The mean field and diffusive limits for weakly interacting diffusions

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JOINT WORK WITH: M. G. DELGADINO (PUC-RIO), G. A. PAVLIOTIS (ICL)

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Outline

Many-particle systems

N indistinguishable interacting particles in $\Omega = \mathbb{R}^d$ etc.

$X_t^i \in \Omega$: location of the i^{th} particle, $i = 1, \ldots, N$.

X_0^i are i.i.d random variables with law $\nu_0 \in \mathcal{P}(\mathbb{R}^d)$.

$$\mathrm{d}X_t^i = -\nabla V(X_t^i) - \frac{1}{N}\sum_{i\neq j}^{N} \nabla W(X_t^i - X_t^j)\,\mathrm{d}t + \sqrt{2\beta^{-1}}\,dB_t^i\,,$$

where $V, W \in C^2(\mathbb{R}^d)$, 1-periodic, even, B_t^i independent Wiener processes.

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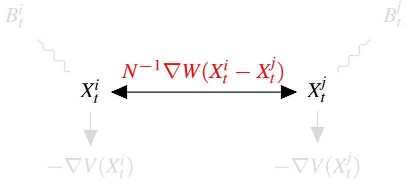

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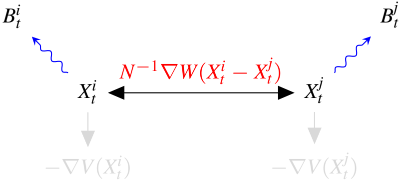

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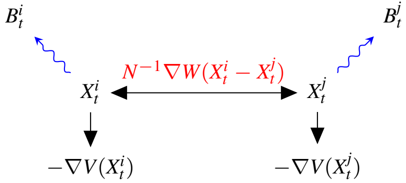

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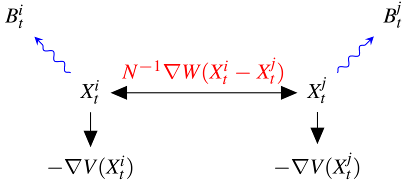

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Applications

Some applications:

- Molecules of a gas
- Opinions of individuals
- Collective motion of agents
- Particles in a granular medium
- Nonlinear synchronizing oscillators
- Liquid crystals

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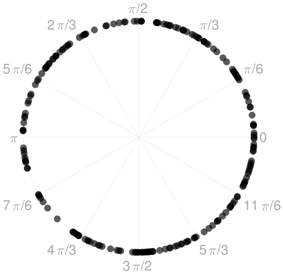
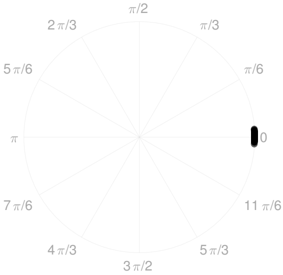


$\beta < \beta_c$, no phase locking



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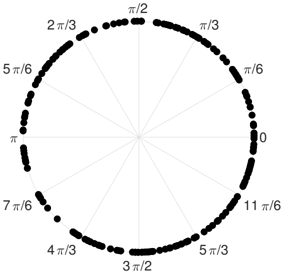

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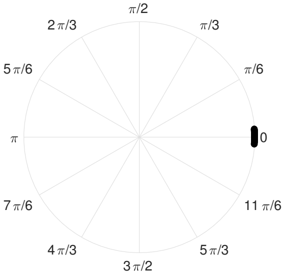

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The Fokker–Planck equation

- Hamiltonian: $H^N(x_1, \ldots, x_N) := \frac{1}{2N}\sum_{i,j} W(x_i - x_j) + \sum_i V(x_i)$
- Associated Fokker–Planck/forward Kolmogorov equation for the law $\nu^N = \mathrm{Law}(X_t^1, \ldots, X_t^N)$:

$$\begin{cases} \partial_t \nu^N &= \beta^{-1}\Delta\nu^N + \nabla\cdot(\nabla H^N \nu^N), \quad (t,x) \in (0,\infty)\times(\mathbb{R}^d)^N \\ \nu^N(0) &= \nu_0^N = \nu_0^{\otimes N} \in \mathcal{P}((\mathbb{R}^d)^N) \end{cases}$$

- Initial data i.i.d.

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Aggregate behaviour: the mean-field limit

Consider the empirical measure : $\nu^{(N)} := \frac{1}{N}\sum_{i=1}^{N} \delta_{X_t^i} \in \mathcal{P}(\mathbb{R}^d)$. Easier to study $\mathbb{E}\left[\nu^{(N)}\right]$:

Theorem (The mean-field limit/propagation of chaos)

As $N \to \infty$, $\mathbb{E}\left[\nu^{(N)}\right]$ converges in weak-$\star$ to $\nu(t, \mathrm{d}x) = \nu(t,x)\,\mathrm{d}x$, which solves (weakly):

$$\partial_t \nu = \beta^{-1}\Delta\nu + \nabla\cdot(\nu(\nabla W \star \nu + \nabla V)) \qquad \text{(McKean–Vlasov equation)}$$

with initial datum $\nu_0 \in \mathcal{P}(\mathbb{R}^d)$.

Another interpretation: $\nu^N \to \nu^{\otimes N}$ as $N \to \infty$.

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   1. Classical: McKean '66, Oelschläger '84, Gärtner '88, Sznitman '91 (coupling)
   2. Rates of convergence: Sznitman '91, Mouhot–Mischler '13, Hauray–Mischler '14, Eberle et al. '17 (coupling)
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The diffusive limit $\varepsilon \to 0$

We place ourselves in the setting:

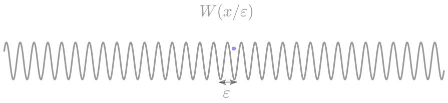
$$\mathrm{d}X_t^{i,\varepsilon} = -\nabla V(\varepsilon^{-1}X_t^{i,\varepsilon}) - \frac{1}{N}\sum_{i\neq j}^{N} \nabla W(\varepsilon^{-1}(X_t^{i,\varepsilon} - X_t^{j,\varepsilon}))\,\mathrm{d}t + \sqrt{2\beta^{-1}}\,dB_t^i$$

with W, V chosen to be 1-periodic.

Let $\rho^{\varepsilon,N} = \mathrm{Law}(X_t^{1,\varepsilon}, \ldots, X_t^{N,\varepsilon})$ and consider the diffusive rescaling

$$\rho^{\varepsilon,N}(x,t) := \varepsilon^{-Nd}\nu^N(\varepsilon^{-1}x, \varepsilon^{-2}t) \in \mathcal{P}((\mathbb{R}^d)^N)\,.$$

Interpretation: zooming out in space and going forward in time.



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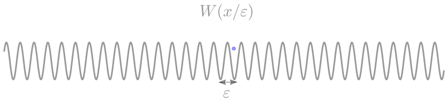

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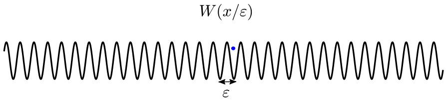

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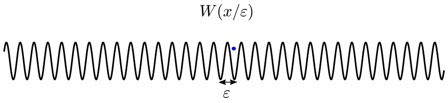

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$\dot{X}_t^i \in \mathbb{T}^d$ and $\dot{B}_t^i$ are $\mathbb{T}^d$-valued Wiener processes.

This is a reversible, ergodic, diffusion process with a unique N-particle invariant measure Gibbs measure

$$M_N(x) = \frac{e^{-H^N(x)}}{\int_{\mathbb{T}^{dN}} e^{-H^N(y)}\ \mathrm{d}y}\,,$$

and the law $\tilde{\nu}^N$ evolves according to

$$\begin{cases} \partial_t \tilde{\nu}^N &= \beta^{-1}\Delta\tilde{\nu}^N + \nabla\cdot(\nabla H^N \tilde{\nu}^N), \quad (t,x)\in(0,\infty)\times(\mathbb{T}^d)^N \\ \tilde{\nu}^N(0) &= \tilde{\nu}_0^N := \sum_{k\in\mathbb{Z}^d} \nu_0^N(k+x) \in \mathcal{P}((\mathbb{T}^d)^N) \end{cases}$$

Periodic rearrangement of ν^N.

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Theorem (The diffusive limit)

Consider $\rho^{\varepsilon,N}$ the solution to the rescaled Fokker–Planck equation with initial data $\rho_0^{\varepsilon,N} \in \mathcal{P}((\mathbb{R}^d)^N)$. Then, for all $t > 0$ the limit

$$\rho^{N,*}(t) = \lim_{\varepsilon\to 0} \rho^{\varepsilon,N}(t)$$

exists. Furthermore, the curve of measures $\rho^{N,*} : [0,\infty) \to \mathcal{P}_{\mathrm{sym}}((\mathbb{R}^d)^N)$ satisfies the heat equation

$$\partial_t \rho^{N,*} = \nabla \cdot (A^{\mathrm{eff},N} \nabla \rho^{N,*}) \, ,$$

with initial data $\rho^{N,*}(0) = \lim_{\varepsilon\to 0} \rho_0^{\varepsilon,N}$ and where the covariance matrix is given by the Kipnis–Varadhan formula

$$A^{\mathrm{eff},N} = \beta^{-1} \int_{(\mathbb{T}^d)^N} (I + \nabla \Psi^N(y)) \, M_N(y) \, \mathrm{d}y \, ,$$

with M_N the Gibbs measure of the quotiented N particle system and $\Psi^N : (\mathbb{T}^d)^N \to (\mathbb{R}^d)^N$ the unique mean zero solution to the associated corrector problem

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The diffusive limit is affected by the properties of the quotiented system on the torus!

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$N \to \infty + \varepsilon \to 0$?

Question: $\lim_{N\to\infty} \rho^{N,*} = ?$.
We already know $\rho^{\varepsilon,N} \to \rho^{\varepsilon,\otimes N}, \quad N \to \infty$ where ρ^{ε} solves the rescaled McKean–Vlasov equation. Another question : $\lim_{\varepsilon\to 0} \rho^{\varepsilon,\otimes N} \to ?$.

Theorem (Delgadino–G–Pavliotis '20)

Assume that the quotiented system has a phase transition at some β_c. Then for $\beta < \beta_c$

$$\lim_{N\to\infty}\lim_{\varepsilon\to 0} \rho^{\varepsilon,N} = \lim_{\varepsilon\to 0}\lim_{N\to\infty} \rho^{\varepsilon,N} .$$

On the other hand if $\beta > \beta_c$, there exists initial data $\rho_0^{\varepsilon,\otimes N}$ such that

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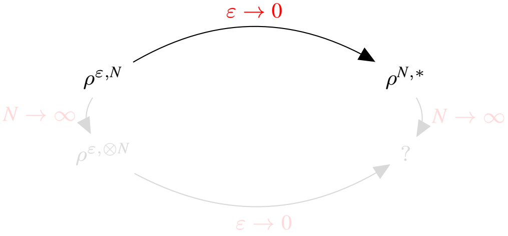

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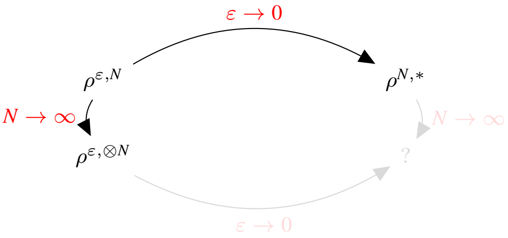

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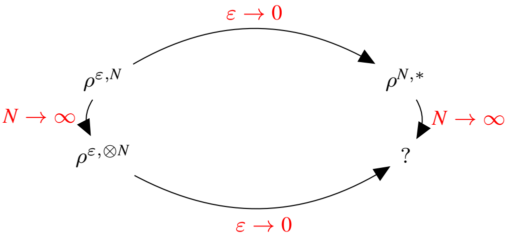

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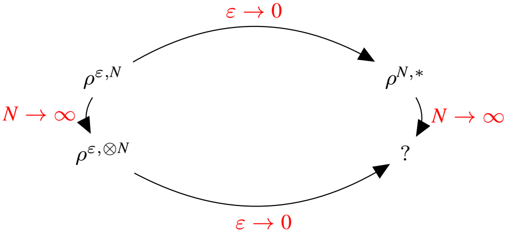

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The space $\mathcal{P}_{\text{sym}}((\mathbb{R}^d)^N)$

Due to the indistinguishability assumption on the particles their joint law is invariant under relabelling of the particles. In probability this is known as exchangeability, i.e., the law $\nu^N \in \mathcal{P}_{\text{sym}}\big((\mathbb{R}^d)^N\big)$.

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Definition (The limit of of $\mathcal{P}_{\text{sym}}((\mathbb{R}^d)^N)$)

Given a family $\{\rho^N\}_{N\in\mathbb{N}}$ such that $\rho^N \in \mathcal{P}_{\text{sym}}\big((\mathbb{R}^d)^N\big)$ we say that

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Another interpretation:

Definition (Empirical measure)

Given some $\rho^N \in \mathcal{P}_{\text{sym}}((\mathbb{R}^d)^N)$ we define its empirical measure $\hat{\rho}^N \in \mathcal{P}(\mathcal{P}(\mathbb{R}^d))$ as follows:

$$\hat{\rho}^N := T_N \# \rho^N ,$$

where $T^N : (\mathbb{R}^d)^N \to \mathcal{P}(\mathbb{R}^d)$ is the measurable mapping $(x_1, \ldots, x_N) \mapsto N^{-1} \sum_{i=1}^N \delta_{x_i}$. Futhermore, given a family $\{\rho^N\}_{N\in\mathbb{N}}$, we have that $\rho^N \to \boldsymbol{X} \in \mathcal{P}(\mathcal{P}(\mathbb{R}^d))$ if and only if $\hat{\rho}^N \rightharpoonup^* \boldsymbol{X}$, i.e tested against $C_b(\mathcal{P}(\mathbb{R}^d))$.

Lemma (de Finneti–Hewitt–Savage)

Given a sequence $\{\rho^N\}_{N\in\mathbb{N}}$, such that $\rho^N \in \mathcal{P}_{\text{sym}}((\mathbb{R}^d)^N)$ for every N, assume that the sequence of the first marginals $\{\rho_1^N\}_{N\in\mathbb{N}} \in \mathcal{P}(\mathbb{R}^d)$ is tight. Then, up to subsequence, not relabelled, there exists $\boldsymbol{X} \in \mathcal{P}(\mathcal{P}(\mathbb{R}^d))$ such that $\rho^N \to \boldsymbol{X}$.

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N-particle free energy, $E^N : \mathcal{P}_{\text{sym}}((\mathbb{R}^d)^N) \to (-\infty, +\infty]$:

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Gradient flow reformulation of the mean field limit

Similar analysis can be carried out for the quotiented system: $\tilde{E}^N, \tilde{E}_{MF}$ with particles living in $\mathbb{T}^d$. We consider the periodic N-particle energy $\tilde{E}^N$ and the periodic mean field energy $\tilde{E}_{MF}$. Then:

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As a consequence, if $\{M_N\}_{N\in\mathbb{N}}$ is the sequence of minimisers of $\tilde{E}^N$ (namely the sequence of Gibbs measures), then any accumulation point $X \in \mathcal{P}(\mathcal{P}(\mathbb{T}^d))$ of this sequence is a minimiser of $\tilde{E}^\infty$.

Theorem (Mean field limit, Carrillo–Delgadino–Pavliotis '19)

Fix some $t > 0$, then,

$$\lim_{N\to\infty} \nu^N(t) = X \in \mathcal{P}(\mathcal{P}(\mathbb{R}^d))$$

Furthermore, we have that the curve $X : [0,\infty) \to \mathcal{P}(\mathcal{P}(\mathbb{R}^d))$ is a gradient flow of E^∞ under the 2-Wasserstein metric $\mathfrak{D}_2$.

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$$\tilde{E}^\infty[\boldsymbol{X}] = \int_{\mathcal{P}(\mathbb{T}^d)} \tilde{E}_{MF}[\rho] \, \mathrm{d}\boldsymbol{X}(\tilde{\nu}) \,.$$

As a consequence, if $\{M_N\}_{N\in\mathbb{N}}$ is the sequence of minimisers of $\tilde{E}^N$ (namely the sequence of Gibbs measures), then any accumulation point $\boldsymbol{X} \in \mathcal{P}(\mathcal{P}(\mathbb{T}^d))$ of this sequence is a minimiser of $\tilde{E}^\infty$.

Theorem (Mean field limit, Carrillo–Delgadino–Pavliotis '19)

Fix some $t > 0$, then,

$$\lim_{N\to\infty} \nu^N(t) = \boldsymbol{X} \in \mathcal{P}(\mathcal{P}(\mathbb{R}^d))$$

Furthermore, we have that the curve $\boldsymbol{X} : [0,\infty) \to \mathcal{P}(\mathcal{P}(\mathbb{R}^d))$ is a gradient flow of E^∞ under the 2-Wasserstein metric $\mathfrak{D}_2$.

Gradient flow reformulation of the mean field limit

Similar analysis can be carried out for the quotiented system: $\tilde{E}^N, \tilde{E}_{MF}$ with particles living in $\mathbb{T}^d$. We consider the periodic N-particle energy $\tilde{E}^N$ and the periodic mean field energy $\tilde{E}_{MF}$. Then:

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Phase transitions

Consider the periodic McKean–Vlasov equation:

$$\begin{cases} \partial_t \tilde{\nu} = \beta^{-1}\Delta\tilde{\nu} + \nabla\cdot(\tilde{\nu}(\nabla W \star \tilde{\nu} + \nabla V)) & (t,x) \in (0,\infty)\times\mathbb{T}^d \\ \tilde{\nu}(0) = \tilde{\nu}_0 = \sum_{k\in\mathbb{Z}^d} \nu_0(k+x)\,. \end{cases}$$

Question: What is a phase transition?

Definition (Phase transition)

The periodic mean field McKean–Vlasov equation is said to undergo a phase transition at some $0 < \beta_c < \infty$ if

1. For $\beta < \beta_c$, there exists a unique steady state.
2. For $\beta > \beta_c$, there exist at least two steady states.

The temperature β_c is referred to as the point of phase transition or the critical temperature.

Example (noisy Kuramoto model)

Let $d = 1$, $W = -\cos(2\pi x)$, and $V = 0$. Then for $\beta \le 2$, $\tilde{\nu}_\infty \equiv 1$ is the unique minimiser of $\tilde{E}_{MF}$ and steady state. For $\beta > 2$, the steady states are given by $\tilde{\nu}_\infty \equiv 1$ and the family of translates of some measure $\tilde{\nu}_\beta^{\min}$. Moreover for $\beta > 2$, $\tilde{\nu}_\beta^{\min}$ (and its translates) are the only minimisers of the periodic mean field energy $\tilde{E}_{MF}$. Thus, $\beta_c = 2$ is the critical temperature.

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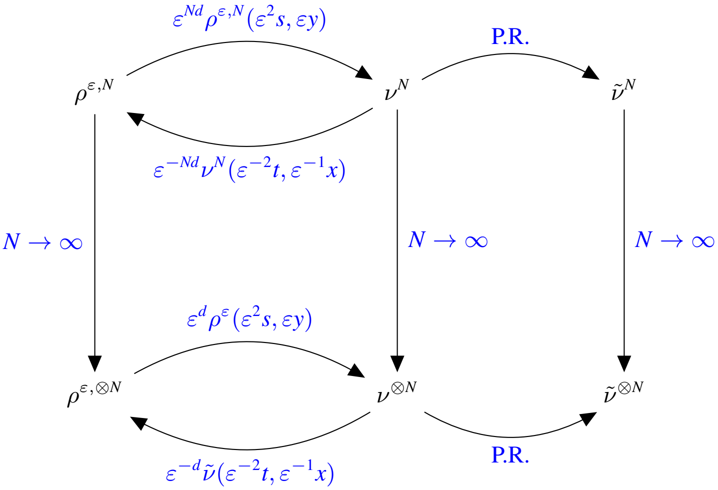
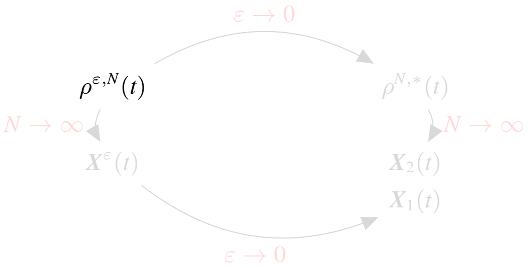


Figure: A schematic of the notation. The P.R. denotes periodic rearrangement.



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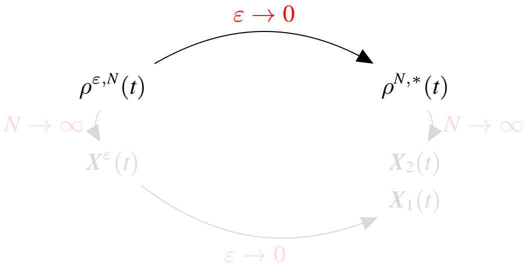

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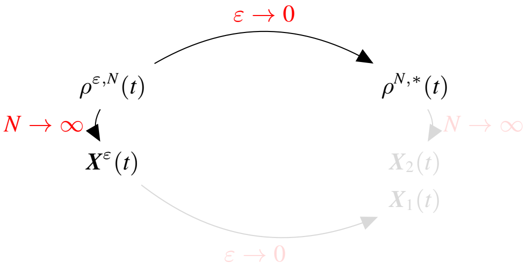

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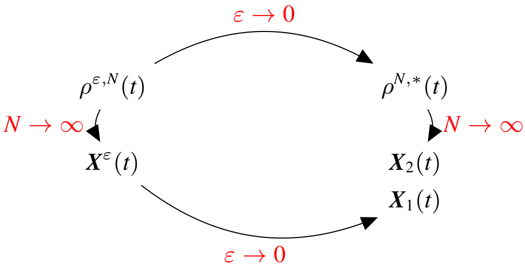

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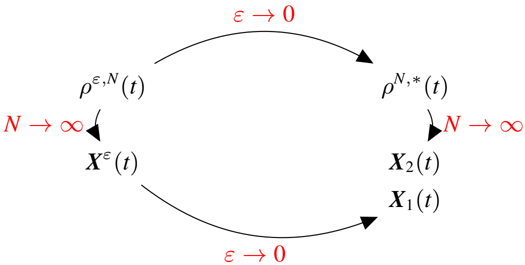

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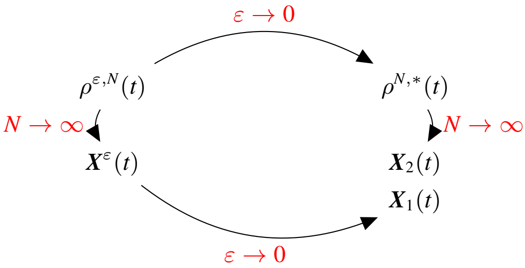

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$N \to \infty$ then $\varepsilon \to 0$

Theorem (Delgadino–G–Pavliotis '20)

Consider the set of initial data given by $\{\rho_0^\varepsilon\}_{\varepsilon>0} \subset \mathcal{P}(\mathbb{R}^d)$*, and consider the periodic rearrangement at scale* $\varepsilon > 0$ *, i.e.*

$$\tilde{\nu}_0^\varepsilon(A) = \varepsilon^d \sum_{k\in\mathbb{Z}^d} \rho_0^\varepsilon(\varepsilon(A+k)) \qquad \text{for } \varepsilon > 0 \,.$$

Assume that there exists $C > 0$*,* $p > 1$ *and* $\tilde{\nu}^* \in \mathcal{P}(\mathbb{T}^d)$ *such that* $\tilde{\nu}^\varepsilon(t)$*, with initial data* $\tilde{\nu}_0^\varepsilon(x)$*, satisfies*

$$\sup_{\varepsilon>0} d_2^2(\tilde{\nu}^\varepsilon(t), \tilde{\nu}^*) \le Ct^{-p} \,.$$

Then,

$$\lim_{\varepsilon\to 0} d_2^2(S_t^\varepsilon \rho_0^\varepsilon, S_t^* \rho_0^*) = 0,$$

where S_t^ε *is the solution semigroup associated to the rescaled PDE on* $\mathbb{R}^d$*,* $\rho_0^* \in \mathcal{P}(\mathbb{R}^d)$ *is the weak-*$*$ *limit of* ρ_0^ε*, and* S_t^* *is the solution semigroup of the heat equation*

$$\partial_t \rho = \nabla \cdot (A_*^{\mathrm{eff}} \nabla \rho),$$

where the covariance matrix

$$A_*^{\mathrm{eff}} = \beta^{-1} \int_{\mathbb{T}^d} (I + \nabla \Psi^*(y)) \;\mathrm{d}\tilde{\nu}^*(y) \,.$$

$N \to \infty$ then $\varepsilon \to 0$

Theorem (Delgadino–G–Pavliotis '20)

$\Psi^* : \mathbb{T}^d \to \mathbb{R}^d$ *is the solution to the associated corrector problem*

$$\nabla \cdot (\tilde{\nu}^* \nabla \Psi^*) = -\nabla \tilde{\nu}^* .$$

Furthermore, assume that $\boldsymbol{X}^\varepsilon(t)$ *is the mean field limit and that* $\lim_{N\to\infty} \rho_0^{\varepsilon,N} = \boldsymbol{X}_0^\varepsilon = \delta_{\rho_0^\varepsilon}$. *Then it holds that:*

$$\boldsymbol{X}_1(t) = \lim_{\varepsilon\to 0} \lim_{N\to\infty} \rho^{\varepsilon,N} = \lim_{\varepsilon\to 0} \boldsymbol{X}(t)^\varepsilon = S_t^* \# \boldsymbol{X}_0 ,$$

where $\boldsymbol{X}_0 = \delta_{\rho_0^*}$.

$\varepsilon \to 0$ then $N \to \infty$

Theorem (Delgadino–G–Pavliotis '20)

Assume that the periodic mean field energy $\tilde{E}_{MF}$ admits a unique minimiser $\tilde{\nu}^{\min}$, then we have that $\rho^{N,*}$ satisfies, for any fixed $t > 0$,

$$\lim_{N\to\infty} \rho^{N,*}(t) = \boldsymbol{X}_2(t) = S_t^{\min}\#\boldsymbol{X}_0,$$

where $\boldsymbol{X}_0 \in \mathcal{P}(\mathcal{P}(\mathbb{R}^d))$ is the limit of $\rho^{N,*}(0)$, and $S_t^{\min} : \mathcal{P}(\mathbb{R}^d) \to \mathcal{P}(\mathbb{R}^d)$ is the solution semigroup of the heat equation

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with $\Psi^{\min} : \mathbb{T}^d \to \mathbb{R}^d$, the solution to the associated corrector problem

$$\nabla \cdot (\tilde{\nu}^{\min} \nabla \Psi^{\min}) = -\nabla \tilde{\nu}^{\min} \,.$$

It follows then, that for any fixed $t > 0$, the solution $\rho^{\varepsilon,N}(t)$ satisfies

$$\boldsymbol{X}_2(t) = \lim_{N\to\infty} \lim_{\varepsilon\to 0} \rho^{\varepsilon,N}(t) = \lim_{N\to\infty} \rho^{N,*}(t) = S_t^{\min}\#X_0 \,.$$

Non-commutativity

- The limit $X_1(t)$ sees the long time behaviour of $\tilde{\nu}$ and thus steady states.
- The limit $X_2(t)$ sees minimisers of $\tilde{E}_{MF}$.

Thus we can break commutativity ahead of the phase transition.

Example (A biased Kuramoto model)

Consider the model with $V = -\eta\cos(2\pi x), W = -\cos(2\pi x)$ *with* $\eta \in (0,1)$*. Then the mean field model on the torus has a phase transition at some* $0 < \beta_c < \infty$*. It has at least two steady states for* $\beta > \beta_c$*,* $\tilde{\nu}^*$ *and* $\tilde{\nu}^{\min}$ *the minimiser of* $\tilde{E}_{MF}$*.*

Additionally, for $\beta > \beta_c$ *and* $\rho_0^{\varepsilon,N} = (\rho_0^\varepsilon)^{\otimes N}$ *such that* $\tilde{\nu}^* = \sum_{k\in\mathbb{Z}^d} \varepsilon^d \rho_0^\varepsilon(\varepsilon x)$ *we have that*

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Sketch of proof for $N \to \infty$ followed by $\varepsilon \to 0$

- Pass to the mean field limit (using CDP'19) to obtain $\boldsymbol{X}^{\varepsilon}(t)$.
- For the associated mean field SDE on the torus consider a moving corrector problem:

$$\nabla \cdot (\tilde{\mu}^{\varepsilon}(t)\nabla\chi) = -\nabla(\tilde{\mu}^{\varepsilon}), \quad \tilde{\mu}^{\varepsilon}(t) \sim \exp(-\beta(W \star \tilde{\nu}(t) - V))$$

  and obtain time-dependent estimates:

$$\|\chi_i\|_{C^m(\mathbb{T}^d)} \lesssim 1$$

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$$\begin{aligned}
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References

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References

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