Wulff shapes in $\mathbb{H}^1$

Manuel Ritoré
(joint work with Julián Pozuelo)

International Sub-Riemannian Geometry Seminar
University of Jyväskylä
May 29, 2020

UNIVERSIDAD DE GRANADA

GOBIERNO DE ESPAÑA MINISTERIO DE CIENCIA E INNOVACIÓN

AGENCIA ESTATAL DE INVESTIGACIÓN

UNIÓN EUROPEA
Fondo Europeo de Desarrollo Regional

Andalucía se mueve con Europa

GHAIA

Introduction

Wulff shapes in $\mathbb{R}^n$

Consider a convex body (compact with interior points) K with $0 \in \text{int}(K)$ and its associated Minkowski content $M(E, K) = \liminf_{r \to 0}(|E + rK| - |E|)/r$. If E has C^1 boundary S

$$M(E, K) = \int_S h_K(N)\, dS = \int_S ||N||_{K,*} dS, \qquad (*)$$

where N is a unit normal to ∂E and $h_K(u) = ||u||_{K,*} = \sup_{v \in K}\langle u, v\rangle$ is the support function of K. $|| \cdot ||_{K,*}$ is also referred to as the dual norm.

(*) is an anisotropic energy used to model the shape of an equilibrium crystal minimizing Gibbs' free energy (1875). The solution to the problem for polyhedra was described by Wulff (1895).

Introduction

Wulff shapes in $\mathbb{R}^n$

The problem is to minimize $M(E, K)$ in the class of sets E with given volume.

The solutions are translations and dilations of K. This is proven from the Brunn-Minkowski inequality

$$|E + rK|^{1/n} - |E|^{1/n} \geq r|K|^{1/n}$$

Taking limits, since $M(K, K) = n|K|$,

$$\frac{M(E, K)}{|E|^{(n-1)/n}} \geq \frac{M(K, K)}{|K|^{(n-1)/n}}.$$

K is known as the Wulff shape for the functional (*)

Introduction

Wulff shapes in $\mathbb{R}^n$

- The functional (*) is used in crystallography (Gibbs free energy). Wulff gave a construction to obtain K from the support function
- Use of Brunn-Minkowski to obtain a solution by Dinghas (1944)
- Mathematical problem considered by Taylor (1978), Fonseca (1991) and Fonseca-Müller (1991)

Introduction

The Heisenberg group $\mathbb{H}^1$

$(\mathbb{R}^3, *)$, where $*$ is the product

$$(z,t) * (z',t') := (z+z', t+t'+\operatorname{Im}(z\bar{z}')), \qquad (z = x+iy),$$

(z,t), $(z',t') \in \mathbb{R}^3 \equiv \mathbb{C} \times \mathbb{R}$

A basis of left invariant vector fields is given by

$$X = \frac{\partial}{\partial x} + y\,\frac{\partial}{\partial t}, \qquad Y = \frac{\partial}{\partial y} - x\,\frac{\partial}{\partial t}, \qquad T = \frac{\partial}{\partial t}.$$

X, Y generate the horizontal distribution $\mathcal{H}$, $\langle \cdot, \cdot \rangle$ is the Riemannian metric so that X, Y, T is orthonormal basis, D Levi-Civita connection, ∇ pseudo-hermitian connection (metric with $\operatorname{Tor}(U,V) = 2\langle J(U), V\rangle\, T$, J)

Introduction

Sub-Finsler norms in $\mathbb{H}^1$

We start with a given convex body $K \subset \mathbb{R}^2$ such that $0 \in \text{int}(K)$. We define

$$||u||_K = \inf\{\lambda \geq 0 : u \in \lambda K\}$$

We assume $||\cdot||_K$ strictly convex and that $\{||\cdot||_K = 1\}$ is C^2 outside 0. The dual norm is

$$||u||_{K,*} = \sup_{||v||_K \leq 1} \langle u, v \rangle$$

The projection $\pi_K(u)$ is defined as the only vector (strict convexity) such that

$$\langle \pi_K(u), u \rangle = ||u||_{K,*}.$$

It satisfies $||\pi_K(u)||_K = 1$ when $u \neq 0$

Introduction

Sub-Finsler norms in $\mathbb{H}^1$

The planar norm $||\cdot||_K$ is extended to a left-invariant norm in $\mathcal{H}$

$$(||fX + gY||_K)_p = ||(f(p), g(p))||_K$$

Sub-Finsler perimeter in $\mathbb{H}^1$

Let $E \subset \mathbb{H}^1$ be a measurable set, $||\cdot||_K$ the left-invariant norm associated to $K \subset \mathbb{R}^2$, and $\Omega \subset \mathbb{H}^1$ an open subset. We say that E has locally finite K-perimeter in Ω if for any relatively compact open set $V \subset \Omega$ we have

$$|\partial E|_K(V) = \sup \left\{ \int_E \operatorname{div}(U)\, d\mathbb{H}^1 : U \in \mathcal{H}_0^1(V), ||U||_{K,\infty} \leq 1 \right\} < +\infty.$$

$|\partial E|_K(V)$ is the K-perimeter of E in V. If K is the closed unit disc centered at 0 this is the classical sub-Riemannian perimeter

Introduction

Sub-Finsler perimeter in $\mathbb{H}^1$

Riesz Representation Theorem implies that $|\partial E|_K$ extends to a Radon measure on Ω and the existence of a $|\partial E|_K$-measurable horizontal vector field ν_K in Ω so that

$$\int_\Omega \operatorname{div}(U)\, d\mathbb{H}^1 = \int_\Omega \langle U, \nu_K\rangle\, d|\partial E|_K$$

for any U horizontal of class C^1 with compact support.

Given K, K', the measures $|\partial E|_K$, $|\partial E|_{K'}$ are absolutely continuous with respect to each other. Moreover, Radon-Nikodym's Theorem implies

$$|\partial E|_K = ||\nu_{K'}||_{K,*}|\partial E|_{K'}, \quad \nu_K = \frac{\nu_{K'}}{||\nu_{K'}||_{K,*}}$$

Introduction

Sub-Finsler perimeter in $\mathbb{H}^1$

In particular, if E has boundary S of class C^1 or Euclidean Lipschitz

$$
\begin{aligned}
|\partial E|_K(V) = \int_{S\cap V} ||\nu_h||_{K,*} d|\partial E|_D &= \int_{S\cap V} ||N_h||_{K,*}\, dS \\
&= \int_{S\cap V} \langle N_h, \pi_K(N_h)\rangle\, dS.
\end{aligned}
$$

where ν_h is the horizontal unit normal, D is the closed unit disc centered at 0 and so $d|\partial E|_D$ is the classical sub-Riemannian measure, and dS is the Riemannian measure of S.

Introduction

Problem

Minimize K-perimeter under a volume constraint

Previous work

- A.P. Sánchez, Ph.D. Thesis, Tufts U., 2017.
- Work in progress by V. Franceschi, R. Monti, A. Righini, and M. Sigalotti

First variation of perimeter for C^2 boundaries

We drop the subscript K

Theorem (First variation of perimeter)

Let S be an oriented C^2 surface immersed in $\mathbb{H}^1$, U be a C^2 vector field with compact support on S, normal component $u = \langle U, N\rangle$ and $\{\varphi_s\}_{s\in\mathbb{R}}$ the associated flow. Let $\eta = \pi(\nu_h)$. Then we have

$$\frac{d}{ds}\bigg|_{s=0} A(\varphi_s(S)) = \int_S \big(\operatorname{div}_S \eta - 2\langle N, T\rangle\langle J(N_h), \eta\rangle\big)\, u\, dS + \int_S \operatorname{div}_S \left(||N_h||_* U^\top - u\eta^\top \right) dS,$$

where div_S is the Riemannian divergence in S, and the superscript $\top$ indicates the tangent projection to S.

Proof as in Ritoré-Rosales (2008)

First variation of perimeter for C^2 boundaries

Lemma

Let S be a C^2 surface immersed in $\mathbb{H}^1$ with unit normal N horizontal unit normal ν_h. Let $Z = -J(\nu_h)$. Then we have

$$\operatorname{div}_S \eta - 2\langle N, T\rangle\langle J(N_h), \eta\rangle = \langle D_Z\eta, Z\rangle.$$

Corollary

Let S be an oriented C^2 surface immersed in $\mathbb{H}^1$. Let U be a C^2 vector field with compact support on $S \setminus S_0$, normal component $u = \langle U, N\rangle$ and $\{\varphi_s\}_{s\in\mathbb{R}}$ the associated flow. Let $\eta = \pi(N_h)$. Then we have

$$\frac{d}{ds}\bigg|_{s=0} A(\varphi_s(S)) = \int_S u\langle D_Z\eta, Z\rangle\, dS,$$

K-mean curvature H_K

We let $H_K = \langle D_Z\eta, Z\rangle$.

First variation of perimeter for C^2 boundaries

The mean curvature can be computed along a horizontal curve γ parameterized by arc-length as

$$\langle \frac{D}{ds} \pi(J(\dot{\gamma})), \dot{\gamma} \rangle,$$

where D/ds is the covariant derivative along the curve γ.

Corollary (uniqueness of horizontal curves)

Let S be a C^2 oriented surface immersed in $\mathbb{H}^1$ with mean curvature H_K. Let $\gamma : I \to S \setminus S_0$ be a horizontal curve in the regular part of S parameterized by arc-length with $\gamma(s) = (x_1(s), x_2(s), t(s))$. Then $z(s) = (x_1, x_2)$ satisfies a differential equation of the form

$$\ddot{z} = F(\dot{z}),$$

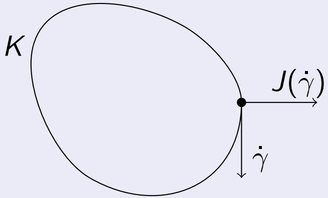
for some smooth function F.

First variation of perimeter for C^2 boundaries

Lemma

Let $z : I \to \mathbb{R}^2$ be a unit speed clockwise parameterization of a translation of the unit sphere of $||\cdot||_K$ in $\mathbb{R}^2$. Let γ be a horizontal lifting of z. Then γ satisfies the equation

$$1 = \langle \frac{D}{ds}\pi(J(\dot\gamma)), \dot\gamma\rangle.$$



$$\pi(J(\dot\gamma)) = \gamma \Rightarrow \frac{D}{ds}\pi(J(\dot\gamma)) = \dot\gamma \Rightarrow \langle \frac{D}{ds}\pi(J(\dot\gamma)), \dot\gamma\rangle = 1.$$

First variation of perimeter for C^2 boundaries

Theorem

Let $||\cdot||_K$ be a smooth, strictly convex, left-invariant norm in $\mathbb{H}^1$. Let γ be a horizontal curve satisfying equation

$$\langle \frac{D}{ds}\pi(J(\dot{\gamma})), \dot{\gamma}\rangle = H,$$

for some constant $H \geq 0$. Then γ is either a horizontal straight line if $H = 0$ or the horizontal lifting of a dilation and traslation of a unit speed clockwise parameterization of the circle $||\cdot||_K = 1$ in case $H > 0$.

Theorem

Let S be a C^2 surface without singular points and constant mean curvature $H_K > 0$. Then S is foliated by horizontal liftings of translations of the circle $||\cdot||_K = 1/H_k$.

The Wulff shape

Definition

Let $K \subset \mathbb{R}^2$ be a convex body, and consider a clockwise-oriented L-periodic parameterization $\gamma : \mathbb{R} \to \mathbb{R}^2$ of the Lipschitz curve $||\cdot||_K = 1$. For any $u \in \mathbb{R}$ consider the horizontal lifting $\Gamma_{\gamma(u)} : \mathbb{R} \to \mathbb{H}^1$ of the curve $t_{-\gamma(u)}(\gamma)$ with initial point $(0,0,0)$.
The set $\mathbb{B}_K$ is defined as

$$\mathbb{B}_K = \bigcup_{u \in [0,L)} \Gamma_{\gamma(u)}([u, u+L]).$$

We shall refer to $\mathbb{B}_K$ as the *Wulff shape* associated to the left-invariant norm $||\cdot||_K$. Its boundary $\mathbb{S}_K = \partial \mathbb{B}_K$ will be called the *Wulff sphere*.

The Wulff shape

Proposition (geometric properties of the Wulff shape)

Let $K \subset \mathbb{R}^n$ be a convex body with $0 \in \operatorname{int}(K)$. We consider the set

$$K_0 = \bigcup_{p \in \partial K} (-p + K).$$

Then we have

1. $0 \in K_0$.
2. K_0 is a convex body.
3. K_0 is the difference body $K - K$ of K. In particular, K_0 is centrally symmetric.
4. If K is centrally symmetric then $K_0 = 2K$.
5. We have
$$\bigcup_{p \in \partial K} (-p + K) = \bigcup_{p \in \partial K} (-p + \partial K).$$

The Wulff shape

The centrally symmetric case

If K is centrally symmetric we have the additional properties

- The projection of $\mathbb{B}_K$ to the plane $t = 0$ is $K_0 = 2K$
- $\mathbb{B}_K$ is symmetric with respect to a horizontal plane (Euclidean symmetry)

The general case

- The projection of $\mathbb{B}_K$ to the plane $t = 0$ is the difference body $K - K$, that it is centrally symmetric
- $\mathbb{B}_K$ is not necessarily symmetric with respect to a horizontal plane
- $\mathbb{B}_K$ is the union of two graphs g_1, g_2 of class C^2 outside the poles defined over K_0. Moreover $g_1 < g_2$ on $\text{int}(K_0)$ and $g_1 = g_2$ on ∂K_0. The graph of the function $g = (g_1 + g_2)/2$ separates $\mathbb{S}_K$ into two pieces $\mathbb{S}_K^+$ and $\mathbb{S}_K^-$.

The Wulff shape

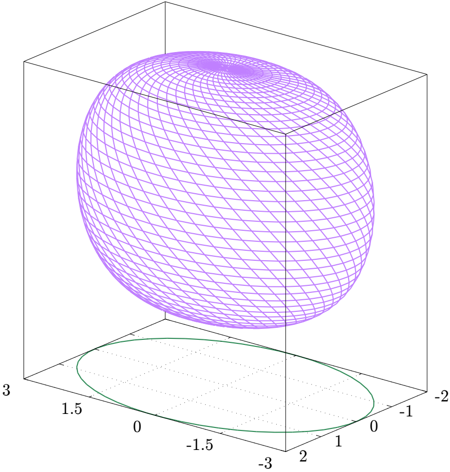


Figure: The Wulff shape associated to the norm $||\cdot||_a = ((x_1/a_1)^2 + (x_2/a_2)^2)^{1/2}$ with $a = (1, 1.5)$. Observe that the projection to the horizontal plane $t = 0$ is an ellipse with semiaxes of lengths 2 and 3.

The Wulff shape

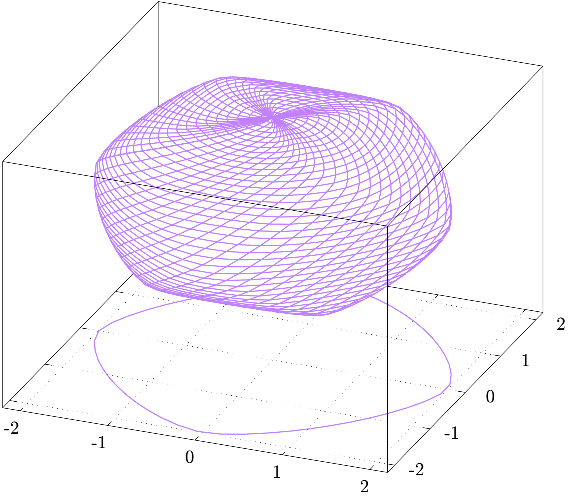


Figure: The Wulff shape $\mathbb{S}_{K_r}$ for the r-norm, $r = 1.5$. The horizontal curve is the projection of the equator to the plane $t = 0$. Since the r-norm is symmetric, the Wulff shape projects to the set $||\cdot||_r \leq 2$ in the $t = 0$ plane.

The Wulff shape

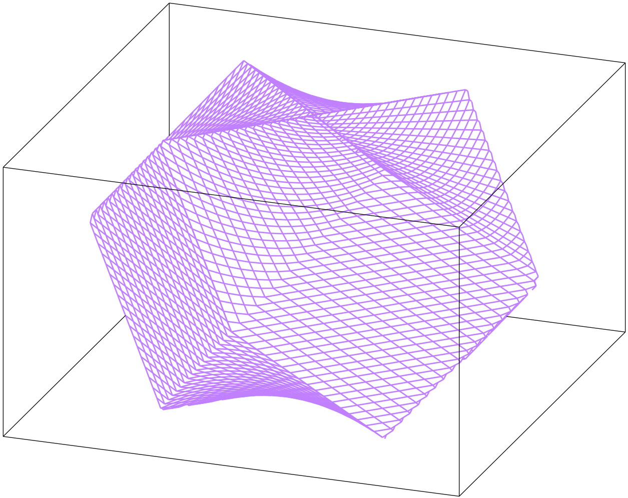


Figure: The ball $\mathbb{B}_1$ obtained as Hausdorff limit of the Wulff shapes $\mathbb{B}_{K_r}$ of the r-norm when r converges to 1

The Wulff shape

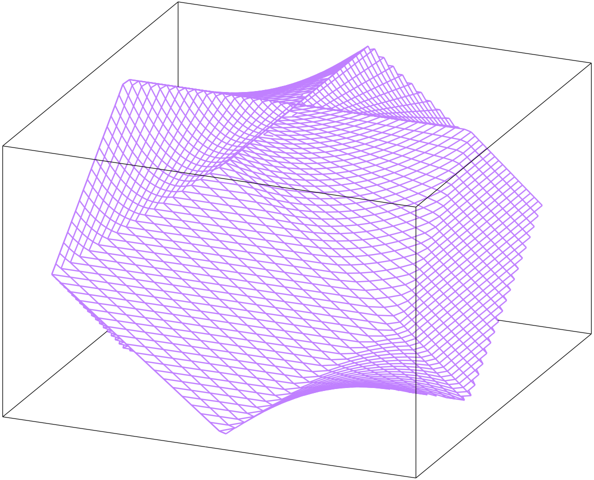


Figure: The ball $\mathbb{B}_\infty$ obtained as Hausdorff limit of the Wulff shapes $\mathbb{B}_{K_r}$ of the r-norm when r converges to ∞

The Wulff shape

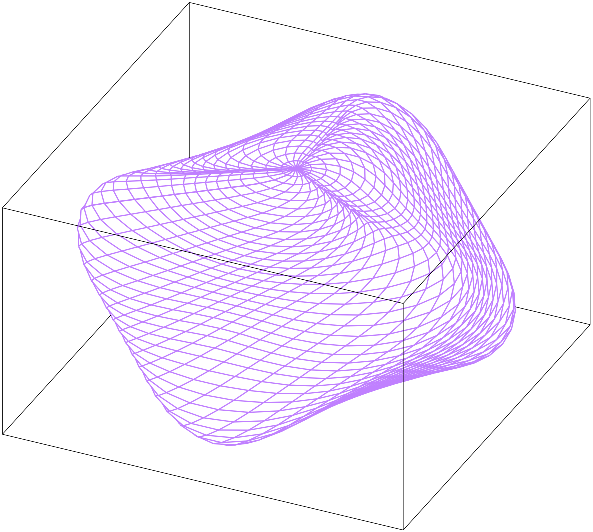


Figure: The Wulff shape $\mathbb{B}_{T,r}$ for the norm $||\cdot||_{T,r}$, with $r = 2$.

The Wulff shape

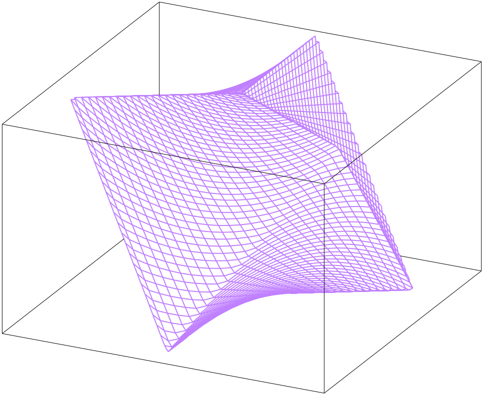


Figure: The ball $\mathbb{B}_T$ obtained as limit of the Wulff shapes $\mathbb{B}_{T,r}$ when $r \to \infty$.

The Wulff shape (regularity properties)

Parameterization of the Wulff shape

Given any convex body $K \subset \mathbb{R}^2$ with $0 \in \text{int}(K)$ we parameterize ∂K as

$$\gamma(s) = \big(x(s), y(s)\big) = r(s)\,\big(\sin(s), \cos(s)\big), \quad s \in \mathbb{R}.$$

where $r(s) = \rho(\sin(s), \cos(s))$ and ρ is the radial function of K.

Then we have the following parameterization of $\mathbb{S}_K$.

$$\begin{aligned} x(u,v) &= r(u+v)\sin(u+v) - r(v)\sin(v), \\ y(u,v) &= r(u+v)\cos(u+v) - r(v)\cos(v), \\ t(u,v) &= r(v)r(u+v)\big(\sin(v)\cos(u+v) - \cos(v)\sin(u+v)\big) \\ &\qquad + \int_v^{u+v} r^2(\xi)\,d\xi. \end{aligned}$$

Regularity properties follow from this expression. Also convergence in Hausdorff distance of Wulff shapes

Minimization properties

Theorem

Let $||\cdot||_K$ be the norm associated to an strictly convex body $K \subset \mathbb{R}^2$ with C^2 boundary. Let $r > 0$ and $h : rK_0 \to \mathbb{R}$ a C^0 function. Consider a subset $E \subset \mathbb{H}^1$ with finite volume and K-perimeter such that

$$\text{graph}(h) \subseteq E \subset rK_0 \times \mathbb{R}.$$

Then

$$|\partial E|_K \geq |\partial \mathbb{B}_E|_K,$$

where $\mathbb{B}_E$ is the Wulff shape with the same volume as E.

Minimization properties

Proof

Let $g_r : rK_0 \to \mathbb{R}$ the function defined by $g_r(x) = r^2 g(\frac{1}{r}x)$, where g is the equatorial function separating $\mathbb{S}_K$. Let D be the graph of g_r, that D separates $r\mathbb{S}_K$ into two parts $r\mathbb{S}_K^+$ and $r\mathbb{S}_K^-$. Let W^+ and W^- the vector fields in $rK_0 \times \mathbb{R} \setminus L$ defined by translating vertically the vector fields

$$\pi_K(\nu_0)\big|_{r\mathbb{S}_K^+}, \quad \pi_K(\nu_0)\big|_{r\mathbb{S}_K^-},$$

respectively. Here ν_0 is the horizontal unit normal to $\mathbb{S}_K$.

Minimization properties

Proof

As a first step in the proof we show: if $F \subset rK_0 \times \mathbb{R}$ is a set of finite volume and K-perimeter so that rel $\text{int}(D) \subset \text{int}(F)$, then the inequality

$$\tfrac{1}{r}|F| \leq \int_D \langle W^+ - W^-, N_D \rangle dD + |\partial F|_K$$

holds, where N_D is the Riemannian normal pointing down and dD is the Riemannian measure of D. Equality holds if and only if $W^+ = \pi_K(\nu_h)$ $|\partial_K F|$-a.e. on $F^+ = F \cap \{t \geq g_r\}$ and $W^- = \pi_K(\nu_h)$ $|\partial_K F|$-a.e. on $F^- = F \cap \{t \leq g_r\}$. Here ν_h is the horizontal unit normal to F.
The proof is done by applying the divergence theorem to W^+ in F^+ and W^- in F^-, taking into account that

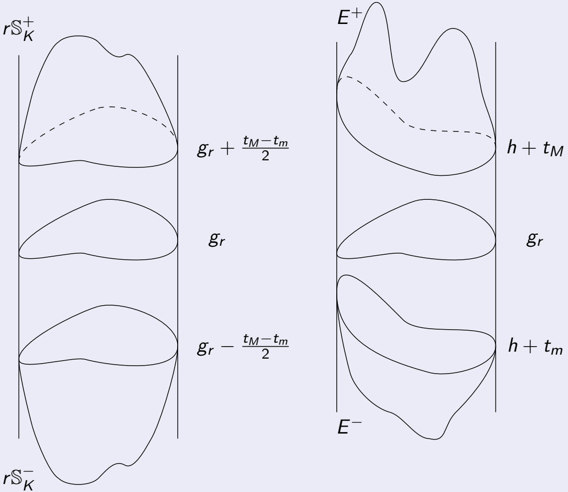
$$\text{div}\, W^{\pm} = \frac{1}{r}.$$

A suitable modification of $W^{\pm}$ must be done near the vertical line t-axis

Minimization properties

Proof

We apply the previous construction to these sets



Minimization properties

Proof

To get

$$\tfrac{1}{r}(|r\mathbb{B}_K| + |rK_0|(t_M - t_m)) = \int_D \langle W^+ - W^-, N_D \rangle dD$$
$$+ (t_M - t_m) \int_{\partial(rK_0)} \|\nu_0\|_* d\partial(rK0) + |\partial(r\mathbb{B}_K)|_K.$$

and

$$\tfrac{1}{r}(|E| + |rK_0|(t_M - t_m)) \leq \int_D \langle W^+ - W^-, N_D \rangle dD$$
$$+ (t_M - t_m) \int_{\partial(rK_0)} \|\nu_0\|_* d\partial(rK_0) + |\partial E|_K.$$

Proof

Hence

$$|\partial E|_K \geq |\partial(r\mathbb{B}_K)|_K + \tfrac{1}{r}(|E| - |r\mathbb{B}_K|).$$

Let $f(\rho) = |\partial(\rho\mathbb{B}_K)|_K + \frac{1}{\rho}(|E| - |\rho\mathbb{B}_K|)$. Since $\rho\mathbb{B}_K$ has mean curvature $\frac{1}{\rho}$, the first variation of the perimeter implies that the Wulff shape $\rho\mathbb{B}_K$ is a critical point of $|\partial\cdot|_K - \frac{1}{\rho}|\cdot|$ for any variation. Therefore $|\partial(\rho\mathbb{B}_K)|_K' - \frac{1}{\rho}|\rho\mathbb{B}_K|' = 0$. Hence

$$f'(\rho) = -\tfrac{1}{\rho^2}(|E| - |\rho\mathbb{B}_K|).$$

So the only critical point of f corresponds to the value ρ_0 so that $|\rho_0\mathbb{B}_K| = |E|$. Since the function $\rho \mapsto |\rho B_K|$ is strictly increasing and takes its values in $(0, +\infty)$, we obtain that $f(\rho)$ is a convex function with a unique minimum at ρ_0. Hence

$$|\partial E|_K \geq f(r) \geq f(\rho_0) = |\partial(\rho_0\mathbb{B}_K)|_K.$$

Open question

Is $\mathbb{B}_K$ a global K-perimeter minimizing set under a volume constraint?

Thanks for your attention