ depends only on $\gamma$, $E_0$ and $S_0$.

Open problems: appearance of moments ? uniform in time estimates ?
*Partial answer:*

## Proposition (Chen, 2010 and 2011)

*Let* $\gamma \in (0,1]$. *Consider* $f_{in} \in L^1_{2s}(\mathbb{R}^3)$ *for any* $s > 1$ *satisfying* $0 \leqslant f_{in} \leqslant 1$ *a.e.*
*Let* $f$ *be the weak solution to the LFD equation. Then,*

- *for any* $0 < t_0 < T < +\infty$, *we have*

$$f \in \mathcal{C}^\infty([t_0, T]; \mathcal{S}(\mathbb{R}^3)),$$

  *where* $\mathcal{S}(\mathbb{R}^3)$ *is the Schwartz space.*

- *for any multi-index* $\alpha \in \mathbb{N}^3$, *any* $s > 1$ *and any* $0 < t_0 < T < +\infty$,

$$\sup_{t_0 \leqslant t \leqslant T} \|\partial_\alpha f(t)\|_{L^2_{2s}(\mathbb{R}^3)} \leqslant \begin{cases} C & \text{if } 0 \leqslant |\alpha| \leqslant 3, \\ C^{|\alpha|-2}(|\alpha|-4)! & \text{if } |\alpha| \geqslant 4, \end{cases}$$

  *where* $C$ *only depends on* $\gamma$, $s$, $t_0$, $T$ *and* $f_{in}$.
  *In particular,* $f(t,\cdot)$ *is analytic in* $\mathbb{R}^3$ *for any* $t > 0$.

Open problems: appearance of moments ? uniform in time estimates ?

Difficulty: Evolution of $L^2$-moments enter naturally in the evolution of $L^1$-moments.

Typically, resuming the arguments done in the classical Landau case, we get something like

$$\frac{d}{dt}\int_{\mathbb{R}^3} f\,\langle v\rangle^s\,dv + K_s\int_{\mathbb{R}^3}\langle v_*\rangle^{s+\gamma}\,f_*(1-\varepsilon f_*)dv_* \leqslant C_{s,1}\int_{\mathbb{R}^3} f\,\langle v\rangle^s dv,$$

- Notice: A positive lower bound on $1-\varepsilon\, f_*$ would be very helpful here.

Given a solution $f(t,v)$ to the LFD equation, we introduce

$$\boldsymbol{m}_s(t) = \int_{\mathbb{R}^3} f(t,v)\langle v\rangle^s dv, \qquad \mathbb{M}_s(t) = \int_{\mathbb{R}^3} f^2(t,v)\langle v\rangle^s dv, \qquad s\in\mathbb{R}.$$

$$\mathbb{D}_s(t) = \left\|\nabla\left(f(t,\cdot)\langle v\rangle^{\frac{s}{2}}\right)\right\|_2^2, \qquad t\geqslant 0.$$

## Proposition

*If $0\leqslant f_0\in L^1_{s_0}(\mathbb{R}^3)$, for some $s_0>2$ satisfies (3) and $f=f(t,v)$ is a weak solution to the LFD equation that preserves mass and energy. Then, for some constants $C_{s,1}, C_{s,2}, C_{s,3}>0$ and $K_s>0$ depending only on $\boldsymbol{m}_2(0)$, $\gamma$ and $s$, it holds*

$$\frac{d}{dt}\boldsymbol{m}_s(t) + K_s\boldsymbol{m}_{s+\gamma}(t) \leqslant K_s\mathbb{M}_{s+\gamma}(t) + C_{s,1}\boldsymbol{m}_s(t)\,, \qquad s>2\,.$$

$$\frac{1}{2}\frac{d}{dt}\mathbb{M}_s(t) + K_0\mathbb{D}_{s+\gamma}(t) \leqslant C_{s,2}\mathbb{M}_{s+\gamma}(t) + C_{s,3}\boldsymbol{m}_{2+\gamma}(t)\mathbb{M}_{s+\gamma-2}(t),$$

*where $K_0$ comes from the ellipticity of $\boldsymbol{\Sigma}_\varepsilon[f]$. All constants are independent of $\varepsilon>0$.*

# Simple observation

$$\left\| f\langle\cdot\rangle^{\frac{s+\gamma-2}{2}} \right\|_{L^2} \leqslant \| f\langle\cdot\rangle^{\frac{s+\gamma}{2}-\frac{5}{2}} \|_{L^1}^{1-\theta} \| f\langle\cdot\rangle^{\frac{s+\gamma}{2}} \|_{L^6}^{\theta} \qquad \theta = \frac{3}{5}$$

Estimating the last $L^6$-norm with Sobolev's inequality, we obtain that

$$\left\| f\langle\cdot\rangle^{\frac{s+\gamma-2}{2}} \right\|_{L^2} \leqslant C \left\| f\langle\cdot\rangle^{\frac{s+\gamma}{2}-\frac{5}{2}} \right\|_{L^1}^{\frac{2}{5}} \left\| \nabla ( f\langle\cdot\rangle^{\frac{s+\gamma}{2}} ) \right\|_{L^2}^{\frac{3}{5}},$$

i.e.

$$\mathbb{M}_{s+\gamma-2}(t) \leqslant C\, \boldsymbol{m}_{\frac{s+\gamma-5}{2}}(t)^{\frac{2}{5}}\, \mathbb{D}_{s+\gamma}(t)^{\frac{3}{5}}.$$

Similar argument

$$\mathbb{M}_{s+\gamma}(t) \leqslant C \boldsymbol{m}_{\frac{s+\gamma}{2}}(t)^{\frac{4}{5}}\, \mathbb{D}_{s+\gamma}(t)^{\frac{3}{5}}.$$

We are only interested in the behaviour of $\boldsymbol{m}_s(t)$ and $\mathbb{M}_s(t)$.

## Uniform in time estimates

Investigate the evolution of

$$\mathcal{E}_s(t) := \boldsymbol{m}_s(t) + \mathbb{M}_s(t), \qquad t \geqslant 0, \quad s \in (s_\gamma, 9-\gamma]$$

and a control of the mixed terms $\boldsymbol{m}_{2+\gamma}(t)\mathbb{M}_{s+\gamma-2}(t)$ give

### Theorem (Alonso, Bagland, L. 2019)

*Consider* $0 \leqslant f_0 \in L^1_{s_\gamma}(\mathbb{R}^3)$*, with* $s_\gamma = \max\{2 + \frac{3\gamma}{2}, 4-\gamma\}$ *satisfying* (3)*. Then, for any* $\varepsilon \in (0, \varepsilon_0]$ *there exists a weak solution* $f$ *to the LFD equation such that:*

*(i)* ***(Generation)*** *For any* $t_0 > 0$*,* $k \in \mathbb{N}$*, and* $s > 0$*, there exists a constant* $C_{t_0} > 0$ *such that*

$$\sup_{t \geqslant t_0} \|f(t)\|_{H^k_s} \leqslant C_{t_0}.$$

*The constant* $C_{t_0}$ *depends, in addition to* $t_0$*, on* $\|f_0\|_{L^1_2}$*,* $S_{in}$*,* $k$*,* $s$*,* $\gamma$*. In particular,*

$$f \in \mathcal{C}^\infty\left([t_0, +\infty); \mathcal{S}(\mathbb{R}^3)\right), \qquad \forall\, t_0 > 0\,.$$

*(ii)* ***(Propagation)*** *Furthermore, if* $\|f_0\|_{H^k_s} < \infty$ *and* $f_{in} \in L^1_{s'}(\mathbb{R}^3)$ *for sufficiently large* $s' > 0$*, the choice* $t_0 = 0$ *is valid with constant depending on such initial regularity.*

The constants are all independent of $\varepsilon \in [0, \varepsilon_0]$.

# Fundamental observation

## Corollary

*Consider* $0 \leqslant f_0 \in L^1_{s_\gamma}(\mathbb{R}^3)$ *satisfying* (3). *Then, for any solution* $f(t) = f_\varepsilon(t)$ *to* (2) *given by Theorem 1, it holds*

$$\sup_{t \geqslant t_0} \|f(t)\|_\infty \leqslant C_{t_0}\,, \qquad \forall\, t_0 > 0.$$

*The constant* $C_{t_0}$ *only depends on* $M(f_0)$, $E(f_0)$, $S_0$, $s$, *and* $t_0$.

*Consequently, for any* $\kappa_0 \in (0,1)$ *there exists* $\varepsilon_\star > 0$ *depending only on* $\kappa_0$, $M(f_0)$, $E(f_0)$, *and* $S_0$, *such that*

$$\inf_{v \in \mathbb{R}^3} \big(1 - \varepsilon\, f(t,v)\big) \geqslant \kappa_0, \qquad \forall\, \varepsilon \in (0, \varepsilon_\star),\ t \geqslant 1. \tag{4}$$

- This is the lower bound which would had turn useful for the moment estimates.....

For $\varepsilon < \varepsilon_\star$, solution is uniformly far away from the degenerate steady state

$$F_\varepsilon(v) = \varepsilon^{-1} \mathbf{1}_{R_\varepsilon}.$$

## Convergence to equilibrium : non quantitative result

**Theorem**

*Consider $0 \leqslant f_0 \in L^1_{s_\gamma}(\mathbb{R}^3)$ satisfying (3) and let $f = f(t,v)$ be the previously obtained solution to the LFD equation. Let $\mathcal{M}_\varepsilon$ be the unique Fermi-Dirac statistics satisfying*

$$\int_{\mathbb{R}^3} f_0(v) \begin{pmatrix} 1 \\ v \\ |v|^2 \end{pmatrix} dv = \int_{\mathbb{R}^3} \mathcal{M}_\varepsilon(v) \begin{pmatrix} 1 \\ v \\ |v|^2 \end{pmatrix} dv = \begin{pmatrix} \varrho \\ 0 \\ 3\varrho E \end{pmatrix}$$

*with $\varepsilon < \frac{4\pi}{3} \frac{(5E)^{\frac{3}{2}}}{\varrho}$. Then,*

$$\lim_{t\to\infty} \|f(t) - \mathcal{M}_\varepsilon\|_{L^1_2} = 0.$$

Idea of the proof. Consider a sequence $\{t_n\}_{n\in\mathbb{N}}$ of positive real numbers with $\lim_n t_n = \infty$. The family $\{f(t_n)\}_{n\in\mathbb{N}}$ is relatively compact in $H^1_p(\mathbb{R}^3)$ for any $p \geqslant 0$. One can extract a subsequence, still denoted $\{t_n\}_n$, and $F_\infty \in H^1_p(\mathbb{R}^3)$ such that

$$\lim_{n\to\infty} \|f(t_n) - F_\infty\|_{H^1_p} = 0.$$

It remains to show that $F_\infty = \mathcal{M}_\varepsilon$. **Ref.:** Carrillo, Laurençot & Rosado (2009).

## Quantitative convergence to equilibrium

How to make the convergence quantitative. We will need some additional assumption on $E, \varrho$ of the form

$$\varepsilon \leqslant c \, \frac{4\pi}{3} \frac{(5E)^{\frac{3}{2}}}{\varrho}$$

We will combine

- Close-to-equilibrium analysis (Spectral gap estimate)
- Far from equilibrium analysis (Entropy/entropy production estimate).

# The linearized operator

Consider $0 \leqslant f_0 \in L^1_{s_0}(\mathbb{R}^3)$, with $s_0 > 2$, satisfying (3). Let $\varepsilon \in (0, \varepsilon_0]$ and $f = f(t,v)$ be the previously obtained weak solution to the LFD equation, and let $\mathcal{M}_\varepsilon$ be the unique Fermi-Dirac statistics with same mass, momentum and energy as $f(t,\cdot)$.

We introduce the fluctuation $\quad f = \mathcal{M}_\varepsilon + g$. Then,

$$\partial_t g = \mathscr{L}_\varepsilon g + \mathbf{\Gamma}_\varepsilon(g)\,,$$

where

$$\mathscr{L}_\varepsilon(g) = \nabla \cdot \int_{\mathbb{R}^3} a(v - v_*)\left[\mathfrak{m}_* \nabla g - \mathfrak{m}\,(\nabla g)_*\right] dv_*$$
$$+ \nabla \cdot \int_{\mathbb{R}^3} a(v - v_*)\Big[g_*\,(1 - 2\varepsilon\mathcal{M}_\varepsilon)_* \nabla \mathcal{M}_\varepsilon - g\,(1 - 2\varepsilon \mathcal{M}_\varepsilon)(\nabla \mathcal{M}_\varepsilon)_*\Big] dv_*\,.$$

with $\mathfrak{m}(v) = \mathcal{M}_\varepsilon(v)(1 - \varepsilon\mathcal{M}_\varepsilon(v))$.

# Spectral gap for the linearized operator

**Theorem**

*There exists an explicit $\varepsilon^\dagger > 0$ such that, for any $\varepsilon \in (0, \varepsilon^\dagger)$ there exists $k_\varepsilon^\dagger > 0$ such that for any $k > k_\varepsilon^\dagger$ the linearized operator $\mathscr{L}_\varepsilon$ around the Fermi-Dirac statistics $\mathcal{M}_\varepsilon$ generates a $C_0$-semigroup $(\boldsymbol{S}_\varepsilon(t))_{t \geqslant 0}$ in $L^2_k$. Furthermore, for any $g \in L^2_k$,*

$$\left\|\boldsymbol{S}_\varepsilon(t)g - \mathbb{P}_\varepsilon g\right\|_{L^2_k} \leqslant C_0 \exp(-\lambda_\varepsilon\, t)\left\|g - \mathbb{P}_\varepsilon g\right\|_{L^2_k}, \qquad \forall\, t \geqslant 0\,,$$

*for some explicit constant $C_0 > 0$ (independent of $\varepsilon$) where $\lambda_\varepsilon > 0$ is the spectral gap of $\mathcal{L}_\varepsilon$ which can be estimated explicitly. The operator $\mathbb{P}_\varepsilon$ is the spectral projection on $\mathrm{Ker}\big(\mathscr{L}_\varepsilon\big)$.*

This is an extension of the result of Lemou (2000) which makes the spectral gap *quantitative*.

# Spectral gap estimate

Several main steps:

1. Lower bound for the Dirichlet form in natural space $L^2(\mathfrak{m})$.
2. No loss of generality to consider maxwell molecules interactions $\gamma = 0$ by a suitable comparison argument.
3. The Dirichlet form reads

$$\mathcal{D}_{\varepsilon,2}(h) = \frac{1}{2}\int_{\mathbb{R}^6} \mathfrak{m}\,\mathfrak{m}_*|v-v_*|^2\,|\Pi(v-v_*)\,(\nabla h - \nabla h_*)|^2\,dvdv_*, \qquad h \in L^2(\mathfrak{m})$$

and there is $\lambda_\gamma(\varepsilon) > 0$ such that

$$\mathcal{D}_{\varepsilon,2}(h) \geqslant \lambda_\gamma(\varepsilon)\|h\|_{L^2(\mathfrak{m})},$$

$$\text{whenever } \int_{\mathbb{R}^3} h(v)\mathfrak{m}(v)\begin{pmatrix} 1 \\ v \\ |v|^2 \end{pmatrix} dv = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

4. Extend the spectral result and decay of the semigroup to the space $L^2(\langle\cdot\rangle^k)$ by *enlargement and factorisation* method, see Gualdani, Mischler, Mouhot (2018).

# Spectral gap estimate

For the 3rd step: idea of the proof borrowed from computations of Villani-Desvillettes (2000) and Desvillettes (2015) for the *production of entropy* of the Landau operator.

Write $\mathcal{D}_{\varepsilon,2}(h) = \frac{1}{2}\int_{\mathbb{R}^6} \mathfrak{m}\,\mathfrak{m}_* |v - v_*|^2 \left|R_h(v, v_*)\right|^2 dvdv_*$ with

$$R_h(v, v_*) = \Pi(v - v_*)\left(\nabla h - \nabla h_*\right) = \nabla h - \nabla h_* - \lambda_h\,(v - v_*)$$

for some suitable $\lambda_h(v, v_*)$. For any circular permutation $(i, k, k)$ of $(1, 2, 3)$, it holds

$$\Big((v - v_*) \wedge R_h(v, v_*)\Big)_k = (v - v_*)_j\left(\partial_i h - \partial_i h_*\right) - (v - v_*)_i\left(\partial_j h - \partial_j h_*\right).$$

Multiply this vectorial identity by $\varphi^{\ell}(v_*)$ and integrate over $\mathbb{R}^3$ to get a suitable system which is solved with Cramer's rule to express $\partial_j h$ in terms of $R_h$.
Contrast with the approach used for classical Landau where the spectral gap is deduced from the one of the Boltzmann operator in the grazing collision limit (see Baranger-Mouhot, 2003).

## Quantitative convergence to equilibrium

The spectral result and decay of semigroup yields an explicit convergence to $\mathcal{M}_\varepsilon$ for *close-to-equilibrium* initial data.

• Tool to extend this to *far-from-equilibrium* initial data is the entropy production.

**Lemma**

*Fix $\varepsilon > 0$ and let $0 \leqslant f \leqslant \varepsilon^{-1}$ be a function such that*

$$\inf_{v \in \mathbb{R}^3} \big(1 - \varepsilon f(v)\big) = \kappa_0 > 0 .$$

*Then,*

$$\kappa_0^2 \mathscr{D}_0(f) \leqslant 2\mathscr{D}_\varepsilon(f) + \frac{4\varepsilon^2}{\kappa_0} \int_{\mathbb{R}^6} f\, f_* \,|v - v_*|^{\gamma+2}\, |\nabla f(v)|^2 \, dv dv_* ,$$

*where $\mathscr{D}_0(f)$ is the entropy production for the classical Landau operator.*

For the solutions to LFD:

$$\kappa_0^2 \, \mathscr{D}_0(f(t)) \leqslant 2\mathscr{D}_\varepsilon(f(t)) + C_1\varepsilon^2, \qquad \forall\, \varepsilon \in (0, \varepsilon_\star) \,,\ t \geqslant 1,$$

with $\varepsilon \in (0, \varepsilon^\dagger)$.

Notice that the Fermi-Dirac entropy

$$\mathcal{S}_\varepsilon(f) = -\frac{1}{\varepsilon}\int_{\mathbb{R}^3} \left(\varepsilon \log(\varepsilon f) + (1-\varepsilon f)\log(1-\varepsilon f)\right) dv$$

does not converge as $\varepsilon \to 0$ to the Boltzmann entropy

$$H(f) = \int_{\mathbb{R}^3} f \log f dv.$$

However, the *relative entropy* do converge

$$\mathcal{H}_\varepsilon(f|g) := \mathcal{S}_\varepsilon(f) - \mathcal{S}_\varepsilon(g) \underset{\varepsilon\to 0}{\longrightarrow} \mathcal{H}_0(f|g) = H(f) - H(g)$$

if $f, g$ share same mass.
This allows to exploit the entropy production estimate for Landau operator

$$\mathscr{D}_0(f) \geqslant \min\left(\lambda_1 \mathcal{H}_0(f|M_f)\,;\, \lambda_2 \mathcal{H}_0(f|M_f)^{1+\frac{\gamma}{2}}\right).$$

established in Desvillettes-Villani (2000).

# Evolution of the relative entropy

## Theorem

*Consider* $0 \leqslant f_0 \in L^1_{s_0}(\mathbb{R}^3)$, *with* $s_0 > 2$, *satisfying* (3) *and a solution* $f(t,v)$ *to* (2) *with* $\varepsilon \in (0, \varepsilon_0]$ *given by Theorem 1. Then, there exist* $\varepsilon_\star \in (0, \varepsilon_0]$, $\lambda_0 > 0$, *and* $C_0 > 0$ *such that*

$$\frac{d}{dt}\mathcal{H}_\varepsilon(f(t)|\mathcal{M}_\varepsilon) \leqslant -\lambda_0 \min\left(\mathcal{H}_\varepsilon(f(t)|\mathcal{M}_\varepsilon)\,;\,\mathcal{H}_\varepsilon(f(t)|\mathcal{M}_\varepsilon)^{1+\frac{\gamma}{2}}\right) + C_0\varepsilon^{1+\frac{\gamma}{2}},$$

*for any* $\varepsilon \in (0, \varepsilon_\star)$ *and any* $t \geqslant 1$. *As a consequence, there is a positive constant* $C_1 > 0$ *such that*

$$\mathcal{H}_\varepsilon(f(t)|\mathcal{M}_\varepsilon) \leqslant C_1\left((1+t)^{-\frac{2}{\gamma}} + \varepsilon^{1+\frac{2}{\gamma}}\right) \qquad \forall\, t \geqslant 1,\ \ \varepsilon \in (0, \varepsilon_\star).$$

Combining this with the close-to-equilibrium convergence, we easily deduce

### Theorem

*Consider* $0 \leqslant f_0 \in L^1_{s_\gamma}(\mathbb{R}^3) \cap L^2_k(\mathbb{R}^3)$, *with* $s_\gamma = \max\{\frac{3\gamma}{2} + 2, 4 - \gamma\}$ *and* $k > k_\varepsilon^\dagger$, *satisfying* (3). *Let* $\varepsilon \in (0, \varepsilon_0]$ *and* $f$ *be a weak solution to* (2). *Then, there exists* $\varepsilon^\ddagger \in (0, \varepsilon^\dagger)$ *such that for any* $\varepsilon \in (0, \varepsilon^\ddagger)$

$$\|f(t) - \mathcal{M}_\varepsilon\|_{L^1_2} \leqslant C \exp(-\lambda_\varepsilon t), \qquad \forall\, t \geqslant 0\,, \tag{5}$$

*where* $\lambda_\varepsilon > 0$ *is the explicit spectral gap of* $\mathscr{L}_\varepsilon$. *The constant* $C > 0$ *depends also on* $M(f_0)$, $E(f_0)$, $S_0$, $\gamma$ *<u>but not on</u>* $\varepsilon$.

# Comments

- All the obtained estimates are uniform in terms of $\varepsilon$ and allows to recover the results known for Landau equation
- Actually, the only place in which we use the knowledge of the result for Landau operator is the final step which used the estimate of Desvillettes-Villani for $\mathscr{D}_0(f)$.
- It is actually possible to adapt the proof of Desvillettes-Villani for Maxwell-molecules to provide a *direct* estimate

$$\mathscr{D}_\varepsilon(f) \geqslant \lambda_\varepsilon \mathcal{H}_\varepsilon(f|\mathcal{M}_\varepsilon)$$

with $\lambda_\varepsilon > 0$ as soon as $\varepsilon \in (0, \varepsilon_1)$ (Alonso, Bagland, L. work in progress).

# Perspectives

- Investigate the link between the entropy dissipation for Landau-Fermi-Dirac and that of Boltzmann-Fermi-Dirac (link with Cercignani's conjecture for Boltzmann-Fermi-Dirac).
- Landau-Fermi-Dirac for Coulomb interactions.