Tubular neighborhoods in the sub-Riemannian Heisenberg groups

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Introduction

Tubular neighborhoods in $\mathbb{R}^n$

Let $E \subset \mathbb{R}^n$, $r > 0$. Consider the (closed) tubular neighborhood of radius $r > 0$

$$E_r := \{p \in \mathbb{R}^n : d(p, E) \leq r\}.$$

Problem: Look at the regularity of $r \mapsto |E_r|$ for small r and find an expression for the volume $|E_r|$.

Introduction

Previous results

1. (Steiner, 1848): polytopes and uniformly convex sets with smooth boundary in $\mathbb{R}^2$ and $\mathbb{R}^3$
2. (Weyl, 1939): smooth submanifolds
3. (Federer, 1958): sets of positive reach

In any case

$$|E_r| = \sum_{i=0}^{n} a_i r^i$$

is a polynomial whose coefficients depend on the geometry of E: they depend on the Riemann tensor of E when E is a smooth submanifold; they are the well-known curvature measures for sets of positive reach (and convex sets)

Introduction

Previous results

E.g.: if E is a domain with smooth boundary then

$$|E_r| = |E| + \int_0^r \left\{ \int_{\partial E} \left(\prod_{i=1}^{n-1} (1 + t\kappa_i) \right) d(\partial E) \right\} dt.$$

So that the coefficients of $|E_r|$ are the integrals of the symmetric functions of the principal curvatures of ∂E

Applications

Isoperimetric inequalities (Heintze-Karcher type arguments)

Introduction

Problem

If $E \subset \mathbb{H}^n$ is a closed set and d_E is the (Carnot-Carathéodory) distance to E, study the regularity properties of d_E and, when E has smooth boundary, find an explicit expression for the volume of the tubular neighborhood

$$E_r := \{p \in \mathbb{H}^n : d_E(p) < r\}$$

in terms of r and geometric terms of ∂E

Introduction

Previous results

1. d_E is 1-lipschitz and thus $\mathbb{H}$-differentiable a.e. (Pansu-Rademacher's Theorem)
2. d_S is C^{k-1} when $S \subset \mathbb{H}^1$ is a C^k surface out of the singular set where the tangent plane to S is horizontal (Arcozzi & Ferrari, 2007)
3. Hessian of d_S (Arcozzi & Ferrari, 2008)
4. Steiner's formula for surfaces in $\mathbb{H}^1$ out of the singular set (Balogh, Ferrari, Franchi, Vecchi, Wildrick, 2015, taking iterated divergences of the distance function), (Ferrari, 2007, using the flow of the horizontal gradient of the distance function)
5. C^k regularity of d_S when S is a hypersurface in *special* step-2 Carnot groups (Arcozzi, Ferrari & Montefalcone, 2017)
6. Applications by Arcozzi (2012) and to geometric inequalities by Ferrari & Valdinoci (2009)

Introduction (notation)

The Heisenberg group $\mathbb{H}^n$

$\mathbb{R}^{2n+1} \equiv \mathbb{C}^n \times \mathbb{R}$, with product

$$(z,t) * (w,s) = (z+w, t+s+\sum_{i=1}^{n} \operatorname{Im}(z_i \bar{w}_i)).$$

Horizontal distribution $\mathcal{H}$: generated by left-invariant ($z_j = x_j + iy_j$)

$$X_i := \frac{\partial}{\partial x_i} + y_i \frac{\partial}{\partial t}, \qquad Y_i := \frac{\partial}{\partial y_i} - x_i \frac{\partial}{\partial t}, \qquad i = 1, \ldots, n.$$

Carnot-Carathéodory distance

Take sub-Riemannian h on $\mathcal{H}$ making X_i, Y_i orthonormal

$d(p,q) := \inf\{L(\gamma) : \gamma : I \to \mathbb{H}^n$ piecewise smooth horizontal joining $p, q\}$

Geometric properties

$(\mathbb{H}^n, \mathcal{H}, h)$ is a sub-Riemannian manifold

Reeb vector field $T := \dfrac{\partial}{\partial t}$

$g = \langle \cdot, \cdot \rangle$ Riemannian metric making X_i, Y_i, T orthonormal (extends h)

∇ pseudohermitian connection: the only metric connection with torsion $\mathrm{Tor}(X, Y) = 2\langle J(X), Y\rangle T$ (left-invariant vector fields are parallel). J is the 90-degrees rotation in the horizontal distribution

Metric properties

$(\mathbb{H}^n, d)$ is a complete metric space

Two given points can be joined by a length-minimizing geodesic $\gamma : I \to \mathbb{H}^n$: a horizontal curve satisfying the equation

$$\nabla_{\dot\gamma}\dot\gamma + \lambda J(\dot\gamma) = 0 \qquad (*)$$

$\lambda \in \mathbb{R}$ is the curvature of the geodesic

(*) is explicitly solvable. Solutions are horizontal straight lines when $\lambda = 0$ and lifting of circles in the $t = 0$ hyperplane when $\lambda \neq 0$

Given $p \in \mathbb{H}^n, v \in \mathcal{H}_p, \lambda \in \mathbb{R}$, there exists a unique geodesic $\gamma^\lambda_{p,v}$ satisfying (*) with initial conditions $\gamma^\lambda_{p,v}(0) = p$, $\dot\gamma^\lambda_{p,v}(0) = v$.

Explicit expresions

$$\gamma^{\lambda}_{p,v}(s) = (\alpha(s), \beta(s)),$$

where

$$\begin{aligned} \alpha(s) &= \pi(p) + s\big(F(\lambda s)\, w - G(\lambda s)\, J(w)\big), \\ \beta(s) &= t(p) + |\dot\gamma(0)|^2 s^2 H(\lambda s) + \langle \pi(p), s\big(G(\lambda s)\, w + F(\lambda s)\, J(w)\big)\rangle. \end{aligned}$$

$\pi : \mathbb{H}^n \to \{t = 0\}$ is the standard projection and the functions F, G, H are the analytic ones

$$F(x) := \frac{\sin(x)}{x}, \quad G(x) := \frac{1 - \cos(x)}{x}, \quad H(x) := \frac{x - \sin(x)}{x^2}.$$

Thus $\gamma^{\lambda}_{p,v}(s)$ depends analytically on p, v, s and λ (w are the coordinates of $v = \dot\gamma(0)$ in the basis $(X_i, Y_i)_i$) (Hajłasz-Zimmerman, 2015)

Regularity of the distance function

Singular set of a submanifold

Let $S \subset \mathbb{H}^n$ be an m-dimensional submanifold of class C^1. The singular set S_0 is composed of the points $p \in S$ such that $T_pS \subset \mathcal{H}_p$.

Theorem (–)

Let $S \subset \mathbb{H}^n$ be an m-dimensional submanifold of class C^k, where $k \geq 2$ and $1 \leq m \leq 2n$, and let $K \subset S \setminus S_0$ a compact subset. Then there exists an open neighborhood Ω of K such that the distance function d_S is of class C^k in Ω $(\Omega \cap S_0 = \emptyset)$.

Key points in the proof

1. Define a normal exponential map
2. Use the Implicit Function Theorem (standard argument)

Defining a normal exponential map

Given $q \in E$, the tangent cone $\mathrm{Tan}(E,q)$ is the closure of the set of vectors $v \in T_q\mathbb{H}^n$ such that there exists a C^1 curve $\alpha : [0,\varepsilon) \to \mathbb{H}^n$ such that $\alpha(0) = q$, $\dot\alpha(s) = v$ and $\alpha(s) \in E$ for a.e. $s \in [0,\varepsilon)$.

The normal horizontal cone $\mathrm{Nor}_H(E,q)$ is the set

$$\{u \in T_q\mathbb{H}^n : \langle u, v\rangle \leq 0 \ \forall v \in \mathrm{Tan}(E,q) \cap \mathcal{H}_q\}.$$

1. If S is a submanifold then $\mathrm{Tan}(S,q) = T_qS$
2. If q is interior to E then $\mathrm{Tan}(E,q) = T_q\mathbb{H}^n$ and $\mathrm{Nor}_H(E,q) = \{0\}$.
3. If S is a hypersurface and $q \in S \setminus S_0$ then $\mathrm{Nor}_H(S,q) = \{\lambda\nu : \lambda \geq 0\}$, where ν is the outward pointing horizontal unit normal. If $q \in S_0$ then $\mathrm{Nor}_H(S,q) = \{0\}$.

Lemma

Let $E \subset \mathbb{H}^n$ be a closed set. Take $p \notin E$ and $q \in E$ such that $d_E(p) = d(p,q)$. Let $\gamma : [0,a] \to \mathbb{H}^n$ be a length-minimizing geodesic of curvature λ joining q and p. Then

1. $\dot\gamma(0) \in \text{Nor}_H(E,q)$.
2. The curvature λ of the geodesic γ lies in the interval

$$\left[\sup_{\substack{v\in \text{Tan}(E,q)\\ \langle v, T_q\rangle <0}} \frac{-2\langle v, \dot\gamma(0)\rangle}{\langle v, T_q\rangle}, \inf_{\substack{v\in \text{Tan}(E,q)\\ \langle v, T_q\rangle >0}} \frac{-2\langle v, \dot\gamma(0)\rangle}{\langle v, T_q\rangle} \right]$$

Sketch of proof

Let $\alpha : [0, \varepsilon_0) \to E$ be a C^1 curve with $\alpha(0) = q$, $\alpha'(0) = v$ such that $\alpha(s) \in E$ for a.e. s ($v \in \mathrm{Tan}(E, q)$)

We construct a variation of γ by smooth horizontal curves $\gamma_\varepsilon : [0, a] \to \mathbb{H}^n$ joining $\alpha(\varepsilon)$ and p. Then $L(\gamma_s) \geq d_E(p)$ for a.e. s and $L(\gamma_0) = d_E(p)$. Taking derivative w.r.t. s and evaluating at $s = 0$

$$0 \leq -\langle v, \dot\gamma(0) + \tfrac{\lambda}{2}\, T_q\rangle.$$

Taking $v \in \mathrm{Tan}(E, q) \cap \mathcal{H}_q$ arbitrary, we conclude that $\dot\gamma(0) \in \mathrm{Nor}_H(E, q)$

For any $v \in \mathrm{Tan}(E, q)$, we have

$$\lambda \, \langle v, T_q\rangle \leq -2\, \langle v, \dot\gamma(0)\rangle,$$

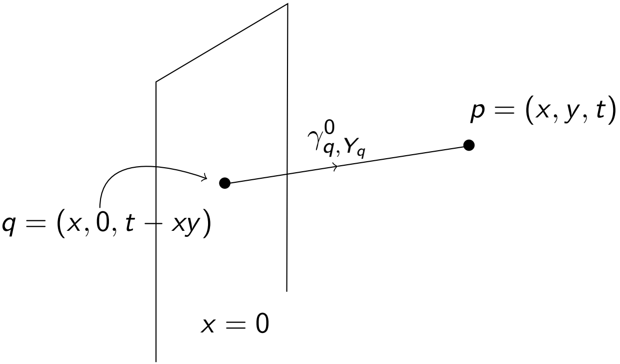
Theorem

Let $E \subset \mathbb{H}^n$ be a closed subset with C^1 boundary S. Let N be the outer unit normal to S and ν the corresponding horizontal unit normal. Take $p \notin E$ and $q \in S$ such that $d_E(p) = d(p, q)$, and consider a minimizing geodesic $\gamma : [0, a] \to \mathbb{H}^n$ of curvature λ connecting q and p. Then

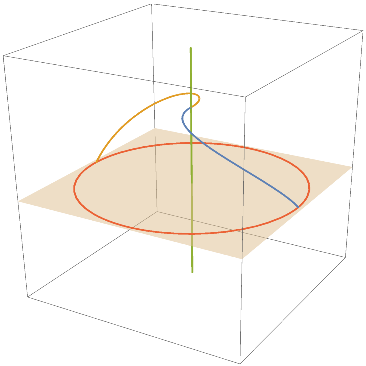
1. q is a regular point of S,
2. $\dot{\gamma}(0) = \nu_q$, and
3. the curvature of γ is given by

$$\lambda = \frac{2\langle N, T\rangle}{|N_h|}(q).$$

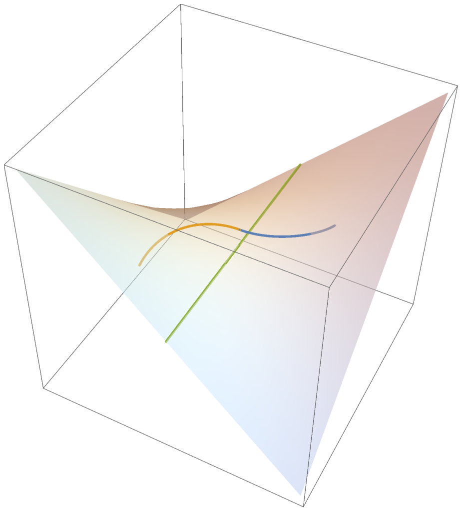
Moreover, in case N is a Euclidean lipschitz vector field, the function λ is locally lipschitz in $S \setminus S_0$.



All length-minimizing geodesics leaving a vertical plane have curvature $\lambda = 0$. An intrinsic graph in the sense of Franchi, Serapioni and Serra-Cassano is thus a metric graph



Length-minimizing geodesics leaving the plane $t = 0$



Length-minimizing geodesics leaving the surface $t = xy$

Theorem

Let $S \subset \mathbb{H}^n$ be an m-dimensional submanifold of class C^1. Let $p \notin S$ and assume that $q \in S$ satisfies $d_S(p) = d(p,q)$.

Then there exists a length-minimizing geodesic $\gamma : [0,a] \to \mathbb{H}^n$ of curvature λ, parameterized by arc-length, joining q and p such that

1. $\dot\gamma(0) \perp T_qS \cap \mathcal{H}_q$.
2. If $q \in S \setminus S_0$, the curvature λ of γ is given by

$$\lambda = -\frac{2\langle E_q, \dot\gamma(0)\rangle}{\langle E_q, T_q\rangle},$$

   where $E_q \in T_qS$ is a unit vector orthogonal to $T_qS \cap \mathcal{H}_q$.

Steiner's formula

Strategy to obtain Steiner's formula

The classical one:

1. Use Federer's coarea formula applied to the distance function
2. Compute the integral on the level sets using Jacobi fields

Jacobi fields (Rumin, 1994; Chanillo & Yang, 2009)

They are produced by variations by geodesics. Taking a smooth family of geodesics $\{\gamma_\varepsilon\}$ with $\gamma_0 = \gamma$ and differentiating the geodesic equation $\nabla_{\dot\gamma_\varepsilon}\dot\gamma_\varepsilon + \lambda_\varepsilon J(\dot\gamma_\varepsilon) = 0$ with respect to ε we get

$$\ddot U + \lambda J(\dot U) + \lambda' J(\dot\gamma) - 2\dot\gamma \langle U, J(\dot\gamma)\rangle\, T = 0.$$

where $U = \partial\gamma_\varepsilon/\partial\varepsilon$, $\lambda' = d\lambda_\varepsilon/d\varepsilon|_{\varepsilon=0}$ and $\dot U = \nabla_{\dot\gamma} U$.

It is a second order differential equation whose solutions are explicitly computable and depends on $U(0)$ and $\dot U(0)$.

If we consider a family of length-minimizing geodesics leaving a given hypersurface, the corresponding Jacobi field satisfies

$$\dot U(0) = \nabla_{U(0)}\nu_h + 2\langle J(\nu_h), U(0)\rangle\, T$$

A horizontal second fundamental form

If S is a hypersurface, the horizontal unit normal ν_h is defined out of the singular set S_0. For $p \in S \setminus S_0$, the endomorphism $A_p : T_pS \cap \mathcal{H}_p \to T_pS \cap \mathcal{H}_p$ defined by:

$$A(u) = -\nabla_u \nu_h - \frac{\langle N, T\rangle}{|N_h|} J(u)_{ht},$$

is selfadjoint. The subscript *ht* denotes the tangent horizontal projection. Its eigenvalues are the horizontal principal curvatures (–,2012). Studied by (Cheng, Chiu, Hwang & Yang, 2015).

Steiner's formula for surfaces in $\mathbb{H}^1$

Let $S \subset \mathbb{H}^1$ be a surface of class C^k, $k \geq 2$, bounding a closed region E, and let $U \subset S$ be an open subset such that $\overline{U} \subset S \setminus S_0$. For $r > 0$ small, the volume of the one-side tubular neighborhood U_r is given by

$$|U_r| = \sum_{i=0}^{4} \int_U \left\{ \int_0^r a_i f_i(\lambda, s)\, ds \right\} dS,$$

where λ is the function $2\langle N, T\rangle/|N_h|$, defined on $S \setminus S_0$, the functions f_i are explicit trigonometric functions, and the coefficients a_i are given by the expressions

$$a_0 = |N_h|, \quad a_1 = |N_h|H, \quad a_2 = -4|N_h|e_1\left(\frac{\langle N, T\rangle}{|N_h|}\right),$$

$$a_3 = -4e_2\left(\frac{\langle N, T\rangle}{|N_h|}\right), \quad a_4 = -4He_2\left(\frac{\langle N, T\rangle}{|N_h|}\right) - 4|N_h|\left(e_1\left(\frac{\langle N, T\rangle}{|N_h|}\right)\right)^2.$$

The functions f_i

$$f_0(\lambda, s) := \cos(\lambda s), \quad f_1(\lambda, s) := F_1(\lambda s)s, \quad f_2(\lambda, s) := F_2(\lambda s)s^2,$$
$$f_3(\lambda, s) := F_3(\lambda s)s^3, \quad f_4(\lambda, s) := F_4(\lambda s)s^4,$$

$$F_1(x) := \frac{\sin(x)}{x}, \quad F_2(x) := \frac{1-\cos(x)}{x^2},$$
$$F_3(x) := \frac{\sin(x) - x\cos(x)}{x^3}, \quad F_4(x) := \frac{2 - 2\cos(x) - x\sin(x)}{x^4},$$

Steiner's formula for hypersurfaces in $\mathbb{H}^n$

Let $S \subset \mathbb{H}^n$, $n \geq 2$, be a hypersurface of class C^k, $k \geq 2$, and let $U \subset S$ be an open subset such that $\overline{U} \subset S \setminus S_0$. For $r > 0$ small, the volume of the tubular neighborhood U_r is given by

$$|U_r| = \int_U \left\{ \int_0^r |\det(B(s))| \, ds \right\} dS,$$

where $B(s)$ is a explicitly computable matrix in terms of Jacobi fields. The function $|\det(B(s))|$ is an analytic function of λ and s multiplied by coefficients involving $\langle N, T\rangle/|N_h|$, $|N_h|$, the horizontal gradient in S of the function $\langle N, T\rangle/|N_h|$ and the principal curvatures of the horizontal second fundamental form.

The matrix B

$$B = \begin{pmatrix} \frac{1}{2}\dot{c}_1 & c_1 & \langle E_1, X_2\rangle & \langle E_1, Y_2\rangle & \dots & \langle E_1, X_n\rangle & \langle E_1, Y_n\rangle \\ \frac{1}{2}\dot{c}_2 & c_2 & \langle E_2, X_2\rangle & \langle E_2, Y_2\rangle & \dots & \langle E_2, X_n\rangle & \langle E_2, Y_n\rangle \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ \frac{1}{2}\dot{c}_i & c_i & \langle E_i, X_2\rangle & \langle E_i, Y_2\rangle & \dots & \langle E_i, X_n\rangle & \langle E_i, Y_n\rangle \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \\ \frac{1}{2}\dot{c}_{2n} & c_{2n} & \langle E_{2n}, X_2\rangle & \langle E_{2n}, Y_2\rangle & \dots & \langle E_{2n}, X_{2n}\rangle & \langle E_{2n}, Y_{2n}\rangle \end{pmatrix}.$$

Taylor's expansion

$$
\begin{aligned}
|U_r| = A(U)r + \frac{1}{2}\bigg(\int_U H dP\bigg) r^2 \\
- \frac{1}{6}\bigg(\int_U \bigg(4e_1\bigg(\frac{\langle N, T\rangle}{|N_h|}\bigg) + (2n+2)\bigg(\frac{\langle N, T\rangle}{|N_h|}\bigg)^2 + |\sigma|^2 - H^2\bigg) dP\bigg) r^3 \\
+ o(r^4),
\end{aligned}
$$

where $e_1 = J(\nu)$, $|\sigma|^2 = \sum_i \kappa_i^2$.

Thanks for your attention