Pansu-Wulff shapes in the Heisenberg group $\mathbb{H}^1$

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(joint work with Julián Pozuelo and Gianmarco Giovannardi)

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Introduction

Wulff shapes in $\mathbb{R}^n$

Given an (asymmetric) norm $||\cdot||_K$ whose unit ball is a convex body $K \subset \mathbb{R}^n$, the area of an oriented Lipschitz boundary S is

$$A_K(S) = \int_S ||N||_{K,*} dS, \qquad (*)$$

where N is the outer unit normal to S and $||u||_{K,*} = \sup_{v \in K} \langle u, v \rangle$ is the dual norm of K.

(*) is an anisotropic energy used to model the shape of an equilibrium crystal minimizing Gibbs' free energy (1875). The solution to the problem for polyhedra was described by Wulff (1895).

Introduction

Wulff shapes in $\mathbb{R}^n$

If $E = \partial S$ and S is regular enough, $A_K(S)$ coincides with the Minkowski content $M(E, K)$ associated to K

$$M(E,K) = \liminf_{r \to 0} \frac{|E + rK| - |E|}{r}.$$

Brunn-Minkowski inequality then implies

$$|E + rK|^{1/n} - |E|^{1/n} \geq r|K|^{1/n}$$

Dividing by r and taking limits when $r \to 0$, since $M(K,K) = n|K|$,

$$\frac{M(E,K)}{|E|^{(n-1)/n}} \geq \frac{M(K,K)}{|K|^{(n-1)/n}}.$$

So K minimizes the functional for given volume. The set K is known as the Wulff shape for the functional (*)

Introduction

Wulff shapes in $\mathbb{R}^n$

- The functional (*) is used in crystallography (Gibbs free energy). Wulff gave a construction to obtain K from the dual norm $||\cdot||_{K,*}$
- Use of Brunn-Minkowski to obtain a solution by Dinghas (1944)
- Mathematical problem considered by Taylor (1978), Fonseca (1991) and Fonseca-Müller (1991)

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Goal: explore a similar problem in the first Heisenberg group $\mathbb{H}^1$

- J. Pozuelo, –, Pansu-Wulff shapes in $\mathbb{H}^1$. Adv. Calc. Var. (to appear).
- G. Giovannardi, –, Regularity of Lipschitz boundaries with prescribed sub-Finsler mean curvature in the Heisenberg group. J. Differential Equations, 2021.

Introduction

The Heisenberg group $\mathbb{H}^1$

$(\mathbb{R}^3, *)$, where $*$ is the product

$$(x, y, t) * (x', y', t') := (x + x', y + x', t + t' + (x'y - xy')),$$

A basis of left invariant vector fields is given by

$$X = \frac{\partial}{\partial x} + y\,\frac{\partial}{\partial t}, \qquad Y = \frac{\partial}{\partial y} - x\,\frac{\partial}{\partial t}, \qquad T = \frac{\partial}{\partial t}.$$

X, Y generate a non-integrable horizontal distribution $\mathcal{H}$ ($[X, Y] = -2T$).

Sub-Finsler norms in $\mathbb{H}^1$

A planar norm $||\cdot||_K$ ($K \subset \mathbb{R}^2$) is extended to a left-invariant norm in $\mathcal{H}$

$$(||fX + gY||_K)_p = ||(f(p), g(p))||_K.$$

Introduction

Sub-Finsler perimeter in $\mathbb{H}^1$

Let $E \subset \mathbb{H}^1$ be a measurable set, $||\cdot||_K$ the left-invariant norm associated to $K \subset \mathbb{R}^2$, and $\Omega \subset \mathbb{H}^1$ an open subset. We say that E has locally finite K-perimeter in Ω if for any relatively compact open set $V \subset \Omega$ we have

$$|\partial E|_K(V) = \sup\left\{ \int_E \operatorname{div}(U)\, d\mathbb{H}^1 : U \in \mathcal{H}_0^1(V), ||U||_{K,\infty} \leq 1 \right\} < +\infty.$$

$|\partial E|_K(V)$ is the K-perimeter of E in V.

Remark

If K is the closed unit disc D centered at 0, $|\partial_D E|$ is the classical sub-Riemannian perimeter of E.

Introduction

Sub-Finsler perimeter in $\mathbb{H}^1$

If E has boundary S of class C^1 or Euclidean Lipschitz

$$|\partial E|_K(V) = \int_{S\cap V} ||N_h||_{K,*}\, dS$$

where N_h is the horizontal projection of the outer unit normal N and dS is the Riemannian measure of S (computed with the Riemannian metric g so that X, Y, T is orthonormal)

Introduction

Problem

Minimize K-perimeter (eventually under a volume constraint): existence, regularity and characterization problems

Previous work

- A.P. Sánchez, Ph.D. Thesis, Tufts U., 2017.
- The case $K = D$ (sub-Riemannian case) extensively considered

Existence

- By Franceschi, Monti, Righini, and Sigalotti, arXiv:2007.11384, following Leonardi and Rigot proof of existence in Carnot groups
- Extended by Pozuelo to sub-Finsler nilpotent groups, arXiv:2103.06630

First variation of perimeter for C^2 boundaries

The singular set

Given a C^1 surface $S \subset \mathbb{H}^1$, its singular set $S_0 \subset S$ is the set of points where T_pS coincides with the horizontal distribution ($N_h = 0$).

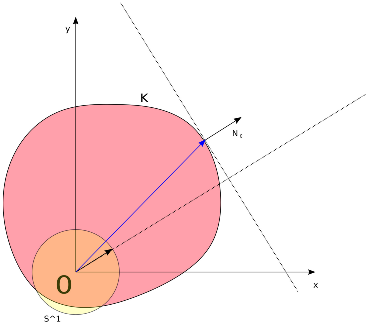
$$A_K(S_0) = \int_{S_0} ||N_h||_{K,*} dS = 0$$

The regular set $S \setminus S_0$ is foliated by horizontal curves

The class C^2_+

We say that a convex body is of class C^2_+ if the boundary of K is C^2 and has positive curvature everywhere.

If K is of class C^2_+, for every $u \neq 0$ there is a unique $\pi_K(u)$ such that $\langle \pi_K(u), u \rangle = ||u||_{K,*}$.



The map π_K. It is geometrically clear that $\pi_K = N_K^{-1}$

Theorem (Pozuelo,–, 2020)

Let S be a C^2 surface in $\mathbb{H}^1$. Let U be a C^2 vector field with compact support on $S \setminus S_0$, and $\{\varphi_s\}_{s\in\mathbb{R}}$ the associated flow. Let $\eta = \pi_K(N_h)$. Then

$$\frac{d}{ds}\bigg|_{s=0} A(\varphi_s(S)) = \int_S \langle U, N\rangle \langle D_Z\eta, Z\rangle \, dS.$$

K-mean curvature H_K

We let $H_K = \langle D_Z\eta, Z\rangle$. This is an ODE along horizontal curves in $S \setminus S_0$

Theorem (Pozuelo,–,2020)

Let S be a C^2 surface without singular points and constant mean curvature $H_K > 0$. Then S is foliated by horizontal liftings of translations of the circle $||\cdot||_K = 1/H_k$.

The Pansu-Wulff shapes

Definition

Let K of class C^2_+ and parametrize ∂K by an L-periodic curve $\gamma : \mathbb{R} \to \mathbb{R}^2$. For any $u \in \mathbb{R}$ consider the horizontal lifting $\Gamma_{\gamma(u)} : \mathbb{R} \to \mathbb{H}^1$ of the curve $t_{-\gamma(u)}(\gamma)$ with initial point $(0,0,0)$. Define

$$\mathbb{B}_K = \bigcup_{u \in [0,L)} \Gamma_{\gamma(u)}([u, u+L]).$$

We shall refer to $\mathbb{B}_K$ as the *Pansu-Wulff shape* associated to the left-invariant norm $||\cdot||_K$. Its boundary $\mathbb{S}_K = \partial\mathbb{B}_K$ will be called the *Pansu-Wulff sphere*.

In the sub-Riemannian case, the corresponding sphere is known as Pansu sphere.

The Pansu-Wulff shapes (regularity properties)

Parameterization of the Wulff shape

Given any convex body $K \subset \mathbb{R}^2$ with $0 \in \mathrm{int}(K)$ we parameterize ∂K as

$$\gamma(s) = \big(x(s), y(s)\big) = r(s)\,\big(\sin(s), \cos(s)\big), \quad s \in \mathbb{R}.$$

where $r(s) = \rho(\sin(s), \cos(s))$ and ρ is the radial function of K.

Then we have the following parameterization of $\mathbb{S}_K$.

$$\begin{aligned} x(u,v) &= r(u+v)\sin(u+v) - r(v)\sin(v), \\ y(u,v) &= r(u+v)\cos(u+v) - r(v)\cos(v), \\ t(u,v) &= r(v)r(u+v)\big(\sin(v)\cos(u+v) - \cos(v)\sin(u+v)\big) \\ &\qquad + \int_v^{u+v} r^2(\xi)\,d\xi. \end{aligned}$$

Regularity properties follow from this expression. Also convergence in Hausdorff distance of Wulff shapes

The Pansu-Wulff shape

Geometric properties

Let $K \subset \mathbb{R}^n$ be a convex body with $0 \in \text{int}(K)$. We consider the set

$$K_0 = \bigcup_{p \in \partial K} (-p + K).$$

Then we have

1. $0 \in K_0$.
2. K_0 is a convex body.
3. K_0 is the difference body $K - K$ of K. In particular, K_0 is centrally symmetric.
4. If K is centrally symmetric then $K_0 = 2K$.
5. We have
$$\bigcup_{p \in \partial K} (-p + K) = \bigcup_{p \in \partial K} (-p + \partial K).$$

The Pansu-Wulff shapes

The centrally symmetric case

If K is centrally symmetric we have the additional properties

- The projection of $\mathbb{B}_K$ to the plane $t = 0$ is $K_0 = 2K$
- $\mathbb{B}_K$ is symmetric with respect to a horizontal plane (Euclidean symmetry)

The general case

- The projection of $\mathbb{B}_K$ to the plane $t = 0$ is the difference body $K - K$, that it is centrally symmetric
- $\mathbb{B}_K$ is not necessarily symmetric with respect to a horizontal plane
- $\mathbb{B}_K$ is the union of two graphs g_1, g_2 of class C^2 outside the poles defined over K_0. Moreover $g_1 < g_2$ on $\text{int}(K_0)$ and $g_1 = g_2$ on ∂K_0. The graph of the function $g = (g_1 + g_2)/2$ separates $\mathbb{S}_K$ into two pieces $\mathbb{S}_K^+$ and $\mathbb{S}_K^-$.

The Pansu-Wulff shapes

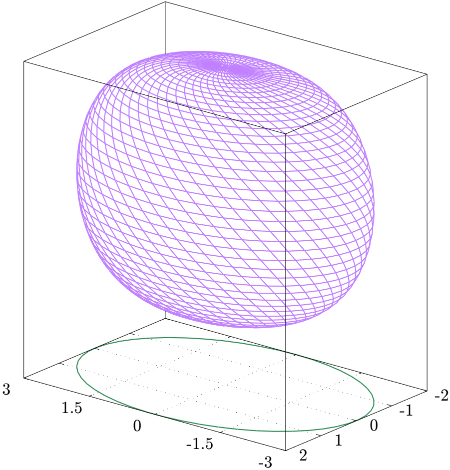


Figure: The Wulff shape associated to the norm $||\cdot||_a = ((x_1/a_1)^2 + (x_2/a_2)^2)^{1/2}$ with $a = (1, 1.5)$. Observe that the projection to the horizontal plane $t = 0$ is an ellipse with semiaxes of lengths 2 and 3.

The Pansu-Wulff shapes

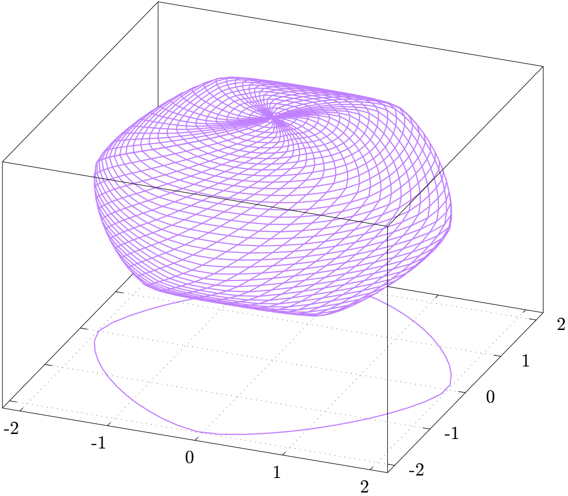


Figure: The Wulff shape $\mathbb{S}_{K_r}$ for the r-norm, $r = 1.5$. The horizontal curve is the projection of the equator to the plane $t = 0$. Since the r-norm is symmetric, the Wulff shape projects to the set $||\cdot||_r \leq 2$ in the $t = 0$ plane.

The Pansu-Wulff shapes

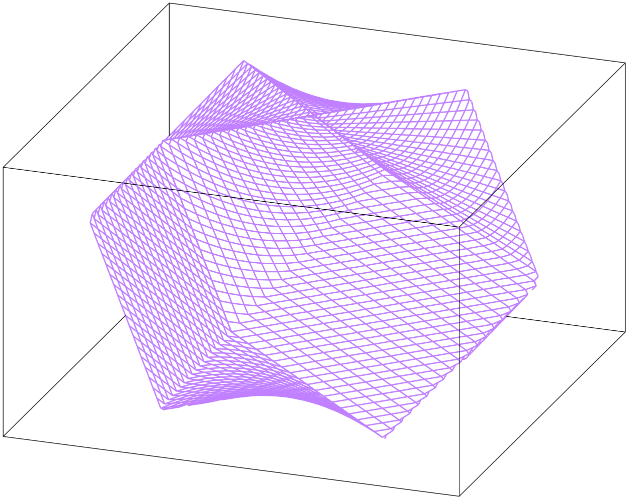


Figure: The ball $\mathbb{B}_1$ obtained as Hausdorff limit of the Wulff shapes $\mathbb{B}_{K_r}$ of the r-norm when r converges to 1

The Pansu-Wulff shapes

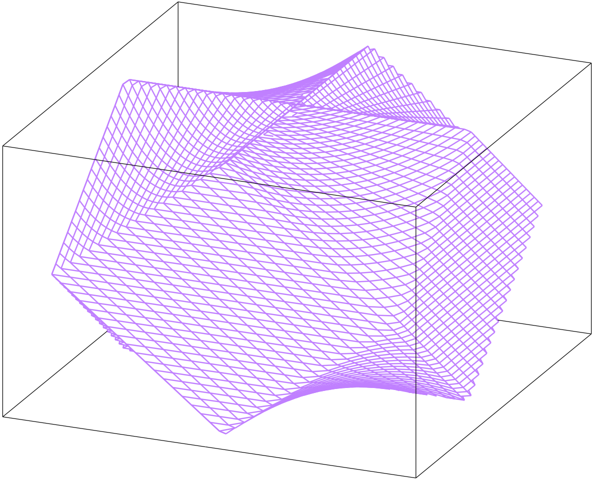


Figure: The ball $\mathbb{B}_\infty$ obtained as Hausdorff limit of the Wulff shapes $\mathbb{B}_{K_r}$ of the r-norm when r converges to ∞

The Pansu-Wulff shapes

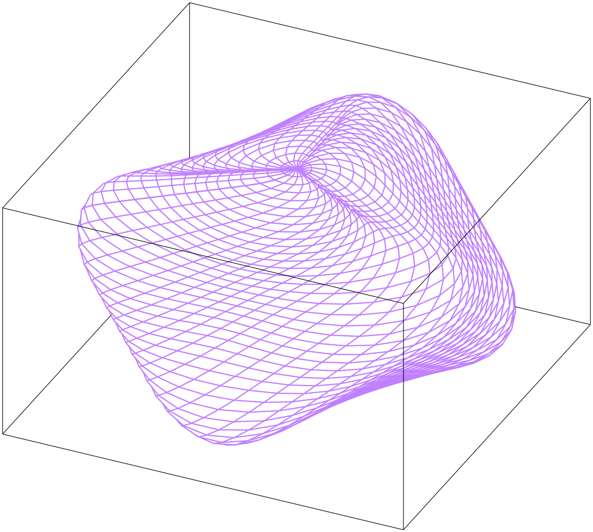


Figure: The Wulff shape $\mathbb{B}_{T,r}$ for the norm $||\cdot||_{T,r}$, with $r = 2$.

The Pansu-Wulff shapes

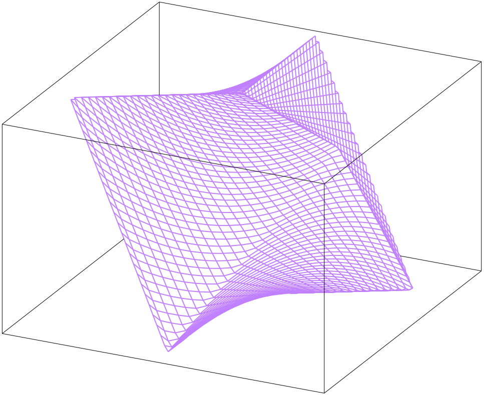


Figure: The ball $\mathbb{B}_T$ obtained as limit of the Wulff shapes $\mathbb{B}_{T,r}$ when $r \to \infty$.

Pansu-Wulff shapes: minimization properties

(Franceschi et al.)

Pansu-Wulff boundaries are the only C^2 stationary points of area under a volume constraint. Proof after Ritoré-Rosales (2005)

Theorem (Pozuelo,–,2020)

Let $||\cdot||_K$ be the norm associated to an strictly convex body $K \subset \mathbb{R}^2$ with C^2 boundary. Let $r > 0$ and $h : rK_0 \to \mathbb{R}$ a C^0 function. Consider a subset $E \subset \mathbb{H}^1$ with finite volume and K-perimeter such that

$$\text{graph}(h) \subseteq E \subset rK_0 \times \mathbb{R}.$$

Then

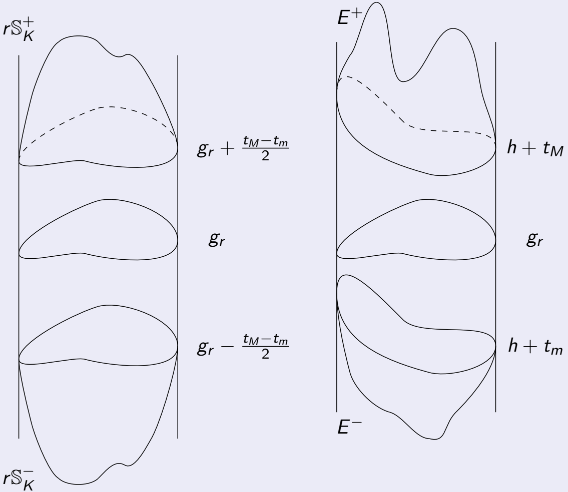
$$|\partial E|_K \geq |\partial \mathbb{B}_E|_K,$$

where $\mathbb{B}_E$ is the Wulff shape with the same volume as E.

Pansu-Wulff shapes: minimization properties

Proof

We apply a calibration argument



Proof

to prove

$$|\partial E|_K - \frac{1}{r}\,|E| \geq |\partial(r\mathbb{B}_K)|_K - \frac{1}{r}|r\mathbb{B}_K|.$$

Calling

$$f(r) = |\partial(r\mathbb{B}_K)|_K + \frac{1}{r}\big(|E| - |r\mathbb{B}_K|\big)$$

we use an old idea by E. Schmidt to get

$$|\partial E|_K \geq f(r) \geq f(\rho_0) = |\partial(\rho_0\mathbb{B}_K)|$$

since the minimum of $f(r)$ is achieved for the ball $\rho_0\mathbb{B}_K$ with the same volume as E.

Regularity properties

Singular set

For C^1 and C^2 surfaces, the arguments by Cheng, Hwang, Malchiodi and Yang (2005,2012) in the sub-Riemannian case can be applied.

Sets with prescribed mean curvature

Given $f \in C(\Omega)$, $E \subset \Omega$ has prescribed K-mean curvature f in $\Omega \Leftrightarrow E$ is a critical point of the functional

$$B \mapsto |\partial E|_K(B) - \int_{E\cap B} f \, d\mathbb{H}^1, \quad \forall\, B \subset \Omega \text{ bounded open set} \tag{1}$$

If $S = \partial E \cap \Omega$ is a Euclidean Lipschitz surface then (1) $\Leftrightarrow E$ is a critical point of the functional

$$B \mapsto A_K(S \cap B) - \int_{E\cap B} f \, d\mathbb{H}^1, \quad \forall\, B \subset \Omega \text{ bounded open set}$$

Remarks

- Perimeter-minimizing sets have prescribed K-mean curvature 0
- Isoperimetric boundaries have prescribed constant K-mean curvature

Regularity of the non-singular set (Giovannardi,–,2021)

Let K be a convex body of class C^2_+, $\Omega \subset \mathbb{H}^1$ an open set and $E \subset \Omega$ a set of prescribed K- mean curvature $f \in C^0(\Omega)$ with Euclidean Lipschitz and $\mathbb{H}$-regular boundary S. Then the horizontal curves Γ of $S \cap \Omega$ are of class C^2.

$\mathbb{H}$-regularity of the boundary means that the surface is locally the level set of a continuous function with non-vanishing continuous horizontal gradient.

Proof

Quite technical and follows from localizing the first variation along horizontal curves of the surface

One of the best Spanish scientists of all times, Ramón y Cajal, wrote his memories in two volumes. The first one, *Story of my childhood and youth*, covers from birth to the age of 40. The second part, *Story of my scientific labour*, the rest of his life.

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Wish Ernst all the best in the beginning of his adult life

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Thanks for your attention