Quantitative regularity for Parabolic De Giorgi classes

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Historical overview : Hölder continuity results

Equations with **bounded merely measurable** coefficients :

1. Elliptic equation : $-\nabla \cdot A(x)\nabla u = 0$

   - [1957, De Giorgi] 19th Hilbert problem : **Analiticity** of

     Local minimizers of $\quad \mathcal{E}(w) = \int_\Omega F(\nabla w)\mathrm{d}x$

     Euler-Lagrange equation $\quad -\nabla \cdot DF(\nabla w) = 0$

     Derive EL wrt ∂_{x_i} $\quad -\nabla \cdot D^2F(\nabla w)\nabla(\partial_{x_i} w) = 0$

     Hölder Continuity $\quad -\nabla \cdot A(x)\nabla u = 0$

2. Parabolic equation : $\partial_t u - \nabla_x \cdot A(t,x)\nabla_x u = 0$

   - [1958, Nash] : Independently, elliptic and parabolic equations
   - [1960, Moser] : New approach

De Giorgi classes :

"Functions satisfying energy estimates which contain enough info"

Elliptic and Parabolic De Giorgi classes :

- [1957, De Giorgi]
- [1967, Ladyzenskaja, Solonnikov, Uralceva]
- [1991, DiBenedetto]

Motivation :

- Allows to deal with less regular A
- For example : Triangular SKT cross-diffusion system

Historical overview

Question : Quantitative Hölder regularity with De Giorgi method ?

	Equation	DG classes
Elliptic	Yes DG	Yes DG
Parabolic	No Vasseur 2016	Not written ?
Kinetic Fokker-Planck	No GIMV 2017	Exists ?

$\rightarrow$ No because non-quantitative step : Intermediate value lemma

Result :
Quantitative parabolic intermediate value lemma

Outline

Outline

Parabolic equations et assumptions

Equation : $u(t,x)$ solution of

$$\partial_t u = \nabla_x \cdot (A\nabla_x u), \quad t \in (T_1, T_2), x \in \Omega,$$

Assumptions :

- Ω bounded open set of $\mathbb{R}^d$
- A measurable function of (t,x)
- $0 < \lambda I \leq A \leq \Lambda I$ (ellipticity condition)

Solution in weak sense : $Q = (T_1, T_2) \times \Omega$

- $\forall \varphi \in C_c^\infty(Q)$,

$$-\int_Q u\partial_t \varphi + \int_Q A\nabla_x u \cdot \nabla_x \varphi = 0.$$

Parabolic equations et assumptions

Equation : $u(t,x)$ solution of

$$\partial_t u = \nabla_x \cdot (A \nabla_x u) + B \cdot \nabla_x u + s, \quad t \in (T_1, T_2), x \in \Omega,$$

Assumptions :

- Ω bounded open set of $\mathbb{R}^d$
- A, B et s measurable functions of (t,x)
- $0 < \lambda I \leq A \leq \Lambda I$ (ellipticity condition)
- $|B| \leq \Lambda$, $s \in L^q$ with $q > \frac{d+2}{2}$

Solution in weak sense : $Q = (T_1, T_2) \times \Omega$

- $\forall \varphi \in C_c^\infty(Q)$,

$$-\int_Q u \partial_t \varphi + \int_Q A \nabla_x u \cdot \nabla_x \varphi - \int_Q B \cdot \nabla_x u \varphi - \int_Q s\varphi = 0.$$

Parabolic De Giorgi classes

"Local energy estimates for $(u-k)_\pm$"

$\mathrm{DG}_\pm^{\gamma_1,\gamma_2}$:

$\forall k \in \mathbb{R}, \quad \forall T_1 \leq s < t \leq T_2, \quad \forall 0 < r < R, \quad \forall x_0$ st $B_R(x_0) \subset \Omega$

$$\int_{B_r(x_0)} (u-k)_\pm^2(t,x)\mathrm{d}x + \gamma_1 \int_s^t \int_{B_r(x_0)} |\nabla_x (u-k)_\pm(\tau,x)|^2 \mathrm{d}x\mathrm{d}\tau$$
$$\leq \int_{B_R(x_0)} (u-k)_\pm^2(s,x)\mathrm{d}x + \frac{\gamma_2}{(R-r)^2} \int_s^t \int_{B_R(x_0)} (u-k)_\pm^2 \mathrm{d}x\mathrm{d}s$$

Remark : {weak sol.} $\subset DG_+ \cap DG_- \subset$ {Hölder continuous fct}

Hölder regularity for solutions

Parabolic cylinder : $Q_r = (-r^2, 0) \times B_r$

Theorem (Interior Hölder regularity)

$$u : Q_2 \to \mathbb{R} \textit{ weak solution } \Rightarrow \begin{cases} u \in C^\alpha(Q_1) \\ \|u\|_{C^\alpha(Q_1)} \leq C\|u\|_{L^2(Q_2)} \end{cases}$$

Remarks :

- α, $C > 0$ universal (d, λ, Λ)
- By rescaling, result still true for $Q' = (s, T_2) \times \Omega'$ and $Q = (T_1, T_2) \times \Omega$ for $T_1 < s < T_2$ and $\Omega' \subset\subset \Omega$.

Hölder regularity for DG classes

Parabolic cylinder : $Q_r = (-r^2, 0) \times B_r$

Theorem (Interior Hölder regularity)

$$u : Q_2 \to \mathbb{R} \ \in DG_+ \cap DG_- \Rightarrow \begin{cases} u \in C^\alpha(Q_1) \\ \|u\|_{C^\alpha(Q_1)} \leq C\|u\|_{L^2(Q_2)} \end{cases}$$

Remarks :

- α, $C > 0$ universal (d, γ_1, γ_2)
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Outline

Reduction of the theorem

Oscillation of u in E :

$$\operatorname*{osc}_E u = \sup_E u - \inf_E u$$

Step 1 : $\displaystyle \operatorname*{\mathbf{osc}}_{\mathbf{Q_r(t_0,x_0)}} \mathbf{u} \leq \mathbf{C r^\alpha \|u\|_{L^2(Q_2)}} \quad \forall (t_0, x_0) \in Q_1 \quad \forall r \in (0, \tfrac{1}{2})$

Use of sol : $u_n(y) = u\left(t_0 + \frac{s}{4^n}, x_0 + \frac{y}{2^n}\right)$ still in $DG_+ \cap DG_-$

Step 2 : **Decrease of the oscillations**

Use of sol : $-u$ still in $DG_+ \cap DG_-$

Step 3 : **Lowering of the maximum :**

"u ≤ 1 with enough mass below 0 $\Rightarrow u \leq 1 - \mu$ *in small cylinder"*

Use of sol : $u_n = 2(u_{n-1} - 1/2)$ still in $DG_+ \cap DG_-$

Step 3 $\Rightarrow$ Step 2 $\Rightarrow$ Step 1 $\Rightarrow$ Theorem

Two main lemmas

First Lemma of De Giorgi

Lemma ($L^2 - L^\infty$ estimate)

There exists $\delta > 0$ st for all $u : Q_2 \to \mathbb{R} \in DG_+$,

$$\|u_+\|_{L^\infty(Q_{1/2})} \leq \delta \|u_+\|_{L^2(Q_1)}$$

Two main lemmas

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Idea of the proof : Construction of a sequence

- $U_k = \int_{Q_k} (u - c_k)_+^2$ such that $c_k : 0 \to \frac{1}{2}$ and $Q_k : Q_1 \to Q_{\frac{1}{2}}$
- Use of DG classes and functional inequalities :

$$U_{k+1} \leq C^k U_k^\alpha \text{ with } \alpha > 1$$

- For U_0 small $U_k \to 0$ when $k \to +\infty$

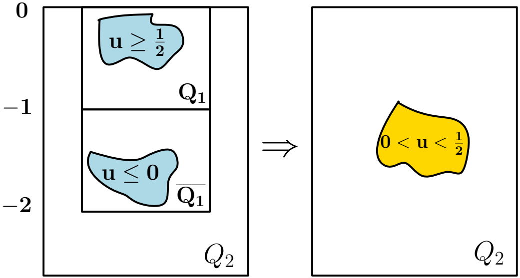
Two main lemmas

Second Lemma of De Giorgi

Lemma (Intermediate value lemma (G. 2020 version))

If $u : Q_2 \to \mathbb{R} \in DG_+$ *st* $u \leq 1$ *then*

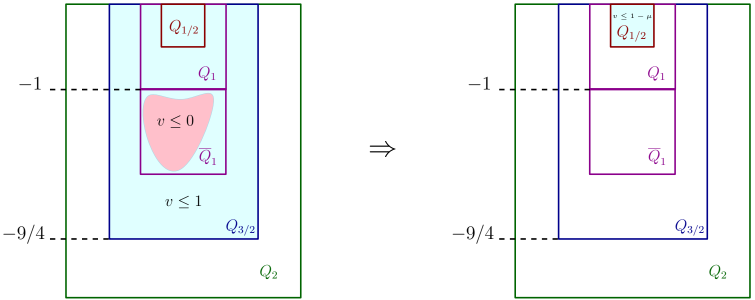
$$|\{u \leq 0\} \cap \overline{Q}_1| |\{u \geq \frac{1}{2}\} \cap Q_1| \leq C |\{0 < u < \frac{1}{2}\} \cap Q_2|^{\frac{1}{6}}$$



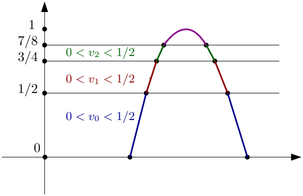
Lemma (Lowering of the maximum)

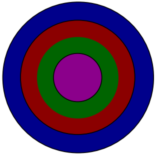
There exists $\mu \in (0,1)$ st for all $v : Q_2 \to \mathbb{R} \in DG_+$,

$$\begin{cases} v \leq 1 \text{ in } Q_{3/2} \\ |\{v \leq 0\} \cap \overline{Q}_1| \geq \frac{|\overline{Q}_1|}{2}, \end{cases} \Rightarrow v \leq 1 - \mu \quad \text{in } Q_{1/2}. \tag{1}$$



- Function in DG_+ : $\begin{cases} v_0 = v \\ v_k = 2\left(v_{k-1} - \frac{1}{2}\right) \end{cases}$
- v_k satisfies assumptions of the lemma
- $A_k = \{0 < v_k < \frac{1}{2}\} = \{1 - \frac{1}{2^k} < v < 1 - \frac{1}{2^{k+1}}\}$ are disjoint





Idea of the proof with De Giorgi lemmas

Lemma ($L^2 - L^\infty$ estimate)

There exists $\delta > 0$ st for all $u : Q_2 \to \mathbb{R} \in DG_+$,

$$\int_{Q_1} u_+^2 \leq \delta \quad \Rightarrow \quad u_+ \leq \frac{1}{2} \quad \textit{in } Q_{\frac{1}{2}}.$$

- 1st case : $\exists k_0$ st $\int_{Q_1} (v_{k_0})_+^2 \leq \delta$

$$\Rightarrow (v_{k_0})_+ \leq \frac{1}{2} \text{ in } Q_{1/2} \quad \Rightarrow v \leq 1 - \frac{1}{2^{k_0+1}} \text{ in } Q_{1/2}$$

- 2nd case : $\forall k$ st $\int_{Q_1} (v_k)_+^2 > \delta$

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If $u : Q_2 \to \mathbb{R} \in DG_+$ *st* $u \leq 1$ *then*

$$|\{u \leq 0\} \cap \overline{Q}_1||\{u \geq \frac{1}{2}\} \cap Q_1| \leq C|\{0 < u < \frac{1}{2}\} \cap Q_2|^{\frac{1}{6}}$$

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- $|\{v_k \geq \frac{1}{2}\} \cap Q_1| = |\{v_{k+1} \geq 0\} \cap Q_1| \geq \int_{Q_1} (v_{k+1})_+^2 > \delta$

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- $|\{v_k \leq 0\} \cap \overline{Q}_1| \geq \frac{|\overline{Q}_1|}{2}$

$$\Rightarrow |\{0 < v_k < \frac{1}{2}\} \cap Q_2| \geq \gamma$$

- 2nd case : $\forall k$ st $\int_{Q_1} (v_k)_+^2 > \delta$

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- $|\{v_k \leq 0\} \cap \overline{Q}_1| \geq \frac{|\overline{Q}_1|}{2}$

$$\Rightarrow |\{0 < v_k < \frac{1}{2}\} \cap Q_2| \geq \gamma \quad \textbf{Contradiction !}$$

Outline

Second Lemma of De Giorgi

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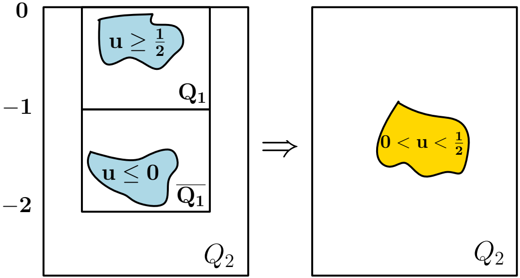

No Jump in time and space

De Giorgi classes DG_+

$$\int_{B_r(x_0)} (u-k)_+^2(t,x)\mathrm{d}x + \gamma_1 \int_s^t \int_{B_r(x_0)} |\nabla_x (u-k)_+(\tau,x)|^2 \mathrm{d}x\mathrm{d}\tau$$
$$\leq \int_{B_R(x_0)} (u-k)_+^2(s,x)\mathrm{d}x + \frac{\gamma_2}{(R-r)^2} \int_s^t \int_{B_R(x_0)} (u-k)_+^2 \mathrm{d}x\mathrm{d}s$$

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1. $(2) \leq (3) + (4)$ no jump in $x \Rightarrow$ stationnary IVL

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1. $(2) \leq (3) + (4)$ no jump in x $\Rightarrow$ stationnary IVL
2. $(1) \leq (3) + (4)$ no increasing jump in t

No Jump in time and space

De Giorgi classes DG_+

$$\int_{B_r(x_0)}(u-k)_+^2(t,x)\mathrm{d}x+\gamma_1\int_s^t\int_{B_r(x_0)}|\nabla_x(u-k)_+(\tau,x)|^2\mathrm{d}x\mathrm{d}\tau$$
$$\leq\int_{B_R(x_0)}(u-k)_+^2(s,x)\mathrm{d}x+\frac{\gamma_2}{(R-r)^2}\int_s^t\int_{B_R(x_0)}(u-k)_+^2\mathrm{d}x\mathrm{d}s$$

1. (2) $\leq$ (3) + (4) no jump in x $\Rightarrow$ stationnary IVL
2. (1) $\leq$ (3) + (4) no increasing jump in t
3. 1+2 $\Rightarrow$ Parabolic IVL

Lemma (Elliptic IVL)

If $u : B_1 \to \mathbb{R} \in H_1(B_1)$, *then*

$$|\{u \leq 0\} \cap B_1||\{u \geq \frac{1}{2}\} \cap B_1| \leq C\|\nabla u_+\|_{L^2(B_1)}|\{0 < u < \frac{1}{2}\} \cap B_1|^{\frac{1}{2}}$$

Proof

$$v(x) = \begin{cases} 0 & \text{if } u(x) \leq 0 \\ u(x) & \text{if } 0 < u(x) < \frac{1}{2} \\ \frac{1}{2} & \text{if } u(x) \geq \frac{1}{2} \end{cases} \in W^{1,1}(B_1)$$

Poincaré inequality :

$$\int_{B_1} |v - \frac{1}{|B_1|} \int_{B_1} v| \leq \int_{B_1} |\nabla v|$$

Stationnary IVL

$$
\begin{aligned}
\int_{B_1} \left|v - \frac{1}{|B_1|}\int_{B_1} v\right| &\geq \int_{\{u\geq\frac{1}{2}\}\cap B_1} \left|v - \frac{1}{|B_1|}\int_{B_1} v\right| \\
&\geq \left|\{u \geq \tfrac{1}{2}\} \cap B_1\right| \left(\tfrac{1}{2} - \tfrac{1}{|B_1|}\textstyle\int_{B_1} v\right) \\
&\geq \left|\{u \geq \tfrac{1}{2}\} \cap B_1\right| \tfrac{1}{|B_1|}\textstyle\int_{B_1} \left(\tfrac{1}{2} - v\right) \\
&\geq \left|\{u \geq \tfrac{1}{2}\} \cap B_1\right| \tfrac{1}{|B_1|}\textstyle\int_{\{u\leq 0\}\cap B_1} \left(\tfrac{1}{2} - v\right) \\
&\geq C\left|\{u \geq \tfrac{1}{2}\} \cap B_1\right|\left|\{u \leq 0\} \cap B_1\right|
\end{aligned}
$$

$$
\begin{aligned}
\int_{B_1} |\nabla v| &\leq \textstyle\int_{\{0<u<\frac{1}{2}\}\cap B_1} |\nabla u_+| \\
\text{Cauchy-Schwarz} &\leq \|\nabla u_+\|_{L^2(B_1)}\left|\{0 < u < \tfrac{1}{2}\} \cap B_1\right|^{\frac{1}{2}}
\end{aligned}
$$

Proof of Parabolic IVL

Lemma (A key inequality)

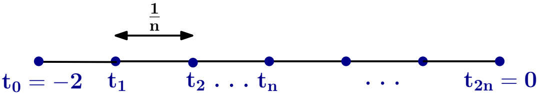
If $v : Q_2 \to \mathbb{R} \in DG_+$ *st* $v \leq 1$ *then* $\forall t_1 < t_2 < t_3 \in [-2, 0]$,

$$|\{v \leq 0\} \cap (t_1, t_2) \times B_1||\{v \geq \frac{1}{2}\} \cap (t_2, t_3) \times B_1|$$
$$\leq C|\{0 < v < \frac{1}{2}\} \cap Q_2|^{\frac{1}{2}} + C(t_3 - t_1)^3$$

Remarks :

- Useful for close times
- The proof uses IVL for H^1 functions, a reverse Hölder inequality, Jensen inequalities, Cacciopoli inequalities, etc

Discretization of the time set : $t_k = \frac{k}{n} - 2$ for $k \in [\![0, 2n]\!]$

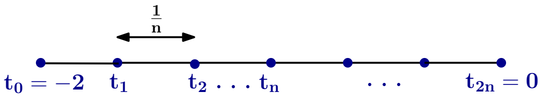


Necessarily by the pigeonhole principle,

1. $\exists i \in [\![1, n]\!]$ st $|v \leq 0,\ (t_{i-1}, t_i)| \geq \frac{|v \leq 0,\ (-2,-1)|}{n}$
2. $\exists j \in [\![n, 2n-1]\!]$ st $|v \geq \frac{1}{2},\ (t_j, t_{j+1})| \geq \frac{|v \geq \frac{1}{2},\ (-1,0)|}{n}$

We need adjacent sets !

Discretization of the time set : $t_k = \frac{k}{n} - 2$ for $k \in [\![0, 2n]\!]$



Necessarily,

1. $\exists i \in [\![1, n]\!]$ st $|v < \frac{1}{2},\ (t_{i-1}, t_i)| \geq \frac{|v \leq 0,\ (-2,-1)|}{2n}$ $\quad (a)$
2. $\exists j \in [\![n, 2n-1]\!]$ st $|v \geq \frac{1}{2},\ (t_j, t_{j+1})| \geq \frac{|v \geq \frac{1}{2},\ (-1,0)|}{2n}$ $\quad (b)$

In this case : $\exists p \in [\![1, 2n-1]\!]$ st p satisfies (a) and (b)

Since $|\{v \leq 0\}| = |\{v < \frac{1}{2}\}| - |\{0 < v < \frac{1}{2}\}|$, we have also

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$$|v < \frac{1}{2},\ (t_1, t_2)||v \geq \frac{1}{2},\ (t_2, t_3)| \\ \leq C|0 < v < \frac{1}{2},\ (-2, 0)|^{\frac{1}{2}} + C(t_3 - t_1)^3$$

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Conclusion :

$$\frac{|v\le 0,\ (-2,-1)|}{2n}\frac{|v\ge\frac{1}{2},\ (-1,0)|}{2n} \le \left|v < \frac{1}{2},\ (t_{p-1},t_p)\right|\left|v \ge \frac{1}{2},\ (t_p,t_{p+1})\right|$$

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$$\le C|0 < v < \frac{1}{2},\ (-2, 0)|^{\frac{1}{2}} + C(t_3 - t_1)^3$$

Conclusion :

$$\begin{aligned} \frac{|v \le 0,\ (-2,-1)|}{2n} \frac{|v \ge \frac{1}{2},\ (-1,0)|}{2n} &\le \left|v < \tfrac{1}{2},\ (t_{p-1}, t_p)\right| \left|v \ge \tfrac{1}{2},\ (t_p, t_{p+1})\right| \\ &\le C|0 < v < \tfrac{1}{2},\ (-2, 0)|^{\frac{1}{2}} + \frac{C}{n^3} \end{aligned}$$

Since $|\{v \le 0\}| = |\{v < \frac{1}{2}\}| - |\{0 < v < \frac{1}{2}\}|$, we have also

Lemma (A key inequality)

If $v : Q_2 \to \mathbb{R} \in DG_+$ st $v \le 1$ then $\forall t_1 < t_2 < t_3 \in (-2,0)$,

$$\begin{aligned} &|v < \frac{1}{2},\ (t_1,t_2)||v \ge \frac{1}{2},\ (t_2,t_3)| \\ &\qquad \le C|0 < v < \frac{1}{2},\ (-2,0)|^{\frac{1}{2}} + C(t_3 - t_1)^3 \end{aligned}$$

Conclusion :

$$\begin{aligned} \frac{|v \le 0,\ (-2,-1)|}{2n} \frac{|v \ge \frac{1}{2},\ (-1,0)|}{2n} &\le \left|v < \tfrac{1}{2},\ (t_{p-1},t_p)\right| \left|v \ge \tfrac{1}{2},\ (t_p,t_{p+1})\right| \\ &\le C|0 < v < \tfrac{1}{2},\ (-2,0)|^{\frac{1}{2}} + \frac{C}{n^3} \end{aligned}$$

$$\left|v \le 0,\ (-2,-1)\right| \left|v \ge \tfrac{1}{2},\ (-1,0)\right| \le Cn^2|0 < v < \tfrac{1}{2},\ (-2,0)|^{\frac{1}{2}} + \frac{C}{n}$$

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Lemma (A key inequality)

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Conclusion :

$$\frac{|v \leq 0,\ (-2,-1)|}{2n} \frac{|v \geq \frac{1}{2},\ (-1,0)|}{2n} \leq \left|v < \frac{1}{2},\ (t_{p-1}, t_p)\right| \left|v \geq \frac{1}{2},\ (t_p, t_{p+1})\right| \leq C|0 < v < \frac{1}{2},\ (-2, 0)|^{\frac{1}{2}} + \frac{C}{n^3}$$

$$\left|v \leq 0,\ (-2, -1)\right| \left|v \geq \frac{1}{2},\ (-1, 0)\right| \leq Cn^2|0 < v < \frac{1}{2},\ (-2, 0)|^{\frac{1}{2}} + \frac{C}{n}$$

$\Rightarrow$ **Optimising in n gives IVL**

Thanks for your attention !