A chemotaxis model treated with duality methods

Laurent Desvillettes, Université de Paris, IMJ-PRG

January 5, 2020

Results obtained in collaboration with

Yong-Jung Kim, KAIST, Daejon, South Korea

Ariane Trescases, CNRS, Toulouse, France

Michael Winkler, Univ. Paderborn, Germany

Changwook Yoon, Korea University, Sejong, South Korea

Chemotaxis

Unknowns: Density of cells $u := u(t,x) \geq 0$; density of chemoattractant $v := v(t,x) \geq 0$

Typical equation satisfied by the chemoattractant (parabolic case):

$$\partial_t v = \varepsilon \, \Delta v - v + u.$$

Keller-Segel class of equations for the cells:

$$\partial_t u = \nabla \cdot \Big(\mu(u,v) \nabla u - \chi(u,v) \nabla v \Big).$$

Here $\mu \geq 0$ is the cell diffusivity and $\chi \geq 0$ is the chemosensitivity.

Problem under study (I)

Possible choice of cell diffusivity and chemosensitivity (Keller-Segel, second paper): $\chi := \chi_0(v)\, u$, $\mu := \mu(v)$, and

$$\chi_0(v) = (\alpha - 1)\, \mu'(v),$$

where α is the ratio of effective body length (i.e., maximum distance between receptors) over the walk length.

When $\alpha = 0$ (gradients of the chemoattractant not measured by the cell), then

$$\partial_t u = \nabla \cdot \Big(\mu(v) \nabla u + \mu'(v)\, u\, \nabla v \Big)$$
$$= \Delta(\mu(v)\, u).$$

This special structure enables to get estimates that are not known to hold in general.

Problem under study (II)

Goal: Find cases in which aggregation can happen (that is, the system does not relax towards a constant steady state) while (weak) solutions exist globally.

Proposed system: For some $c \geq 0$, $k > 0$,

$$\partial_t u = \Delta\Big(\frac{u}{c + v^k}\Big),$$

$$\partial_t v = \varepsilon\,\Delta v - v + u.$$

Homogeneous Neumann boundary conditions on a smooth bounded domain Ω:

$$\nabla u \cdot n = \nabla v \cdot n = 0 \qquad \text{on} \qquad \partial\Omega.$$

Proposition Let $c \geq 0$, $\varepsilon > 0$, $k > 0$, and $\Omega =]a, b[$ be a bounded open interval. Let $u(0, \cdot) \geq 0$ and $v(0, \cdot) \geq c_0 > 0$ lie in $C^{2,\alpha}([a, b])$ for some $\alpha > 0$ (compatible with the Neumann boundary conditions).

Then there exists a unique $(u \geq 0, v \geq 0)$ (global in time) classical solution to the system

$$\partial_t u = \partial_{xx}\Big(\frac{u}{c + v^k}\Big),$$

$$\partial_t v = \varepsilon\, \partial_{xx} v - v + u,$$

with homogeneous Neumann boundary conditions.

Existence of weak solutions in dimension 2 (critical)

Theorem Let $c \geq 0$, $\varepsilon > 0$, $k > 0$, and Ω be a bounded smooth open set of $\mathbb{R}^2$. Let $u(0,\cdot) \geq 0$ lie in $\cup_{p>1} L^p(\Omega)$ and $v(0,\cdot) \geq c_0 > 0$ lie in $L^1(\Omega)$.

Then, there exists a (very) weak (global in time) solution $(u \geq 0, v \geq 0)$ of the system

$$\partial_t u = \Delta\Big(\frac{u}{c + v^k}\Big),$$

$$\partial_t v = \varepsilon\, \Delta v - v + u,$$

with homogeneous Neumann boundary conditions.

Moreover, for all $0 < t_0 < T$,

$v \in L^\infty([0,T]; L^1(\Omega)) \cap L^\infty([t_0,T]; \cap_{p\in[1,\infty[} L^p(\Omega))$,

$u \in L^\infty([0,T]; L^1(\Omega)) \cap L^2([t_0,T]; \cap_{p\in[1,2[} L^p(\Omega))$.

Existence of weak solutions in dimension 3

Theorem Let $c \geq 0$, $\varepsilon > 0$, $k \in]0, 3[$, and Ω be a bounded smooth open set of $\mathbb{R}^3$. Let $u(0, \cdot) \geq 0$ lie in $\cup_{p>6/5} L^p(\Omega)$ and $v(0, \cdot) \geq c_0 > 0$ lie in $L^1(\Omega)$.

Then, there exists a (very) weak (global in time) solution $(u \geq 0, v \geq 0)$ of the system

$$\partial_t u = \Delta\Big(\frac{u}{c + v^k}\Big),$$

$$\partial_t v = \varepsilon\, \Delta v - v + u,$$

with homogeneous Neumann boundary conditions.

Moreover, for all $0 < t_0 < T$,

$v \in L^\infty([0, T]; L^1(\Omega)) \cap L^\infty([t_0, T]; \cap_{p \in [1,3[} L^p(\Omega))$,

$u \in L^\infty([0, T]; L^1(\Omega)) \cap L^2([t_0, T]; \cap_{p \in [1, \frac{6}{3+k}[} L^p(\Omega))$.

Ingredients of the proof (I)

Existence of weak solutions

Direct integration of

$$\partial_t u = \Delta\Big(\frac{u}{c + v^k}\Big),$$

$$\partial_t v = \varepsilon\, \Delta v - v + u,$$

leads to the estimates:

$$u \in L^\infty([0,T]; L^1(\Omega)), \qquad v \in L^\infty([0,T]; L^1(\Omega)).$$

Ingredients of the proof (II)

Use of the **semigroup properties** of the heat kernel for

$$\partial_t v = \varepsilon \, \Delta v - v + u$$

leads to the estimates: For all $q < \infty$ in dimension 2 (and 1),

$$||v(t,\cdot)||_{L^q(\Omega)} \leq cst \left(t^{(1/q-1)} + t^{1/q} \right).$$

and for all $q < 3$ in dimension 3,

$$||v(t,\cdot)||_{L^q(\Omega)} \leq cst \left(t^{\frac{3}{2}\,(1/q-1)} + t^{1+\frac{3}{2}\,(1/q-1)} \right).$$

Ingredients of the proof (III)

Duality lemma of Pierre-Schmitt: Observing that

$$\partial_t u - \Delta(A\, u) = 0, \qquad \partial_\nu (A\, u)|_{\partial\Omega} = 0,$$

where $A := (c + v^k)^{-1} \geq 0$ lies in $L^1([0,T] \times \Omega)$, we get the L^2-type estimate

$$\int_0^T \int_\Omega A\, u^2 \, dxdt = \int_0^T \int_\Omega \frac{u^2}{c + v^k} \, dxdt \leq C_T.$$

Ingredients of the proof (IV)

Interpolation of the estimates: For all $q < \infty$ in dimension 2 (and 1),

$$||v(t,\cdot)||_{L^q(\Omega)} \le cst \left(t^{(1/q-1)} + t^{1/q} \right).$$

and for all $q < 3$ in dimension 3,

$$||v(t,\cdot)||_{L^q(\Omega)} \le cst \left(t^{\frac{3}{2}\,(1/q-1)} + t^{1+\frac{3}{2}\,(1/q-1)} \right).$$

with

$$\int_0^T \int_\Omega \frac{u^2}{c+v^k}\, dxdt \le C_T.$$

Result: in dimension 2 (and 1), $u \in L^2([t_0, T]; \cap_{p\in[1,2[} L^p(\Omega))$;

in dimension 3, $u \in L^2([t_0, T]; \cap_{p\in[1,\frac{6}{3+k}[} L^p(\Omega))$.

Ingredients of the proof (V)

Maximal regularity ensures that in dimension 2,

$$\partial_t v \in \cap_{p\in[1,2[} L^p([t_0, T] \times \Omega);$$

$$\partial_{x_i x_j} v \in \cap_{p\in[1,2[} L^p([t_0, T] \times \Omega));$$

and in dimension 3,

$$\partial_t v \in \cap_{p\in[1,\frac{6}{3+k}[} L^p([t_0, T] \times \Omega));$$

$$\partial_{x_i x_j} v \in \cap_{p\in[1,\frac{6}{3+k}[} L^p([t_0, T] \times \Omega)).$$

Ingredients of the proof (VI)

Maximal regularity estimates give the **compactness** for an approximating sequence of solutions for v.

This is enough to **pass to the limit** in an approximating sequence for the (unique nonlinear) term $\Delta\left(\frac{u}{c+v^k}\right)$ of the system

$$\partial_t u = \Delta\left(\frac{u}{c+v^k}\right),$$

$$\partial_t v = \varepsilon\, \Delta v - v + u,$$

Instability of homogeneous steady states

Proposition: Let Ω be a smooth bounded subset of $\mathbb{R}^n$, and $\mu_1 > 0$ be the principal eigenvalue of the Laplace operator $-\Delta$ on Ω
Let $c \geq 0$, $k > 1$, $\bar{u} = \bar{v} > 0$, and suppose that $\bar{u} > u_1 := \left(\frac{c}{k-1}\right)^{\frac{1}{k}}$.

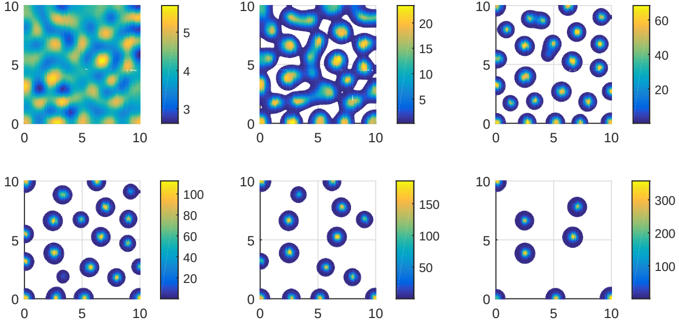
If $\varepsilon < \varepsilon_1(\bar{u})$, then the constant state $(u, v) = (\bar{u}, \bar{v})$ is linearly unstable.
If $\varepsilon > \varepsilon_1(\bar{u})$, then $(\bar{u}, \bar{v})$ is linearly asymptotically stable.

Here

$$\varepsilon_1(\bar{u}) := \frac{(k-1)\bar{v}^k - c}{\mu_1(c + \bar{v}^k)} > 0.$$

Numerical illustration

Concentration of u at different times (dimension 2, case of a linearly unstable steady state)



Perspectives and open problems

- Existence of strong solutions for $n = 2$, "critical dimension"?
- Informations about the existence or nonexistence of non homogeneous steady states? (work in progress with C. Yoon)
- Better mathematical theory when growth of cells is included in the system, appearance of waves?
- Possibility to recover the model from a diffusive limit of a kinetic equation?