Central WENO reconstructions

from their origin to the most recent developments and applications

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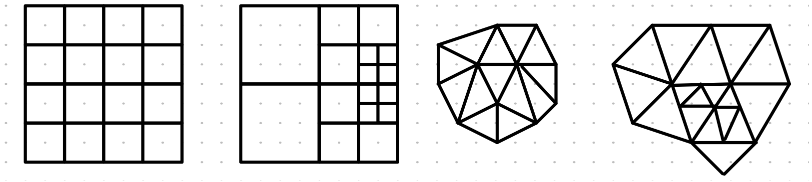
Goal

$$\frac{\partial \mathbf{u}}{\partial t} + \nabla_{\mathbf{x}} \cdot \mathbf{f}(\mathbf{u}) = \mathbf{s}(\mathbf{u})$$

- High order accurate finite volume schemes for hyperbolic conservation (and balance) laws
- multidimensional case
- little restrictions from grid type (structured, unstructured, locally adapted as in quad-tree, etc)



Outline

High order FV schemes

The CWENO(Z) paradigm

Uniform grid reconstructions

Unstructured, AMR and well-balanced computations

Time advancement in finite volume schemes

- Cell averages

$$\overline{u}_j(t) = \frac{1}{|\Omega_j|} \int_{\Omega_j} u(t,x) \mathrm{d}x$$

- Method of lines: compute the cell average of the PDE on each Ω_j and get the coupled system of ODEs

$$\frac{\mathrm{d}}{\mathrm{d}t}\overline{u}_j(t) = \mathbf{L}_j(\overline{u}_\star(t))$$

where $\mathbf{L}_j$ is the spatial discretization

- ADER schemes: integrate in $\Omega_j \times [t_n, t_{n+1}]$ to obtain

$$\overline{u}_j^{n+1} = \overline{u}_j^{n+1} + \mathrm{K}_j \left(u_\star(t^n, x) \right)$$

with K_j depending on a local high order representation of the solution at time t^n

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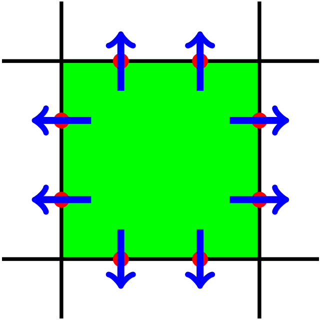
High-order flux integration

fluxes = Quadrature ∘ R. Solver ∘ Reconstruction

$$\int_{\Omega_j} \nabla \cdot \mathbf{f}(u(t,x))\mathrm{d}x = \int_{\partial\Omega_j} \mathbf{f}(u(t,\gamma)) \cdot \mathbf{n}(\gamma)\mathrm{d}\gamma = \sum_k \int_{I_{jk}} \mathbf{f}(u(t,\gamma)) \cdot \mathbf{n}(\gamma)\mathrm{d}\gamma$$

where I_{jk} is the intersection of Ω_j and the neighbour Ω_k.

$$\int_{I_{jk}} \mathbf{f}(u(t,\gamma)) \cdot \mathbf{n}(\gamma)\mathrm{d}\gamma$$
$$= |I_{jk}| \sum_q w_q \mathbf{f}(u(t,x_q)) \cdot \mathbf{n}(x_q)$$
$$= |I_{jk}| \sum_q w_q \mathcal{F}\Big(u(t,x_q)^{\text{in}}, u(t,x_q)^{\text{out}}; \mathbf{n}(x_q)\Big)$$



where $\mathcal{F}$ is a compatible Riemann Solver and $u^{\text{in/out}}$ denote suitable point value reconstructions.

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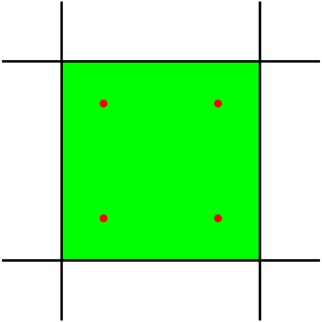
High-order source terms

sources = Quadrature ∘ Reconstruction

- use quadrature rule

$$\int_{\Omega_j} \mathbf{s}(u(t,x))\mathrm{d}x$$
$$= |\Omega_j| \sum_q w_q \mathbf{s}(u(t, x_q))$$

⇒ need also inner point value reconstructions



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Point-value reconstructions

Finite volume schemes

- store cell averages
- need point-value reconstructions
  (possibly at very many locations in each cell)

p-wise const → p-wise $\mathbb{P}_G$

cell averages in Ω_j and neighbours —RECONSTRUCTION→ $R_j(x)\ \forall x \in \Omega_j$

ENO	very large stencils
WENO	need to know the evaluation point beforehand
CWENO	$\to$ this talk
MOOD	$\to$ comparison in *M.S., R. Loubere – JCP, 2018*

High order FV schemes

The CWENO(Z) paradigm

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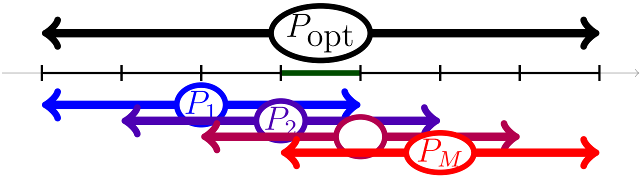
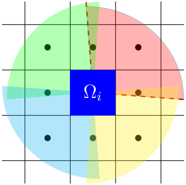
Essentially non-oscillatory reconstructions

Given the cell averages $\overline{u}_i$, $i \in \mathcal{S}_{\text{opt}}$, for a bounded function $u(x)$,





P_{opt} s.t. $\forall i \in \mathcal{S}_{\text{opt}}$:
$$\frac{1}{|\Omega_i|}\int_{\Omega_i} P_{\text{opt}}(x)\mathrm{d}x = \overline{u}_i$$

1. has very good accuracy in smooth regions
2. is (wildly) oscillatory if a discontinuity is present in its stencil
3. is best replaced by a (lower accuracy) non-oscillatory alternative, e.g. one of the P_k's defined on substencils

The CWENO master equation

Given $P_{\text{opt}} \in \mathbb{P}_G$ and $P_1, \dots, P_M \in \mathbb{P}_g$, $g < G$

freely choose $d_0, \dots, d_M \in (0,1)$ such that $\sum_0^M d_k = 1$

and set $P_0(x) = \frac{1}{d_0}\left(P_{\text{opt}}(x) - \sum_{k=1}^M d_k P_k(x)\right)$

$$\forall x \in \text{cell}: \; P_{\text{opt}}(x) = d_0 \, P_0(x) + \sum_{k=1}^{M} d_k P_k(x) \qquad \text{(linear)}$$

nonlinear weights

$$\omega_i = \frac{\alpha_i}{\sum \alpha_k}$$

$$\alpha_i = \frac{d_i}{(\mathrm{OSC}[P_i] + \varepsilon)^\ell}$$

$$\forall x \in \text{cell}: \; R(x) = \omega_0 \, P_0(x) + \sum_{k=1}^{M} \omega_k P_k(x) \qquad \text{(nonlinear)}$$

Levy, Puppo, Russo SISC, 2000

"Essentially non-oscillatory" property

$$\text{OSC}[P] := \sum_{k\geq 1} \Delta x^{2k-1} \int (d^k P/\mathrm{d}x^k)^2$$

smooth data $\text{OSC}[P] = (u')^2 \Delta x^2 + \text{lower order terms}$

discontinuous data $\text{OSC}[P] \asymp 1$

1. **assume that at least one candidate polynomial has a smooth stencil**, that is $\text{OSC}[P_{\hat{k}}] \ll 1$
2. if P_k is oscillatory, then $\text{OSC}[P_k] \asymp 1$
3. the computation of the nonlinear weights

$$\alpha_k = \frac{d_k}{(\text{OSC}[P_k] + \epsilon)^\ell} \qquad \omega_k = \frac{\alpha_k}{\sum_j \alpha_j}$$

yields $\alpha_{\hat{k}} \gg \alpha_k$ and, after the renormalization,

$$\boldsymbol{\omega_{\hat{k}} \approx 1 \quad \text{while} \quad \forall k \neq \hat{k} : \omega_k \approx 0}$$

so that $R \approx P_{\hat{k}}$ which is not oscillatory.

Accuracy of WENO-like reconstructions

Assume that the stencils are chosen such that

$$|P_{\text{opt}}(\vec{x}) - u(\vec{x})| = \mathcal{O}(\Delta x^{G+1}) \quad \text{and} \quad |P_k(\vec{x}) - u(\vec{x})| = \mathcal{O}(\Delta x^{g+1})$$

$$\text{CWENO}(P_{\text{opt}}; P_1, \ldots, P_M) = \sum_{k=0}^{m} \omega_k P_k \neq \sum_{k=0}^{m} d_k P_k = P_{\text{opt}}$$

The reconstruction error can be written as

$$\underbrace{u(\vec{x}) - R(\vec{x})}_{=\mathcal{O}(\Delta x^{G+1})?} = \underbrace{(u(\vec{x}) - P_{\text{opt}}(\vec{x}))}_{\mathcal{O}(\Delta x^{G+1})} + \sum_{k=0}^{m} (d_k - \omega_k) \underbrace{(P_k(\vec{x}) - u(\vec{x}))}_{\mathcal{O}(\Delta x^{g+1})}$$

Accuracy on smooth data depends on

$$d_k - \omega_k = \mathcal{O}(\Delta x^{G-g})$$

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Comparison with WENO

$$\text{given } \hat{x} \in \text{cell}: \; P_{\text{opt}}(\hat{x}) = \sum_{k=1}^{M} d_k(\hat{x}) P_k(\hat{x}) \qquad \text{(WENO)}$$

$$\forall x \in \text{cell}: \; P_{\text{opt}}(x) = d_0 \, P_0(x) + \sum_{k=1}^{M} d_k P_k(x) \qquad \text{(CWENO)}$$

In CWENO:

- ✓ d_k need not be x-dependent
- ✓ d_k always exist (trivially)
- ✓ d_k can be chosen independently of the mesh
- ✓ compute ω_k once per cell, not once per reconstruction point
- ✗ P_{opt} must be explicitly computed

Cravero, Puppo, M.S., Visconti Math. Comp. (2018)

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CWENOZ reconstruction

$$\forall x \in \text{cell}: \; {\color{red}P_{\text{opt}}(x)} = {\color{blue}d_0}\; P_0(x) + \sum_{k=1}^{M} {\color{blue}d_k} {\color{red}P_k(x)} \qquad \text{(linear)}$$

~~$$\omega_i = \frac{\alpha_i}{\sum \alpha_k} \qquad \alpha_i = \frac{d_i}{(\text{OSC}[P_i] + \varepsilon)^{\ell}}$$~~

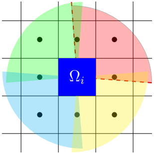
$$\boxed{\omega_i = \frac{\alpha_i}{\sum \alpha_k} \qquad \alpha_i = d_i \left[1 + \left(\frac{{\color{red}\tau}}{\text{OSC}[P_i] + \varepsilon} \right)^{\ell} \right]}$$

$$\forall x \in \text{cell}: \; {\color{red}R(x)} = {\color{red}\omega_0}\; P_0(x) + \sum_{k=1}^{M} {\color{red}\omega_k} P_k(x) \qquad \text{(nonlinear)}$$

In CWENOZ: $\tau = {\color{red}\lambda_0 \text{OSC}[P_{\text{opt}}]} + \sum_{k=1}^{M} \lambda_k \text{OSC}[P_k]$

Cravero, M.S., Visconti SINUM (2019)

Example



- $P_{\text{opt}} \in \mathbb{P}_2(x,y)$ on the central 3×3 stencil
- $P_{NE} \in \mathbb{P}_1(x,y)$ on the 2×2 North-East sub-stencil
- $P_{SE}, P_{NW}, P_{SW} \in \mathbb{P}_1(x,y)$ similarly on 2×2 sub-stencils

On a 2d uniform Cartesian grid with $\Delta x = \Delta y = h$,

$$\mathrm{OSC}[P_{NE}^{(1)}] = (u_x^2 + u_y^2)h^2 + \left(+u_x u_{xx} + \tfrac{2}{3}u_x u_{xy} + \tfrac{2}{3}u_y u_{xy} + u_y u_{yy}\right)h^3 + \mathcal{O}(h^4)$$

$$\mathrm{OSC}[P_{NW}^{(1)}] = (u_x^2 + u_y^2)h^2 + \left(-u_x u_{xx} + \tfrac{2}{3}u_x u_{xy} - \tfrac{2}{3}u_y u_{xy} + u_y u_{yy}\right)h^3 + \mathcal{O}(h^4)$$

$$\mathrm{OSC}[P_{SE}^{(1)}] = (u_x^2 + u_y^2)h^2 + \left(+u_x u_{xx} - \tfrac{2}{3}u_x u_{xy} + \tfrac{2}{3}u_y u_{xy} - u_y u_{yy}\right)h^3 + \mathcal{O}(h^4)$$

$$\mathrm{OSC}[P_{SW}^{(1)}] = (u_x^2 + u_y^2)h^2 + \left(-u_x u_{xx} - \tfrac{2}{3}u_x u_{xy} - \tfrac{2}{3}u_y u_{xy} - u_y u_{yy}\right)h^3 + \mathcal{O}(h^4)$$

$$\mathrm{OSC}[P_{\text{opt}}] = (u_x^2 + u_y^2)h^2 + \mathcal{O}(h^4)$$

In general, $\sum_{k=0}^{m} \lambda_k = 0, \quad \Rightarrow \quad \boxed{\tau = \lambda_0 \mathrm{OSC}[P_{\text{opt}}] + \sum_{k=1}^{4} \lambda_k \mathrm{OSC}[P(1)_k] = \mathcal{O}(h^3)}$

but the symmetries allow an even better definition of τ:

$$\tau = \mathrm{OSC}[P_{NE}^{(1)}] + \mathrm{OSC}[P_{NW}^{(1)}] + \mathrm{OSC}[P_{SE}^{(1)}] + \mathrm{OSC}[P_{SW}^{(1)}] - 4\mathrm{OSC}[P_{\text{opt}}] = \mathcal{O}(h^4)$$

Taylor expansions of multidimensional OSC

Using the multi-index notation $\boldsymbol{\beta} = (\beta_1, \dots, \beta_d)$,

$$OSC[q] := \sum_{|\boldsymbol{\beta}|\geq 1} \Delta\vec{x}^{2\boldsymbol{\beta}-\mathbf{1}} \int_{\Omega_0} (\partial_{\boldsymbol{\beta}} q(\vec{x}))^2 \mathrm{d}x.$$

On smooth data, independently on the mesh:

Proposition

Let $\mathcal{S}$ be a stencil including Ω_0 and let $q(\vec{x})$ be a polynomial with $\deg q(\vec{x}) \geq g$ approximating a regular function $u(\vec{x})$, then

$$OSC[q] = \langle \vec{v}(q), C\vec{v}(q) \rangle = B_g + R[q]$$

- C depends on $\mathcal{S}$
- $R[q] = o(B_g)$
- B_g depends on g but not on $q(\vec{x})$ (and thus not on $\mathcal{S}$).

Cravero, M.S., Visconti SINUM (2019)

Accuracy results

Theorem

Assume that

- $P_1(\vec{x}), \dots, P_M(\vec{x}) \in \mathbb{P}_g$ *and* $P_{\text{opt}}(\vec{x}) \in \mathbb{P}_G$ *in the* CWENOZ *scheme*
- $g \geq G/2$

... (technical), τ-dependent, sufficient conditions on ϵ, ℓ so that, on smooth data, the CWENOZ *scheme achieves the optimal order $G+1$.*

Corollaries

- can always find ℓ, ϵ for optimal convergence
- in any case, the smaller is τ, the smaller ϵ and smaller ℓ are needed to achieve optimal convergence.

From before:

1. always take $\tau = \sum_{k\geq 0} \lambda_k \text{OSC}[P_k]$ with $\sum_k \lambda_k = 0$
2. if possible, optimize your choice of λ_k for your grid/stencils

Cravero, M.S., Visconti SINUM (2019)

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CWENOZ optimal τ in 1D (uniform grids)

Let $I_k = \text{OSC}[P_k]$ and $I_0 = \text{OSC}[P_{\text{opt}}]$:

CWENOZ3 $\forall t \in \mathbb{R}$:

$$\hat{\tau}_3 = |tI_1 + tI_2 - 2tI_0| = \mathcal{O}(\Delta x^4)$$

instead of $\tau_3 = \mathcal{O}(\Delta x^3)$ without using I_0.

CWENOZ5 $\forall t \in \mathbb{R}$:

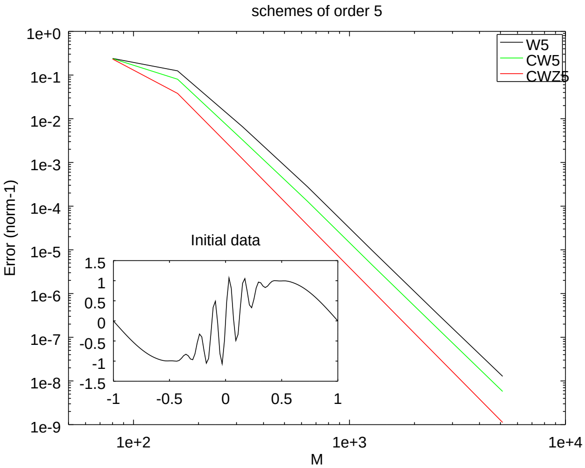
$$\hat{\tau}_5 = |tI_1 + 4tI_2 + tI_3 - 6tI_0| = \mathcal{O}(\Delta x^6)$$

instead of $\tau_5 = \mathcal{O}(\Delta x^5)$ without using I_0.

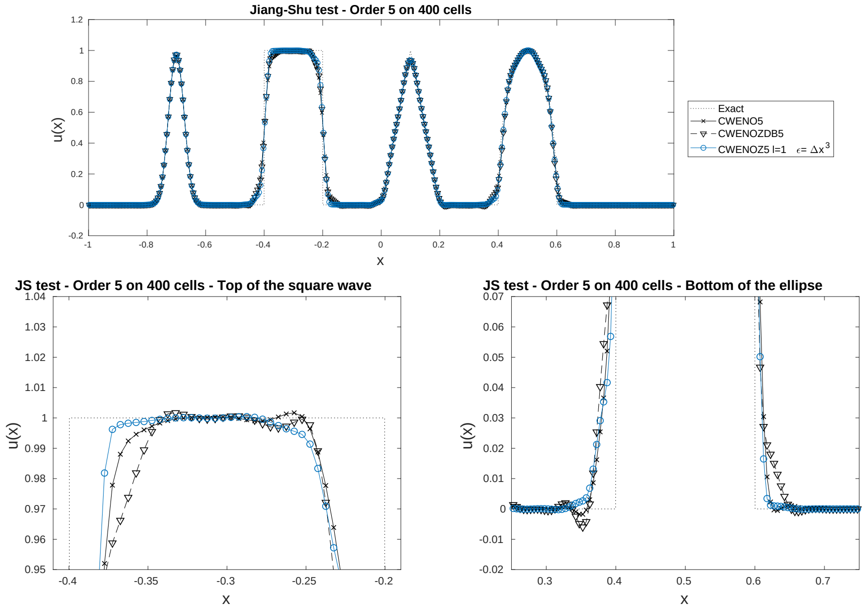
higher orders the optimal definition for WENOZ is also optimal for CWENOZ.

Is CWENOZ really better?

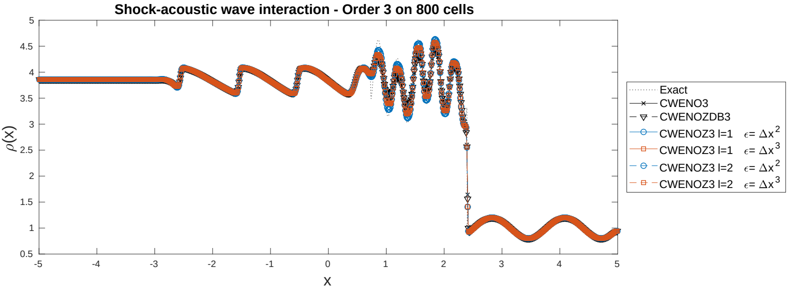
$$u_t + u_x = 0$$

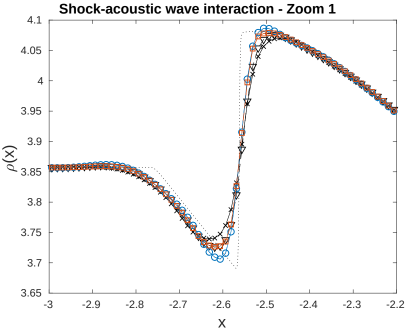


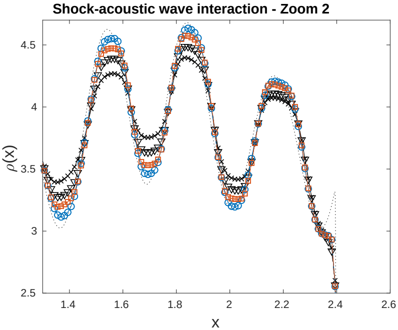
Jiang-Shu linear transport test with CWENOZ5



A 1D Euler computation







"Adaptive order" WENO

- we have been able to include, in a reconstruction with target order G, polynomials of degree at least $G/2$.
- yet, it would be beneficial to include e.g. a $\mathbb{P}_2$ in a CWENO7...

Existing "adaptive order" WENO

- 😉 are really hierarchic CWENO or hierarchic CWENOZ
- example WAO(7,5,3) by Balsara, Garain, Shu (2016)

$$\text{CWENOZ}\left(\text{CWENOZ}\left(P^{(6)}; P^{(3)}_{1,2,3,4}\right); \text{CWENOZ}\left(P^{(4)}; P^{(2)}_{1,2,3}\right)\right)$$

- ☹ Hierarchic $\Rightarrow$ multiple nonlinear weights computations

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CWENO(Z) with high degree gap

Let us consider on (or more) polynomials with very low degree:

$$\boxed{\mathrm{CWENOZ}(P_{\mathrm{opt}}; P_1, \dots, P_M; Q) = \omega_0 P_0 + \sum_{i=1}^{M} \omega_k P_k + \boldsymbol{\omega} Q}$$

where

$$P_0 = \frac{1}{d_0}\left[P_{\mathrm{opt}} - \sum_{i=1}^{M} d_i P_i - \boldsymbol{\delta} Q\right]$$

and

$$\deg P_{\mathrm{opt}} = 2g \qquad \deg P_k \geq g \qquad \deg Q = \boldsymbol{\gamma} < g$$

Accuracy on smooth data depends on

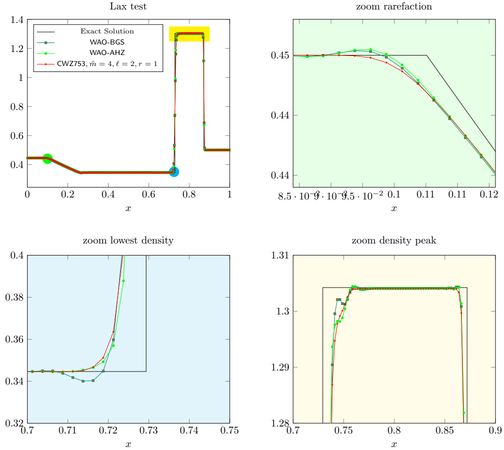
$$d_k - \omega_k = \mathcal{O}(\Delta x^{2g-g}) \qquad \delta - \omega = \mathcal{O}(\Delta x^{2g-\gamma})$$

Theorem [MS, Visconti, arXiv: 1910.03559]

Optimal convergence with one single non-linear weight computation can be achieved if $\delta = \mathcal{O}(\Delta x^{g-\gamma})$.

Lax shock tube with $\mathrm{CWZ}(7,5,3)$ and $\mathrm{WENO-AO}$

	Core i7-6600U @ 2.60GHz			Core i3-2100T @ 2.50GHz		
Cells	CWZ753	WAO753	ARBO753	CWZ753	WAO753	ARBO753
200	3.108 s	+10.15%	+16.04%	10.82 s	+9.61%	+11.67%
400	12.11 s	+13.81%	+15.31%	43 s	+9.00%	+10.32%
800	47.92 s	+13.16%	+19.70%	172.2 s	+9.22%	+9.93%



M.S., Visconti arxiv:1910.03559

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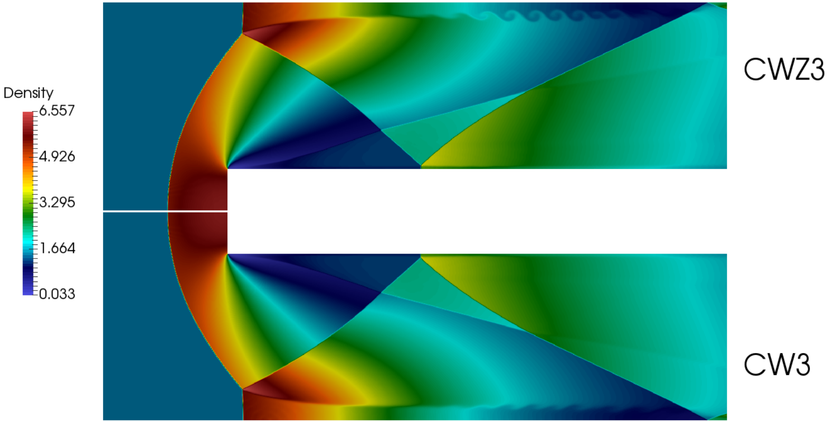
Open-source implementation

claw1dArena

- Downloadable[1] from zenodo.org
  DOI 10.5281/zenodo.2641724
- GPL licence
- Developed with numerical experimentation in mind:
  - C++ implementation with very few required libraries
  - choose conservation law at compile time
  - choose any combination of reconstruction, timestepper, numerical flux, well-balancing, discretization parameters, etc at run-time

[1] The next release will contain also the "adaptive order CWENOZ"

Forward-facing step at $t = 2.4$ with 1M dofs



Cravero, M.S., Visconti SINUM (2019)

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Attractive features of CWENOZ

- linear coefficients d_k are not accuracy-bound (and always exist)
  - ➔ easy to apply to unstructured and AMR meshes
  - ➔ no need to employ dimensional splitting
- one computation of nonlinear weights ω_k and one polynomial evaluation per reconstruction point (vs early polynomial evaluation and one ω_k computations per reconstruction point)
  - ➔ better suited if many reconstruction points per cells are employed (even on uniform grids)

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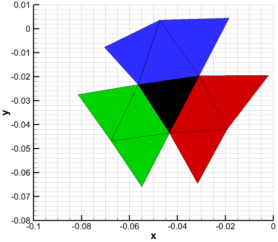
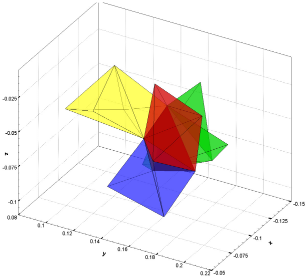
2D and 3D reconstructions for simplicial meshes

- CWENO type
- one large central stencil for a polynomial of degree ≥ 2
- three/four directionally biased stencils for $\mathbb{P}_1$ polynomials
- finite volume schemes, ALE framework
- also employed as a-posteriori subcell limiter for DG





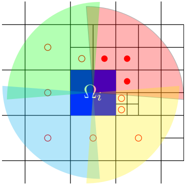
Dumbser, Boscheri, M.S., Russo J. Sci. Comput., 2017

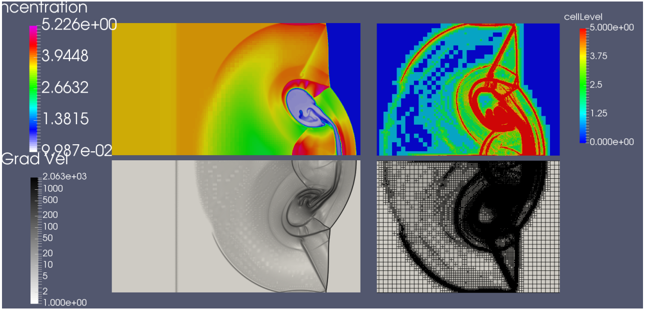
Boscheri, M.S., Dumbser Comm. Comput. Phys., 2019

CWENO3 in quad-tree AMR

- $P_{\text{opt}} \in \mathbb{P}_2(x, y)$
- $P_1, \dots, P_4 \in \mathbb{P}_1(x, y)$

with the depicted stencils





M.S., Coco, Russo J. Sci. Comput., 2016

High order accurate shallow water computations

h	water thickness
q	discharge
Z	bottom topography

$$\rightsquigarrow \begin{cases} h_t + q_x = 0 \\ q_t + (q^2/h + \frac{1}{2}gh^2)_x = -g\,hZ_x \end{cases}$$

- useful schemes are well-balanced,
  i.e. preserve the steady states at machine precision
  (or at least the still-water ones)
- the approach to well-balancing based on the hydrostatic reconstruction and the Richardson extrapolation of the trapezoidal rule requires

order	3	5	7	9
inner rec. points	1	3	7	15

Cravero, Puppo, M.S., Visconti Math. of Comp., 2018

Cravero, M.S., Visconti SINUM (2019)

Shallow water equations using CWENO(Z)

Convergence test on a smooth solution

	CW3		CWZ3		CW5		CWZ5	
N	error	rate	error	rate	error	rate	error	rate
32	9.37e-03		6.65e-03		4.01e-04		2.58e-04	
64	1.44e-03	2.70	7.48e-04	3.15	1.73e-05	4.54	9.72e-06	4.73
128	1.56e-04	3.21	6.40e-05	3.55	5.74e-07	4.91	3.15e-07	4.95
256	1.57e-05	3.31	7.17e-06	3.16	1.81e-08	4.98	9.99e-09	4.98
512	1.83e-06	3.10	8.80e-07	3.03	5.70e-10	4.99	3.13e-10	5.00
1024	2.29e-07	3.00	1.10e-07	3.00	1.79e-11	5.00	9.80e-12	5.00

	CW7		CWZ7		CW9		CWZ9	
N	error	rate	error	rate	error	rate	error	rate
16	1.30e-03		1.63e-03		7.02e-04		6.88e-04	
32	7.25e-05	4.17	6.22e-05	4.71	2.82e-05	4.64	2.38e-05	4.85
64	6.70e-07	6.76	7.44e-07	6.39	1.22e-07	7.85	1.17e-07	7.67
128	5.02e-09	7.06	6.68e-09	6.80	3.44e-10	8.47	3.15e-10	8.54
256	3.91e-11	7.00	5.37e-11	6.96	7.43e-13	8.86	6.65e-13	8.89
512	3.07e-13	6.99	4.25e-13	6.98				

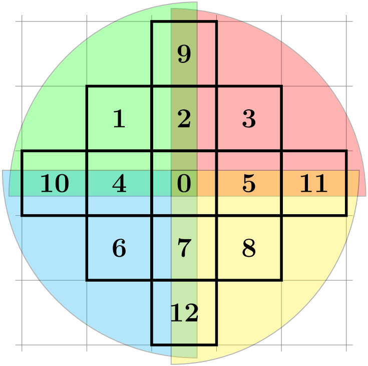
Well-balancing test (lake at rest with random bottom)

	$\|\Delta(h+z)\|_\infty$				$\|q\|_\infty$			
	N=100	N=200	N=400	N=800	N=100	N=200	N=400	N=800
CW9	8.88e-16	1.33e-15	2.22e-15	2.66e-15	7.44e-16	1.43e-15	1.82e-15	2.51e-15
CW7	1.93e-15	3.31e-15	7.13e-15	1.62e-14	2.12e-15	3.05e-15	7.15e-15	1.64e-14
CW5	1.26e-15	2.59e-15	5.33e-15	1.00e-14	1.74e-15	3.08e-15	5.32e-15	9.94e-15
CW3	7.61e-16	1.90e-15	3.27e-15	5.48e-15	1.90e-15	3.56e-15	4.78e-15	7.66e-15

See Puppo, MS - J Sci Comput (2016) for non-uniform grid examples

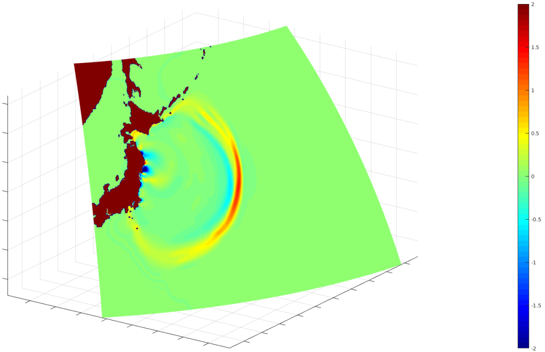
SWE: well-balancing in 2D at order 3 and 4

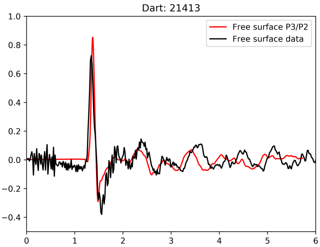
- CWENO reconstruction of order 3
- novel CWENO reconstruction of order 4 with stencils as shown:
  - ➔ $P_{\mathrm{opt}} \in \mathbb{P}_3(x, y)$
  - ➔ $P_1, \dots, P_4 \in \mathbb{P}_1(x, y)$
    or
    $P_1, \dots, P_4 \in \mathbb{P}_2(x, y)$

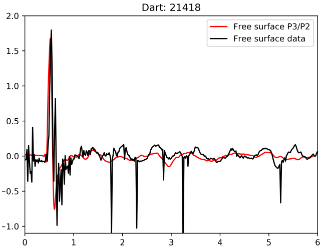


Castro, M.S. Int. J. Numer. Meth. Fluids, 2018

Tohoku tsunami simulation







Euler equation with gravity (1D and 2D)

- the well-balancing technique requires high order accurate pressure averages:

$$\overline{p}_j = \mathcal{Q}_{\Omega_j}\left[(\gamma-1)\left(E_j(x) - \tfrac{1}{2}\rho_j(x)|\vec{v}_j(x)|^2\right)\right]$$

  - ➔ $\mathcal{Q}$ is a gaussian quadrature rule
  - ➔ need total energy $E_j(x)$, density $\rho_j(x)$ and velocity $\vec{v}_j(x)$ at quadrature nodes of $\mathcal{Q}$

- special well-balanced quadrature (Gaussian $\times$) Richardson extrapolation of trapezoidal

$\Rightarrow$ very many reconstruction points per cell

$\Rightarrow$ CWENO reconstructions were employed for efficiency

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Klingenberg, Puppo, M.S. SIAM J. Sci. Comput., 2019

Summary

CWENO and CWENOZ family of reconstructions

- much more versatile than WENO
- multi-D, AMR, unstructured, many reconstruction points, ...
- may include very low order candidates to better control spurious oscillations

CWENOZ reconstructions

- better accuracy than CWENO on smooth flows
- on par with CWENO on discontinuous flows
- now we have the optimal definition of τ

Note: present results can be used to analyze all reconstructions based on the “CWENO master equation”, like the “WENO” reconstructions by Zhu, Qiu, Balsara and collaborators.

Thank you for your kind attention!

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